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STABLE COMBINATION TESTS

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Abstract: This short note proposes a stable combination test as a natural extension of the Cauchy combination test. Similarly to the latter, the proposed test is simple to compute, enjoys good size, and has asymptotically optimal power even when the individual tests are not independent. Our simulations demonstrate that the proposed stable combination test may improve the Cauchy combination test if its parameters are chosen carefully.

Key words and phrases: Additive combination test, multiple hypothesis testing, stable distribution.

Liu and Xie (2020) recently proposed the Cauchy combination test (CCT), which originated from the observation that the standard Cauchy distribution is closed under convex combinations (Pillai and Meng (2016)). One of the most fascinating features of the CCT is its robustness to dependency among individual tests, despite its simple form. The standard Cauchy distribution is a member of the stable distribution family, which is also asymptotically closed under convex combinations. In this brief note, we investigate whether the favorable properties of the CCT can be extended to a stable combination test (SCT).

Let the p-values $p_1, \ldots, p_n$ be uniform under the global null hypothesis $\boldsymbol{H_0} = \bigcap_{i=1}^n H_i$. The SCT statistic is defined as

$$T_{n;\alpha,\beta}(\boldsymbol{p}) = a_{n;\alpha} \sum_{i=1}^{n} w_i W_{i;\alpha,\beta}, \tag{0.1}$$

where w_i are positive constants such that $\sum_{i=1}^n w_i = 1$, $a_{n;\alpha} = (\sum_{j=1}^n w_j^\alpha)^{-1/\alpha}$ is the normalizing factor, and $W_{i;\alpha,\beta} = F^{-1}(1 - p_i|\alpha,\beta)$. Here, $F(\cdot|\alpha,\beta)$ is the distribution function of $\boldsymbol{S}(\alpha,\beta)$, which is the stable distribution with stability, skewness, scale, and location parameters α, β, 1, and 0, respectively, following Nolan's (2020) S1 parametrization. We consider $0 < \alpha < 2$, and $-1 < \beta \leq 1$ for

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$\alpha \neq 1$ and $\beta = 0$ for $\alpha = 1$ to ensure strict stability.

Under $\boldsymbol{H_0}$, it is well known that $T_{n;\alpha,\beta}(\boldsymbol{p})$ follows $\boldsymbol{S}(\alpha,\beta)$ for any n if the p_i are independent. When the p_i are allowed to be dependent, an asymptotic relationship can be established under the following assumptions.

Assumption 1. *Let A be any finite union of disjoint intervals of the form $(a,b]$ that do not contain zero, and let A^c be the complementary set of A in the reals. The sequence $\{W_{i;\alpha,\beta}\}_{i=1}^n$ satisfies*

$$\sup_{1\le p<q<r\le n}\left|\Pr\left[\bigcap_{p\le i\le r}(a_{n;\alpha}w_iW_{i;\alpha,\beta}\in A^c)\right]-\right.$$
$$\left.\Pr\left[\bigcap_{p\le i\le q}(a_{n;\alpha}w_iW_{i;\alpha,\beta}\in A^c)\right]\Pr\left[\bigcap_{q\le i\le r}(a_{n;\alpha}w_iW_{i;\alpha,\beta}\in A^c)\right]\right|\underset{n\to\infty}{\longrightarrow}0.$$

Assumption 2. *$\{W_{i;\alpha,\beta}\}_{i=1}^n$ is ρ-mixing with $\sum_{j=1}^\infty \rho(2^j) < +\infty$.*

Assumption 3. *Let $\Delta(r)$ be an arbitrary partitioning of the set $\{1,2,\ldots,n\}$ into r segments, and $0 = m_0 \le m_1 \le \cdots \le m_r = n$. For every $\varepsilon > 0$, the sequence $\{W_{i;\alpha,\beta}\}_{i=1}^n$ satisfies*

$$\limsup_{n\to\infty}\left[\inf_{\Delta(r)}\sum_{q=1}^r\sum_{m_{q-1}<i<j\le m_q}\Pr\left(w_i|W_{i;\alpha,\beta}|>a_{n;\alpha}^{-1}\varepsilon, w_j|W_{j;\alpha,\beta}|>a_{n;\alpha}^{-1}\varepsilon\right)\right]\underset{r\to\infty}{\longrightarrow}0.$$

Assumptions 1 and 2 address the long-range dependence. Assumption 3 limits the short-range dependence by assuming that large values cannot be clustered into a small segment (Beirlant et al. (2006)). The size of the SCT can be controlled asymptotically using the following theorem, which is a direct consequence of Theorems 4.1 and 4.2 of Jakubowski and Kobus (1989).

Theorem 1. *Suppose Assumptions 1 and 3 are satisfied when $0 < \alpha < 1$ or Assumptions 2 and 3 are satisfied when $1 \le \alpha < 2$. Under $\boldsymbol{H_0}$, as $n \to \infty$,*

$$T_{n;\alpha,\beta}(\boldsymbol{p}) = a_{n;\alpha}\sum_{i=1}^n w_iW_{i;\alpha,\beta} \xrightarrow{d} \boldsymbol{S}(\alpha,\beta).$$

The asymptotic optimal power property of the CCT can also be extended to the SCT under sparse alternatives. Similarly to Liu and Xie (2020), let $\boldsymbol{X} = [X_1, X_2, \ldots, X_n]^T$ be the collection of individual test statistics with $\mathrm{E}[\boldsymbol{X}] = \boldsymbol{\mu}$ and $\mathrm{Cov}(\boldsymbol{X}) = \boldsymbol{\Sigma}$. We assume that X_i is marginally normal with variance one, and $p_i = p(X_i) = 2[1-\Phi(|X_i|)]$, where Φ is the standard normal distribution function. The global null is $\boldsymbol{H_0} : \boldsymbol{\mu} = \boldsymbol{0}$.

Assumption 4. *Let* $S = \{1 \le i \le n : \mu_i \ne 0\}$ *be the collection of indices of false null hypotheses. Let* $S^c = \{1 \le i \le n : \mu_i = 0\}$ *and* $S_+ = \{1 \le i \le n : \mu_i > 0\}$, *and, without loss of generality, assume* $|S_+| \ge |S|/2$.

1. *For* $i \in S^c$, *the* p_i *are uniformly distributed, and* $\{W_{i;\alpha,\beta}\}_{i\in S^c}$ *satisfies the requirements in Theorem* 1 *with* $a_{|S^c|;\alpha}$.
2. *The number,* $|S|$, *of elements in* S *is* n^γ, *with* $0 < \gamma < \min(0.5/\alpha, 0.5)$.
3. *For* $i \in S$, $|\mu_i| = \mu_0 = \sqrt{2r\log n}$ *and* $\sqrt{r} + \sqrt{\gamma} > \max(\sqrt{\alpha}, 1)$.
4. *There exists a positive constant* c_0 *such that* $\min_{i=1}^n w_i \ge c_0 n^{-1}$. *The sum of the weights is given by* $\sum_{j\in S} w_j = n^{\gamma-1}$.

The sparse alternative assumption in Part 2 is common in the multiple testing field. Part 3 requires the magnitude of the nonzero signals to be large enough to ensure that the test statistic is arbitrarily large. Part 4 helps keep the contribution of $\max_{i\in S} p_i$ under control, which is necessary because $F^{-1}(1 - \max_{i\in S} p_i)$ can be negative. The following theorem establishes the asymptotic optimal power of the SCT. The proof is presented in the Supplementary Material.

Theorem 2. *Let* $0 < \alpha < 2$ *and* $-1 < \beta \le 1$. *Under Assumption* 4, *for any significance level* s, *the power of the SCT converges to one as* $n \to \infty$.

Remark 1. The Supplementary Material presents simulation results for the SCT with various choices of stability and skewness parameters, α and β, respectively. Interestingly, a larger β means better size and power, in general, regardless of the strength of the dependency. In contrast, the effect of α is mixed. In particular, when the dependency is relatively strong, $\alpha \approx 0.9$ or 1 seem to work best, whereas $\alpha \approx 1.5$ or 1.7 have the best size and power when the dependence is weak (Model 1 and Model 2 with $\rho = 0.2$). This suggests that considering a different α may offer a possible improvement over the CCT, especially when the dependence is relatively weak. A systematic guide for the choices of α and β and detailed theoretical comparisons with the CCT shall be pursued in a future study.

Supplementary Material

The online Supplementary Material presents a proof for Theorem 2 and our simulation setting and results.

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