Arc-Sin Transformation for Binomial Sample Proportions in Small Area Estimation

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Supplementary Material

The supplemental material provides the proves of Theorems 1-3. For this purpose, we first introduce several results which help to proving to the lemmas and theorems hereafter. We define $Z^* = Z - X\beta \sim N(0, V)$.

$$\hat{\theta}_i^B = (1 - B_i)\nu_i' Z + B_i x_i' \beta = a_{1i}(A)' Z^* + x_i' \beta, \tag{S0.1}$$

$$\hat{\theta}_i^{EB} = (1 - \hat{B}_i)\nu_i' Z + \hat{B}_i x_i' \hat{\beta} = (a_{1i}(\hat{A}) + a_{2i}(\hat{A}))' Z^* + x_i' \beta, \tag{S0.2}$$

where $a_{1i}(A) = (1 - B_i)\nu_i$, $a_{2i}(A) = B_i(x_i'(X'V^{-1}X)^{-1}X'V^{-1})'$, ν_i being the n-dimansional vector of which i-th component is one while others are zero.

The above $a_{1i}(A)$ and $a_{2i}(A)$ are used throughout our proofs and we have under the regularity conditions R1–R3, for large m,

$$a_{1i}'Va_{1i} = (1-B_i)^2\nu_i'V\nu_i = \frac{A^2}{A+D_i} = A - g_{1i}(A), \tag{S0.3}$$

$$a_{2i}'Va_{2i} = B_i^2x_i'(X'V^{-1}X)^{-1}x_i = g_{2i}(A) = O(m^{-1}), \tag{S0.4}$$

$$a_{2i}'Va_{1i} = B_i(1-B_i)x_i'(X'V^{-1}X)^{-1}x_i = O(m^{-1}), \tag{S0.5}$$

$$i_uVa_{1i} = i_uA\nu_i, \tag{S0.6}$$

where $D_i = 1/(4n_i)$ and $i_u = \sqrt{-1}$. Note that regularity conditions are given in the main manuscript.

We also prove two more lemmas with some additional notations for proofs of theorems. Specifically, let

$$\begin{aligned}
\hat{A}_+ =& \hat{A}(Z^* + i_uVa_{1i}),\ \hat{A}_- = \hat{A}(Z^* - i_uVa_{1i}),\\
\hat{\theta}_{i+}^B =& \hat{\theta}_i^B(Z^* + i_uVa_{1i}),\ \hat{\theta}_{i-}^B = \hat{\theta}_i^{EB}(Z^* - i_uVa_{1i}),\\
\hat{\theta}_{i+}^{EB} =& \hat{\theta}_i^{EB}(Z^* + i_uVa_{1i}),\ \hat{\theta}_{i-}^{EB} = \hat{\theta}_i^{EB}(Z^* - i_uVa_{1i}),
\end{aligned}$$

where $i_u = \sqrt{-1}$.

Lemma 2. *Let $Z^* \equiv Z - X\beta \sim N(0,V)$, then we have under the regularity conditions R1-R3,*

(i) $E[(\hat{A}_+ - A)^2] = V_A + o(m^{-1})$,

(ii) $E[(\hat{A}_+ - A)(\hat{\theta}_{i+}^{EB} - \hat{\theta}_{i+}^B)] = i_uV_AB_i(1-B_i) + o(m^{-1})$,

(iii) $E[\hat{\theta}_{i+}^{EB} - \hat{\theta}_{i+}^{B}] = O(m^{-1})$,

(iv) $E[(\hat{\theta}_{i+}^{EB} - \hat{\theta}_{i+}^{B})^2] = g_{2i}(A) + g_{3i}(A) - V_A(1-B_i)^2 B_i^2 + o(m^{-1})$,

(v) $E[\hat{A}_+ - A] = O(m^{-1})$,

where $E[(\hat{A} - A)^2] = V_A + o(m^{-1})$.

Lemma 3. *Under the regularity conditions R1-R3, we have for large* m,

(i) $E[(x_i'\hat{\beta} - x_i'\beta)^4] = O(m^{-2})$,

(ii) $E[x_i'\hat{\beta} - x_i'\beta] = 0$,

(iii) $E[(\hat{A} - A)(x_i'\hat{\beta} - x_i'\beta)] = 0$.

Lemmas 2 and 3 are shown in S1.2 and S1.3, respectively.

S1 Proofs of Lemmas

We now prove Lemmas 1–3 in this section. Lemma 1 is provided in the body of the main manuscript. Hereafter, $a_{1i}^{(j)}(A)$ and $a_{2i}^{(j)}(A)$ denote $\partial^j a_{1i}/\partial A^j \Big|_A$ and $\partial^j a_{2i}/\partial A^j \Big|_A$, respectively.

Let some n dimensional random vector $W_n \sim N(0, \Sigma)$ with non-singular matrix Σ and let $f(W_n)$ be some integrable function such that $f(W_n) \in \mathbb{R}$. Then

S1.1 Lemma 1

Suppose that $W_n \sim N(0, \Sigma)$ with non-singular matrix Σ and let $f(W_n)$ be some integrable function such that $f(W_n) \in \mathbb{R}$. Then, we get

$$
\begin{aligned}
&E[\exp(i_u c' W_n) f(W_n)] \\
&\quad = \exp\left(-\frac{c'\Sigma c}{2}\right) \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \int f(W_n) \exp\left\{-\frac{(W_n - i_u \Sigma c)'\Sigma^{-1}(W_n - i_u \Sigma c)}{2}\right\} dW_n, \\
&\quad = \exp\left(-\frac{c'\Sigma c}{2}\right) E[f(W_n + i_u \Sigma c)],
\end{aligned}
$$

where c denotes some n-dimensional vectors of which components are all constants.

Lemma 1 then follows from the above noting $\cos(x) = (\exp(i_u x) + \exp(-i_u x))/2$ and $\sin(x) = (\exp(i_u x) - \exp(-i_u x))/2i_u$.

S1.2 Lemma 2

From the assumption, (S0.1)–(S0.5) and the dominated convergence theorem, defining $r = r(Z^*, i_u V a_{1i})$, it follows that

$$
\hat{A}_+ - A = \hat{A}(Z^*) + r = \hat{A} + r, \tag{S1.1}
$$

$$
\begin{aligned}
\hat\theta_{i+}^{EB} - \hat\theta_{i+}^{B} &= \{a_{1i}(\hat A_+) - a_{1i}(A) + a_{2i}(\hat A_+) - a_{2i}(A) + a_{2i}\}(Z^* + i_u V a_{1i}) \\
&= \left\{(\hat A - A + r)a_{1i}^{(1)}(A) + \frac{1}{2}(\hat A - A + r)^2 a_{1i}^{(2)}(A^*)\right\}(Z^* + i_u V a_{1i}) \\
&\quad + \{(\hat A - A + r)a_{2i}^{(1)}(A^*) + a_{2i}(A)\}'(Z^* + i_u V a_{1i}) \\
&= (\hat\theta_i^{EB} - \hat\theta_i^{B}) + (\hat A - A)(a_{1i}^{(1)})'(i_u V a_{1i}) + R^*, \qquad \text{(S1.2)}
\end{aligned}
$$

where A^* lies betewen A and $\hat A$. In the above, R^* is satisfying that $E[R^*] = O(m^{-1})$ and $E[(R^*)^2] = O(m^{-2})$ from the Cauchy–Schwartz inequality and the assumption on r.

Using the assumption of $\hat A$, (S1.1) and Cauchy-Schwarz inequality,

$$
\begin{aligned}
E[\hat A_+ - A] &= E[\hat A(Z^*) - A] + E[r] = O(m^{-1}), \qquad \text{(S1.3)} \\
E[(\hat A_+ - A)^2] &= E[(\hat A(Z^*) - A)^2] + E[r^2] + 2E[(\hat A(Z^*) - A)r] \\
&= E[(\hat A(Z^*) - A)^2] + o(m^{-1}). \qquad \text{(S1.4)}
\end{aligned}
$$

This leads to parts (i) and (v).

Next, we prove parts (iii) and (iv). To this end, we use (S1.2).

$$
\begin{aligned}
E[\hat\theta_{i+}^{EB} - \hat\theta_{i+}^{B}] &= E[\hat\theta_i^{EB} - \hat\theta_i^{B} + (\hat A - A)(a_{1i}^{(1)})'(i_u V a_{1i}) + R^*], \qquad \text{(S1.5)} \\
E[(\hat\theta_{i+}^{EB} - \hat\theta_{i+}^{B})^2] &= E[(\hat\theta_i^{EB} - \hat\theta_i^{B})^2 + \{(\hat A - A)(a_{1i}^{(1)})'(i_u V a_{1i}) + R^*\}^2] \\
&\quad + 2E[(\hat\theta_i^{EB} - \hat\theta_i^{B})\{(\hat A - A)(a_{1i}^{(1)})'(i_u V a_{1i}) + R^*\}], \\
&\qquad \text{(S1.6)}
\end{aligned}
$$

where R^* is such that $E[(R^*)^2] = O(m^{-2})$.

Using the result of Kackar and Harville (1981), the Cauchy–Schwartz inequality and (S0.3), (S1.5) and (S1.6) can be rewritten as

$$\text{(S1.5)} = i_u B_i(1-B_i)E[\hat{A}-A] + O(m^{-1}) = O(m^{-1}),$$

$$\text{(S1.6)} = g_{2i}(A) + g_{3i}(A) - V_A B_i^2(1-B_i)^2 + o(m^{-1}).$$

The above equalities follow from the result $(a_{1i}^{(1)})'(i_u V a_{1i}) = i_u B_i(1-B_i)$ due to (S0.6) and some results of Prasad and Rao (1990) and Datta and Lahiri (2000).

Finally, we prove part (ii). With a proof similar as above, (S1.1) and (S1.2) yield the following.

$$\begin{aligned} &E[(\hat{A}_+ - A)(\hat{\theta}_{i+}^{EB} - \hat{\theta}_{i+}^{B})] \\ &\quad = E\left[(\hat{A}-A+r)\left\{\hat{\theta}_i^{EB} - \hat{\theta}_i^{B} + (\hat{A}-A)(a_{1i}^{(1)})'(i_u V a_{1i}) + R^*\right\}\right], \\ &\quad = i_u V_A B_i(1-B_i) + o(m^{-1}). \qquad \text{(S1.7)} \end{aligned}$$

This leads to Lemma 2.

S1.3 Lemma 3

We first prove part (i). Using $Z^* = Z - X\beta \sim N(0, V)$, we obtain

$$\begin{aligned} E[(x_i'\tilde{\beta} - x_i'\beta)^{2l}] &= E[\{x_i'(X'V^{-1}X)^{-1}X'V^{-1}Z^*\}^{2l}] \\ &\le C \sum_{\{(k_1,k_2)\in\{0\}\cup\mathbb{Z}_+ : k_1+k_2=l\}} (x_i'(X'V^{-1}X)^{-1}x_i)^{k_1+k2} \\ &= O(m^{-l}), \end{aligned} \tag{S1.8}$$

where $\tilde{\beta} = \hat{\beta}(A)$, $l \in \mathbb{Z}_+$ and C is some generic positive constants.

From (S1.8), we obtain the following.

$$\begin{aligned} E\left[\left\{\frac{\partial x_i'\tilde{\beta}}{\partial A}\Big|_{A^*}\right\}^8\right] =& E\left[\{x_i'(X'V^{-1}X)^{-1}X'V^{-2}(I - X(X'V^{-1}X)^{-1}X'V^{-1})Z\}^8\Big|_{A=A^*}\right] \\ \le& CE\left[\{x_i'(X'V^{-1}X)^{-1}X'V^{-1}Z\}^8\Big|_{A=A^*}\right] = O(m^{-4}), \end{aligned}$$

where A^* lies between A and $\hat{A}$.

From the above results and the Cauchy–Schwartz inequality,

$$E[(x_i'\hat{\beta} - x_i'\tilde{\beta})^4] \le E[(\hat{A} - A)^8]^{1/2} E\left[\left\{\frac{\partial x_i'\tilde{\beta}}{\partial A}\Big|_{A^*}\right\}^8\right]^{1/2} = o(m^{-2}),$$

where A^* is lying between A and $\hat{A}$.

Consequently, part (i) follows from

$$\begin{aligned} E[(x_i'\hat{\beta} - x_i'\beta)^4] \le& C\{E[(x_i'\hat{\beta} - x_i'\tilde{\beta})^4] + E[(x_i'\tilde{\beta} - x_i'\beta)^4]\} \\ =& O(m^{-2}). \end{aligned} \tag{S1.9}$$

The remaining two parts (ii) (iii) follow immediately from the fact that $x_i'\hat{\beta} - x_i'\beta$ and $\hat{A} - A$ are odd and even functions of Z^* respectively.

S2 Proofs of Theorems

S2.1 Theorem 1

Theorem 1 (i)

We first prove Part (i) of Theorem 1.

The unbiasedness of $\hat{p}_i^B$, that is, $E[\hat{p}_i^B - p_i] = 0$, results in

$$\begin{aligned} E[\hat{p}_i^{EB} - p_i] =& E[\hat{p}_i^{EB} - \hat{p}_i^B] + E[\hat{p}_i^B - p_i], \\ =& E[\hat{p}_i^{EB} - \hat{p}_i^B]. \end{aligned} \tag{S2.1}$$

Let $C_{1i}(A)$ define $\exp(-g_{1i}(A)/2)$ hereafter. Then,

$$\begin{aligned} \text{(S2.1)} =& \frac{1}{2} E[C_{1i}(\hat{A}) \sin(\hat{\theta}_i^{EB}) - C_{1i}(A) \sin(\hat{\theta}_i^B)] \\ =& \frac{1}{2} E\left[(C_{1i}(\hat{A}) - C_{1i}(A)) \sin(\hat{\theta}_i^{EB}) + C_{1i}(A)(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B)) \right], \\ =& \frac{1}{2}(J_1 + J_2), \end{aligned} \tag{S2.2}$$

where $J_1 = E[(C_{1i}(\hat{A}) - C_{1i}(A)) \sin(\hat{\theta}_i^{EB})]$ and $J_2 = C_{1i}(A)E[\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B)]$.

For J_1, from Lemmas 1, 2 (v) and the dominated convergence theorem, we have,

$$\begin{aligned}
J_1 =& E[(C_{1i}(\hat{A}) - C_{1i}(A))(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^{B}) + \sin(\hat{\theta}_i^{B}))] \\
=& E\left[\left\{(\hat{A}-A)C_{1i}^{(1)}(A) + \frac{1}{2}(\hat{A}-A)^2 C_{1i}^{(2)}(A^*)\right\}(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^{B}))\right] \\
&+ E\left[\left\{(\hat{A}-A)C_{1i}^{(1)}(A) + \frac{1}{2}(\hat{A}-A)^2 C_{1i}^{(2)}(A^*)\right\}\sin(\hat{\theta}_i^{B})\right], \\
=& E[(\hat{A}-A)C_{1i}^{(1)}(A)\cos(\hat{\theta}_i^{B} - x_i'\beta)\sin(x_i'\beta)] + O(m^{-1}) = O(m^{-1}).
\end{aligned} \tag{S2.3}$$

where A^* lies between $\hat{A}$ and A. In the above, note that $C_{1i}^{(j)}(A) = \partial^j C_{1i}(A)/\partial A^j\Big|_A$ for $j = 1, 2$.

For the third and fourth equalities in the above, we use the assumption on $\hat{A}$, the fact that $\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^{B})$ convergences to zero in probability and $\sin(\hat{\theta}_i^{B} - x_i'\beta)$ is a odd function of $Z^* = Z - X\beta$.

In addition we use Lemma 2 (iii) and (S0.3) for calculation of J_2.

$$\begin{aligned}
J_2 =& 2C_{1i}(A)E\left[\cos\left(\frac{\hat{\theta}_i^{EB} + \hat{\theta}_i^{B}}{2}\right)\sin\left(\frac{\hat{\theta}_i^{EB} - \hat{\theta}_i^{B}}{2}\right)\right], \\
=& -2C_{1i}(A)\sin(x_i'\beta)E\left[\sin\left(\frac{\hat{\theta}_i^{EB} + \hat{\theta}_i^{B}}{2} - x_i'\beta\right)\sin\left(\frac{\hat{\theta}_i^{EB} - \hat{\theta}_i^{B}}{2}\right)\right], \\
=& -2C_{1i}(A)\sin(x_i'\beta) \\
&\times E\left[\sin(\hat{\theta}_i^{B} - x_i'\beta)\left\{\left(\frac{\hat{\theta}_i^{EB} - \hat{\theta}_i^{B}}{2}\right) - \frac{1}{6}\left(\frac{\hat{\theta}_i^{EB} - \hat{\theta}_i^{B}}{2}\right)^3 \sin\left(\eta\frac{(\hat{\theta}_i^{EB} - \hat{\theta}_i^{B})}{2}\right)\right\}\right] \\
&+ O(m^{-1}) = O(m^{-1}),
\end{aligned} \tag{S2.4}$$

where $|\eta| < 1$.

Note that $\sin(\hat{\theta}_i^{EB} - \hat{\theta}_i^{B})$ is odd function of $Z^* = Z - X\beta$ and $\sin(x) = x - x^3 \sin(\eta x)/6$ with $|\eta| < 1$ for the above calculation. Also, we use Liapounov's inequality with the following result which comes from Lemma 3 (i) and the assumption on $\hat{A}$.

$$E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^{B})^4] \leq E[(B_i - \hat{B}_i)^4 (z_i - x_i'\beta)^4] + E[(x_i'\hat{\beta} - x_i'\beta)^4] = O(m^{-2}). \tag{S2.5}$$

Combining (S2.2)-(S2.4), one gets

$$\text{(S2.1)} = O(m^{-1}).$$

Theorem 1 (ii)

First we use the identity

$$E[(\hat{p}_i^{EB} - p_i)^2] = E[(\hat{p}_i^{B} - p_i)^2] + E[(\hat{p}_i^{EB} - \hat{p}_i^{B})^2]. \tag{S2.6}$$

Next we evaluate $E[(\hat{p}_i^{B} - p_i)^2]$ in the right hand side of (S2.6). By standard results,

$$E[(\hat{p}_i^{B} - p_i)^2] = E[V(p_i|z_i)] = \frac{1}{4} E[V(\sin(\theta_i)|z_i)]; \tag{S2.7}$$

$$\begin{aligned} V(\sin(\theta_i)|z_i) &= E[\sin^2(\theta_i)|z_i] - [E(\sin(\theta_i)|z_i)]^2, \\ &= \frac{1}{2} E[1 - \cos(2\theta_i)|z_i] - [E(\sin(\theta_i)|z_i)]^2. \end{aligned} \tag{S2.8}$$

Equation (2.5) and Corollary 1 provide the results:

$$E[1-\cos(2\theta_i)|z_i] = 1-\exp(-2g_{1i}(A))\cos(2\hat{\theta}_i^B); \tag{S2.9}$$

$$\begin{aligned}[E(\sin(\theta_i)|z_i)]^2 &= \exp(-g_{1i}(A))\sin^2(\hat{\theta}_i^B) \\ &= \frac{1}{2}\exp(-g_{1i}(A))[1-\cos(2\hat{\theta}_i^B)]. \end{aligned} \tag{S2.10}$$

Hence, we get from (S2.8)-(S2.10) and Corollary 1 again,

$$V(\sin(\theta_i)|z_i) = \frac{1}{2}\{1-\exp(-2g_{1i}(A))\cos(2\hat{\theta}_i^B)\} - \frac{1}{2}\exp(-g_{1i}(A))(1-\cos(2\hat{\theta}_i^B)), \tag{S2.11}$$

$$E[V(\sin(\theta_i)|z_i)] = \frac{1}{2}(1-\exp(-g_{1i}(A)))\{1+\exp(-2A+g_{1i}(A))\cos(2x_i'\beta)\}. \tag{S2.12}$$

In the above calculation, we used the result $\hat{\theta}_i^B \sim N(x_i'\beta, A(1-B_i))$.

Combining (S2.7) and (S2.12), we obtain

$$E[(\hat{p}_i^B - p_i)^2] = \frac{1}{8}(1-\exp(-g_{1i}(A)))\{1+\exp(-2A+g_{1i}(A))\cos(2x_i'\beta)\}. \tag{S2.13}$$

Next, we find an asymptotic expansion of the second term $E[(\hat{p}_i^{EB} - \hat{p}_i^B)^2]$ in the right hand side of (S2.6), correct up to the order $O(m^{-1})$ for large m. Let $C_{1i}(A)$ continue to define $\exp(-g_{1i}(A)/2)$,

$$
\begin{aligned}
E[(\hat{p}_i^{EB} - \hat{p}_i^B)^2] =& \frac{1}{4} E\left[(C_{1i}(\hat{A})\sin(\hat{\theta}_i^{EB}) - C_{1i}(A)\sin(\hat{\theta}_i^B))^2 \right], \\
=& \frac{1}{4} E\left[\left\{ (C_{1i}(\hat{A}) - C_{1i}(A))\sin(\hat{\theta}_i^{EB}) + C_{1i}(A)(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B)) \right\}^2 \right], \\
=& \frac{1}{4}(I + II + III), \qquad \text{(S2.14)}
\end{aligned}
$$

where

$$
\begin{aligned}
I =& E\left[(C_{1i}(\hat{A}) - C_{1i}(A))^2 \sin^2(\hat{\theta}_i^{EB}) \right], \\
II =& (C_{1i}(A))^2 E\left[(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B))^2 \right], \\
III =& 2C_{1i}(A) E\left[(C_{1i}(\hat{A}) - C_{1i}(A))\sin(\hat{\theta}_i^{EB})(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B)) \right].
\end{aligned}
$$

We first calculate I using Lemma 1 and Lemma 2 (i):

$$
\begin{aligned}
I =& E\left[\left\{ (\hat{A} - A)C_{1i}^{(1)}(A) + \frac{1}{2}(\hat{A} - A)^2 C_{1i}^{(2)}(A^*) \right\}^2 \sin^2(\hat{\theta}_i^{EB}) \right] \\
=& \frac{1}{2}(C_{1i}^{(1)}(A))^2 E\left[(\hat{A} - A)^2 (1 - \cos(2\hat{\theta}_i^B)) \right] + o(m^{-1}), \\
=& \frac{B_i^4}{8} V_A \exp(-g_{1i}(A))(1 - \cos(2x_i'\beta)\exp(-2A + 2g_{1i}(A))) + o(m^{-1}),
\end{aligned}
\qquad \text{(S2.15)}
$$

where A^* lies between A and $\hat{A}$. We note that the second equality holds due to the dominated convergence theorem, the assumption on $\hat{A}$ and the result that $\sin^2(\hat{\theta}_i^{EB}) - \sin^2(\hat{\theta}_i^B)$ converges to zero in probability.

Next, we prove II;

$$E\left[\left\{\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^{B})\right\}^2\right] = 4E\left[\cos^2\left(\frac{\hat{\theta}_i^{EB}+\hat{\theta}_i^{B}}{2}\right)\sin^2\left(\frac{\hat{\theta}_i^{EB}-\hat{\theta}_i^{B}}{2}\right)\right],$$
$$= E[(1+\cos(\hat{\theta}_i^{EB}+\hat{\theta}_i^{B}))(1-\cos(\hat{\theta}_i^{EB}-\hat{\theta}_i^{B}))],$$
$$= E\left[(1+\cos(\hat{\theta}_i^{EB}+\hat{\theta}_i^{B}))\left\{\frac{(\hat{\theta}_i^{EB}-\hat{\theta}_i^{B})^2}{2} + \frac{(\hat{\theta}_i^{EB}-\hat{\theta}_i^{B})^4}{24}\cos(\eta(\hat{\theta}_i^{EB}-\hat{\theta}_i^{B}))\right\}\right]. \tag{S2.16}$$

In the above calculation, the third equality follows from the fact that $1 - \cos(x) = x^2/2 - x^4\cos(\eta x)/24$ with $|\eta| < 1$.

The results (2.2) and (2.3), given in the main paper, for the untransformed case remind us that $g_{2i}(A) + g_{3i}(A)$ are second-order approximations of $E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^{B})^2]$, and recalling (S2.5). Also, note that $\sin(2(\hat{\theta}_i^{B} - x_i'\beta))$ is odd function of $Z^* = Z - X\beta$ while $\cos(2(\hat{\theta}_i^{B} - x_i'\beta))$ is even function of $Z^* = Z - X\beta$. Moreover, $\cos(\hat{\theta}_i^{EB} + \hat{\theta}_i^{B}) - \cos(2\hat{\theta}_i^{B})$ converges to zero in probability. These above results provide the following.

$$\begin{aligned}
\text{(S2.16)} =& \frac{1}{2}\left\{g_{2i}(A) + g_{3i}(A) + E\left[\cos(\hat{\theta}_i^{EB}+\hat{\theta}_i^{B})(\hat{\theta}_i^{EB}-\hat{\theta}_i^{B})^2\right]\right\} + o(m^{-1}),\\
=& \frac{1}{2}\left\{g_{2i}(A) + g_{3i}(A) + E\left[\cos(2\hat{\theta}_i^{B} - 2x_i'\beta + 2x_i'\beta)(\hat{\theta}_i^{EB}-\hat{\theta}_i^{B})^2\right]\right\} + o(m^{-1}),\\
=& \frac{1}{2}\left\{g_{2i}(A) + g_{3i}(A) + \cos(2x_i'\beta)E\left[\cos(2\hat{\theta}_i^{B} - 2x_i'\beta)(\hat{\theta}_i^{EB}-\hat{\theta}_i^{B})^2\right]\right\} + o(m^{-1}).
\end{aligned} \tag{S2.17}$$

The third equality follows from the result that $\sin(2\hat{\theta}_i^{B} - 2x_i'\beta)(\hat{\theta}_i^{EB} - \hat{\theta}_i^{B})^2$ is a odd function of Z^* with zero mean.

Using Lemma 1 and (S0.3),

$$\begin{aligned}\text{(S2.17)} =& \frac{1}{2}(g_{2i}(A) + g_{3i}(A)) \\ &+ \frac{\exp(-2A + 2g_{1i}(A))}{4}\cos(2x_i'\beta)\Big\{E[(\hat{\theta}_{i+}^{EB} - \hat{\theta}_{i+}^{EB})^2] + E[(\hat{\theta}_{i-}^{EB} - \hat{\theta}_{i-}^{EB})^2]\Big\} \\ &+ o(m^{-1}). \end{aligned} \quad \text{(S2.18)}$$

Lemma 2 (iv) yields

$$\begin{aligned}\text{(S2.18)} =& \frac{1}{2}(g_{2i}(A) + g_{3i}(A)) \\ &+ \frac{\exp(-2A + 2g_{1i}(A))}{2}\cos(2x_i'\beta)\{g_{2i}(A) + g_{3i}(A) - V_A(1 - B_i)^2 B_i^2\} \\ &+ o(m^{-1}), \end{aligned} \quad \text{(S2.19)}$$

where $E[(\hat{A} - A)^2] = V_A + o(m^{-1})$.

Hence,

$$\begin{aligned} II =& \frac{1}{2}\exp(-g_{1i}(A))(g_{2i}(A) + g_{3i}(A))(1 + \cos(2x_i'\beta)\exp(-2A + 2g_{1i}(A))) \\ &- \frac{1}{2}\cos(2x_i'\beta)\exp(-2A + g_{1i}(A))V_A(1 - B_i)^2 B_i^2 + o(m^{-1}). \end{aligned} \quad \text{(S2.20)}$$

We finally calculate III.

$$\begin{aligned} III =& 2C_{1i}(A)E\left[(C_{1i}(\hat{A}) - C_{1i}(A))(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B) + \sin(\hat{\theta}_i^B))(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B))\right], \\ =& 2C_{1i}(A)E\left[\left\{(\hat{A} - A)C_{1i}^{(1)}(A) + \frac{1}{2}(\hat{A} - A)^2 C_{1i}^{(2)}(A^*)\right\}(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B))^2\right] \\ &+ 2C_{1i}(A)E\left[\left\{(\hat{A} - A)C_{1i}^{(1)}(A) + \frac{1}{2}(\hat{A} - A)^2 C_{1i}^{(2)}(A^*)\right\}\sin(\hat{\theta}_i^B)(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B))\right] \\ =& 2C_{1i}(A)C_{1i}^{(1)}(A)E\left[(\hat{A} - A)\sin(\hat{\theta}_i^B)(\sin(\hat{\theta}_i^{EB}) - \sin(\hat{\theta}_i^B))\right] + o(m^{-1}). \end{aligned}$$

(S2.21)

Using the results that $(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)$ and $\cos(2(\hat{\theta}_i^B - x_i'\beta))$ are respectively odd and even functions of $Z^* = Z - X\beta$, we obtain

$$
\begin{aligned}
\text{(S2.21)} =& 2C_{1i}(A)C_{1i}^{(1)}(A) \\
& \times E\left[(\hat{A} - A)\sin(\hat{\theta}_i^B)\left\{(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)\cos(\hat{\theta}_i^B) - \frac{1}{2}(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)^2 \sin(\eta(\hat{\theta}_i^{EB} - \hat{\theta}_i^B))\right\}\right] \\
& + o(m^{-1}), \\
=& C_{1i}(A)C_{1i}^{(1)}(A)\cos(2x_i'\beta)E[(\hat{A} - A)\sin(2(\hat{\theta}_i^B - x_i'\beta))(\hat{\theta}_i^{EB} - \hat{\theta}_i^B)] + o(m^{-1}), \\
=& C_{1i}(A)C_{1i}^{(1)}(A)\cos(2x_i'\beta)\frac{1}{2i_u}\exp(-2A + 2g_{1i}(A)) \\
& \times \left\{E\left[(\hat{A}_+ - A)(\hat{\theta}_{i+}^{EB} - \hat{\theta}_{i+}^B)\right] - E\left[(\hat{A}_- - A)(\hat{\theta}_{i-}^{EB} - \hat{\theta}_{i-}^B)\right]\right\} + o(m^{-1}),
\end{aligned}
\tag{S2.22}
$$

where $|\eta| < 1$. For the last equality, Lemma 1 is used.

From Lemma 2 (ii) and (S2.22), we can rewrite (S2.21) as

$$
-\frac{1}{2}V_A\cos(2x_i'\beta)\exp(-2A + g_{1i}(A))B_i^3(1 - B_i) + o(m^{-1}). \tag{S2.23}
$$

From (S2.15), (S2.20), and (S2.23), (S2.14) can be approximated up to the order of $O(m^{-1})$ as

$$
\begin{aligned}
\text{(S2.14)} =& \frac{1}{8}\exp(-g_{1i}(A))\left\{g_{2i}(A) + g_{3i}(A) + \frac{B_i^4}{4}V_A\right\} \\
& + \frac{1}{8}\cos(2x_i'\beta)\exp(-2A + g_{1i}(A))\left\{g_{2i}(A) + g_{3i}(A) - \frac{B_i^2(B_i - 2)^2}{4}V_A\right\} \\
& + o(m^{-1}).
\end{aligned}
\tag{S2.24}
$$

S2.2 Theorem 2

We first prove part (i).

$$
\begin{aligned}
E[\hat{M}_{1i}] =& \frac{1}{8}E[1-\exp(-g_{1i}(\hat{A}))] + \frac{1}{8}E[\cos(2x_i'\hat{\beta})\exp(-2\hat{A}+g_{1i}(\hat{A}))] \\
& - \frac{1}{8}E[\cos(2x_i'\hat{\beta})\exp(-2\hat{A})], \\
=& \frac{1}{8} - \frac{1}{8}\exp(-g_{1i}(A))E[\exp\{-(g_{1i}(\hat{A})-g_{1i}(A))\}] \\
& + \frac{1}{8}\exp(-2A+g_{1i}(A))E[\cos(2x_i'\hat{\beta})\exp\{-2(\hat{A}-A)+(g_{1i}(\hat{A})-g_{1i}(A))\}] \\
& - \frac{1}{8}\exp(-2A)E[\cos(2x_i'\hat{\beta})\exp\{-2(\hat{A}-A)\}], \\
=& \frac{1}{8} - \frac{1}{8}\left\{\exp(-g_{1i}(A))T_1 + \exp(-2A+g_{1i}(A))T_2 - \exp(-2A)T_3\right\},
\end{aligned}
\tag{S2.25}
$$

where

$$
\begin{aligned}
T_1 =& E[\exp\{-(g_{1i}(\hat{A})-g_{1i}(A))\}]; \\
T_2 =& E[\cos(2x_i'\hat{\beta})\exp\{-2(\hat{A}-A)+(g_{1i}(\hat{A})-g_{1i}(A))\}]; \\
T_3 =& E[\cos(2x_i'\hat{\beta})\exp\{-2(\hat{A}-A)\}].
\end{aligned}
$$

The results of Prasad and Rao (1990) and Datta and Lahiri (2000) lead to the following :

$$
\begin{aligned}
T_1 =& E\left[1-(g_{1i}(\hat{A})-g_{1i}(A))+\frac{1}{2}(g_{1i}(\hat{A})-g_{1i}(A))^2\right] \\
& - E\left[\frac{1}{6}(g_{1i}(\hat{A})-g_{1i}(A))^3\exp\{\eta(g_{1i}(\hat{A})-g_{1i}(A))\}\right],
\end{aligned}
$$

$$=1 + g_{3i}(A) - b_A B_i^2 + \frac{B_i^4}{2} V_A + o(m^{-1}), \tag{S2.26}$$

where $b_A = E[\hat{A} - A] + o(m^{-1})$ and $|\eta| < 1$.

In the above calculation, we use

$$E[(g_{1i}(\hat{A}) - g_{1i}(A))^2] = E[(\hat{A} - A)^2 B_i^4] + o(m^{-1}) = B_i^4 V_A + o(m^{-1})$$

and

$$E[(g_{1i}(\hat{A}) - g_{1i}(A))^3] = o(m^{-1}),$$

the latter following from the dominated convergence theorem.

We next evaluate T_2. Consider some integrable functions $f_1(\cdot)$ and $f_2(\cdot)$. Then Lemma 3 (ii) and (iii) yield

$$\begin{aligned} E[f_1(\hat{A} - A) f_2(x_i'\hat{\beta} - x_i'\beta)] =& Cov(f_1(\hat{A} - A), f_2(x_i'\hat{\beta} - x_i'\beta)) \\ &+ E[f_1(\hat{A} - A)] E[f_2(x_i'\hat{\beta} - x_i'\beta)], \\ =& E[f_1(\hat{A} - A)] E[f_2(x_i'\hat{\beta} - x_i'\beta)]. \end{aligned} \tag{S2.27}$$

Using (S2.27), we obtain as

$$T_2 = \cos(2x_i'\beta) E[\cos(2(x_i'\hat{\beta} - x_i'\beta))] E[\exp\{-2(\hat{A} - A) + (g_{1i}(\hat{A}) - g_{1i}(A))\}].$$

On the set $\mathscr{S} = \{\hat{A} : |\hat{A} - A| < \delta\}$ with some constant value $\delta > 0$, by

Corollary 1, T_2 reduces to

$$
\begin{aligned}
T_2 =& \cos(2x_i'\beta)E[\cos(2(x_i'\hat{\beta} - x_i'\beta))] \\
&\times E[1 - 2(\hat{A} - A) + (g_{1i}(\hat{A}) - g_{1i}(A)) + 2(\hat{A} - A)^2 + \frac{1}{2}(g_{1i}(\hat{A}) - g_{1i}(A))^2 : \mathscr{S}], \\
&- \cos(2x_i'\beta)E[\cos(2(x_i'\hat{\beta} - x_i'\beta))]E[2(\hat{A} - A)(g_{1i}(\hat{A}) - g_{1i}(A)) : \mathscr{S}] \\
&+ \frac{1}{6}\cos(2x_i'\beta)E[\cos(2(x_i'\hat{\beta} - x_i'\beta))] \\
&\times E[\{-2(\hat{A} - A) + (g_{1i}(\hat{A}) - g_{1i}(A))\}^3 \exp\{\eta(-2(\hat{A} - A) + (g_{1i}(\hat{A}) - g_{1i}(A)))\} : \mathscr{S}], \\
=& \cos(2x_i'\beta)\left\{1 - \frac{2g_{2i}(A)}{B_i^2}\right\}\left\{1 - g_{3i}(A) + b_A(B_i^2 - 2) + \frac{(B_i^2 - 2)^2}{2}V_A\right\} + o(m^{-1}), \\
=& \cos(2x_i'\beta)\left\{1 - \frac{2g_{2i}(A)}{B_i^2} - g_{3i}(A) + b_A(B_i^2 - 2) + \frac{(B_i^2 - 2)^2}{2}V_A\right\} + o(m^{-1}),
\end{aligned}
\tag{S2.28}
$$

where $|\eta| < 1$. Note that, in the above calculation, some terms of odd function with zero mean vanish.

In a similar way on $\mathscr{S}$, T_3 reduces to

$$
\begin{aligned}
T_3 =& \cos(2x_i'\beta)E[\cos(2(x_i'\hat{\beta} - x_i'\beta))] \\
&\times E\left[1 - 2(\hat{A} - A) + 2(\hat{A} - A)^2 - \frac{4}{3}(\hat{A} - A)^3 \exp\{2\eta(\hat{A} - A)\} : \mathscr{S}\right], \\
=& \cos(2x_i'\beta)\left(1 - \frac{2g_{2i}(A)}{B_i^2}\right)(1 - 2b_A + 2V_A) + o(m^{-1}), \\
=& \cos(2x_i'\beta)\left(1 - \frac{2g_{2i}(A)}{B_i^2} - 2b_A + 2V_A\right) + o(m^{-1}).
\end{aligned}
\tag{S2.29}
$$

From (S2.25), (S2.26), (S2.28) and (S2.29), we find on $\mathscr{S}$,

$$
\begin{aligned}
E[\hat{M}_{1i} - M_{1i} : \mathscr{S}] = &-\frac{1}{8}\exp(-g_{1i}(A))\left(g_{3i}(A) - b_A B_i^2 + \frac{B_i^4}{2}V_A\right) \\
&- \frac{1}{8}\exp(-2A + g_{1i}(A))\cos(2x_i'\beta)\left\{\frac{2g_{2i}(A)}{B_i^2} + g_{3i}(A) - b_A(B_i^2 - 2) - \frac{(B_i^2-2)^2}{2}V_A\right\} \\
&- \frac{1}{8}\exp(-2A)\cos(2x_i'\beta)\left(2V_A - 2b_A - \frac{2g_{2i}(A)}{B_i^2}\right) \\
&+ o(m^{-1}). \qquad \text{(S2.30)}
\end{aligned}
$$

Then, Part (i) is obtained from (S2.30) with $0 < s < 1$ using a proof similar to that of Das, et al. (2004). Specifically,

$$
\begin{aligned}
|E[\hat{M}_{1i} - M_{1i} - b_M(\lambda)]| =& |E[\hat{M}_{1i} - M_{1i} - b_M(\lambda) : \mathscr{S}] + E[\hat{M}_{1i} - M_{1i} : \mathscr{S}^c]|, \\
\le& o(m^{-1}) + Cm^s\frac{E[(\hat{A} - A)^4]}{\delta^4}. \\
=& o(m^{-1}). \qquad \text{(S2.31)}
\end{aligned}
$$

We next prove part (ii). From the regularity conditions, $M_{2i}(\lambda)$ and $b_M(\lambda)$ are of the order $O(m^{-1})$ for large m and these are every bounded continuous functions with a finite λ. Continuous mapping theorem and dominated convergence theorem provide the following with $s < 1$:

$$
\begin{aligned}
|E[M_{2i}(\hat{\lambda}) - M_{2i}(\lambda)]| =& |E[M_{2i}(\hat{\lambda}) - M_{2i}(\lambda) : \mathscr{S}] + E[M_{2i}(\hat{\lambda}) - M_{2i}(\lambda) : \mathscr{S}^c]| \\
\le& o(m^{-1}) + Cm^s\frac{E[(\hat{A} - A)^4]}{\delta^4} = o(m^{-1}).
\end{aligned}
$$

where C is some positive constants.

Similarly, we get

$$E[b_M(\hat{\lambda}) - b_M(\lambda)] = o(m^{-1}).$$

The results follow.

S2.3 Theorem 3 (i)

From Lemma 3 (i) and assumption on $\hat{A}$, with some constant value $s > 0$,

$$\begin{aligned} E[(\hat{M}_i^0(\hat{\lambda}) - M_i(\lambda))^4] =& E[(\hat{M}_i^0(\hat{\lambda}) - M_i(\lambda))^4 : \mathscr{S}] + E[(\hat{M}_i^0(\hat{\lambda}) - M_i)^4 : \mathscr{S}^c] \\ \leq& O(m^{-2}) + Cm^{4s}\frac{E[(\hat{A} - A)^8]}{\delta^8} = O(m^{-2\vee 4(s-1)}). \end{aligned} \tag{S2.32}$$

where $C(> 0)$ is some constants.

We then get from (S2.32),

$$\begin{aligned} P(\hat{M}_i^0(\hat{\lambda}) \leq 0) \leq& P(|\hat{M}_i^0(\hat{\lambda}) - M_i| \geq M_i), \\ \leq& \frac{E[(\hat{M}_i^0(\hat{\lambda}) - M_i)^4]}{M_i^4} = O(m^{-2\vee 4(s-1)}). \end{aligned} \tag{S2.33}$$

We now let $\mathscr{M}$ define a set such that $\{\hat{M}_i^0 > 0\}$. Then the result (S2.33) and Theorem 2 lead to the following result with $0 < s < 3/5$.

$$\begin{aligned} |E[\hat{M}_i - M_i]| =& |E[\hat{M}_i - M_i : \mathscr{M}] + E[\hat{M}_i - M_i : \mathscr{M}^c]| \\ \leq& |E[\hat{M}_i^0(\hat{\lambda}) - M_i]| + Cm^s P(\hat{M}_i^0(\hat{\lambda}) \leq 0), \\ =& o(m^{-1}) + O(m^{(s-2)\vee(5s-4)}) = o(m^{-1}). \end{aligned}$$

We thus get part (i).

Bibliography

Das, K., Jiang, J., and Rao, J. N. K. (2004). Mean squared error of empirical predictor. *The Annals of Statistics*, **32**, 818-840.

Datta, G.S. and Lahiri, P. (2000), A unified measure of uncertainty of estimated best linear unbiased predictions in small area estimation problems, *Statist.Sinica*, **10**, 613-627.

Kackar, R. N. and Harville, D. A. (1981). Unbiasedness of two-stage estimation and prediction procedures for mixed linear models. *Communications in statistics-theory and methods* **10**: 1249-1261.

Prasad, N.G.N. and Rao, J.N.K. (1990). The estimation of the mean squared error of small area estimators, *Journal of the American Statistical Association*, **85**, 163-171.