Supplementary Material for Efficient Experimental Plans for Second-Order Event-Related Functional Magnetic Resonance Imaging

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Proofs of Theorems and Lemmas

0.1 Proof of Theorem 1

Let G be a directed graph whose vertex set is a collection of all s-ary $(t-1)$-tuples $(d_1, d_2, \ldots, d_{t-1})$, so there are totally s^{t-1} vertices. In addition, there are $\lambda^{d_1 d_2 \cdots d_{t-1} d'_{t-1}}$ directed edges from $(d_1, d_2, \ldots, d_{t-1})$ to $(d'_1, d'_2, \ldots, d'_{t-1})$ if $d'_i = d_{i+1}$ for all $i = 1, 2, \ldots, t-2$. By definition, G is a $t-$dimensional De Bruijn frequency graph based on Λ.

It is well-known in graph theory that a directed graph has an Eulerian circuit if it is connected and each vertex whose in-degree is equal to its out-degree. Now, we claim that each vertex in G has equal in-degree and out-degree, and G is connected. For each $x \in Z_s$, $\lambda^{a_1 \cdots a_{t-1} x}$ represents the number of directed edges from the vertex $(a_1, a_2, \ldots, a_{t-1})$ to vertex $(a_2, \ldots, a_{t-1}, x)$ in G. Therefore, the total number of edges incident to $(a_1, a_2, \ldots, a_{t-1})$ is its in-degree $\sum_{x=0}^{s-1} \lambda^{a_1 \cdots a_{t-1} x}$. Similarly, the out-degree of $(a_1, a_2, \ldots, a_{t-1})$ is $\sum_{x=0}^{s-1} \lambda^{x a_1 \cdots a_{t-1}}$. Obviously, each vertex in G has equal in-degree and out-degree because of $\sum_{x=0}^{s-1} \lambda^{a_1 \cdots a_{t-1} x} = \sum_{x=0}^{s-1} \lambda^{x a_1 \cdots a_{t-1}}$ for all $a_i \in Z_s$.

By definition, a directed graph is connected if any two vertices are connected by at least one directed path. Let $\boldsymbol{x} = (x_1x_2\dots x_{t-1})$ and $\boldsymbol{y} = (y_1y_2\dots y_{t-1})$ be any two vertices in G. According to the assumption that $\lambda^{a_1\cdots a_t} \geq 1$ for all $a_i \in Z_s$, it guarantees that the existence of any t-tuple subsequence of the sequence $x_1x_2\cdots x_{t-1}y_1y_2\cdots y_{t-1}$. It implies that there exists at least one directed path from $\boldsymbol{x}$ to $\boldsymbol{y}$, whose edges are $x_1x_2\cdots x_{t-1}y_1$, $x_2\cdots x_{t-1}y_1y_2$, $\cdots$, and $x_{t-1}y_1y_2\cdots y_{t-1}$. Thus, G is connected, and it has an Eulerian circuit.

Let $\boldsymbol{e} = e_1e_2\cdots e_n$ be an Eulerian circuit in G, where $n = \sum_{a_i\in Z_s} \lambda^{a_1\cdots a_t}$ is the total number of edges in G. Let $\boldsymbol{A}_{t\times n}$ be the first t rows of the circulant matrix with the first row $e_1e_2\dots e_n$. Then each column is a t-tuple subsequence of $\boldsymbol{e}$. Since there is a one-to-one correspondence between n columns and edges, by definition, $\boldsymbol{A}_{t\times n}$ is a $CAOA(n,t,s,t,b)$.

0.2 Proof of Proposition 1

Let $\mathscr{S} = (\mathscr{S}^{\alpha,\beta})_{2\times 2}$ be the difference matrix of V. By definition, $\lambda^{\alpha,\beta}_{r_1,r_2} = \#\{g \in V_\alpha | \{r_1,r_2\} \subseteq S^{\alpha,\beta}_g\}$.

(i) Consider the block sub-matrix $(\mathscr{S}^{\alpha,\beta}|\mathscr{S}^{\beta,\beta})$. For each $x \in V_\beta$, there exists exactly one element g such that $g - x \equiv r_1 \pmod n$, because $\mathscr{S}$ is a Latin square. Assume that $\lambda^{\beta,\beta}_{r_2-r_1} \neq 0$, there exists a pair $x, y \in V_\beta$ such that $x - y \equiv r_2 - r_1 \pmod n$. Then $r_2 - r_1 \equiv x - y \equiv (g - r_1) - y \pmod n$, so $g - y \equiv r_2 \pmod n$. Thus, for each (x,y)-pair, there is a unique $g \in V_\alpha$ (or V_β) such that $\{r_1, r_2\} \subseteq S^{\alpha,\beta}_g$ (or $S^{\beta,\beta}_g$). If $\lambda^{\beta,\beta}_{r_2-r_1} = 0$, then it can be showed that $\lambda^{\alpha,\beta}_{r_1,r_2} = \lambda^{\beta,\beta}_{r_1,r_2} = 0$ via contradiction. Thus $\lambda^{\alpha,\beta}_{r_1,r_2} + \lambda^{\beta,\beta}_{r_1,r_2} = \lambda^{\beta,\beta}_{(r_2-r_1)}$.

(ii) Consider the block sub-matrix $(\mathscr{S}^{\alpha,\alpha}|\mathscr{S}^{\alpha,\beta})$. For each pair (r_1, r_2), there are $\lambda^{\alpha,\beta}_{r_1,r_2}$ rows having neither r_1 nor r_2 in $\mathscr{S}^{\alpha,\alpha}$. It is equal to the total number of rows minus the number of rows contains r_1 or r_2. By inclusion-and-exclusion principle, it follows that $\lambda^{\alpha,\beta}_{r_1,r_2} = |V_\alpha| - (\lambda^{\alpha,\alpha}_{r_1} + \lambda^{\alpha,\alpha}_{r_2} - \lambda^{\alpha,\alpha}_{r_1,r_2})$.

Now, if $|V_\alpha| = |V_\beta| = n/2$ and $\lambda^{\alpha,\alpha}_{r_1} = n/4$ for $1 \leq r_1 \leq k$, it can be verified that $\lambda^{\alpha,\beta}_{r_1,r_2} = \lambda^{\alpha,\alpha}_{r_1,r_2}$ and $\lambda^{\alpha,\alpha}_{r_1,r_2} + \lambda^{\beta,\beta}_{r_1,r_2} = n/4$ for $1 \leq r_1 < r_2 \leq k$ by the previous two propositions.

0.3 Proof of Lemma 1

Let $\boldsymbol{A} = (a_{i,j})_{n\times k}$. Assume that $\delta^{c_0,c_1,\cdots,c_m}_{r_1,r_2,\cdots,r_m} = \gamma \neq 0$, then there are γ column vectors of $\boldsymbol{A}_{g_0,g_1,\cdots,g_m}$ equal $(c_0, c_1, \cdots, c_m)^T$. For each column vector corresponding to $(c_0, c_1, \cdots, c_m)^T$, says the ωth column, we have $a_{g_i,\omega} = c_i$ for $i = 0, 1, \cdots, m$. By Definition 3, it follows that $\omega \in V_{c_i} + (g_i - 1)$. So we have $\omega - (g_i - 1) \in V_{c_i}$ for each i. Let $r_i = g_i - g_0$ for $i = 1, 2, \cdots, m$. For each $i \neq 0$, the difference of $\omega-(g_0-1) \in V_{c_0}$ and $\omega-(g_i-1) \in V_{c_i}$ equals $[\omega - (g_0 - 1)] - [\omega - (g_i - 1)] = g_i - g_0 = r_i$. This implies that $\omega - (g_0 - 1) \in \{g \in V_{c_0} | r_i \in \boldsymbol{S}^{c_0,c_i}_g$ for $i = 1, 2, \cdots, m\}$. By the definition of $\delta^{c_0,c_1,\cdots,c_m}_{r_1,r_2,\cdots,r_m}$, the total number of distinct ω is equal to γ. In a similar manner, it can be shown that the result also holds for $\delta^{c_0,c_1,\cdots,c_m}_{r_1,r_2,\cdots,r_m} = 0$ by contradiction. This complies the proof.

0.4 Proof of Lemma 2

Since V is a $(n, k, 2; \Phi_1, \Phi_2)$-HCDS, V can be represented as $\{V_0, V_1\}$ which is a partition of Z_n. In addition, its difference matrix $\mathscr{S}$ is a block matrix consists of $\mathscr{S}^{\alpha,\beta}$ for $\alpha, \beta \in \{0, 1\}$. Now we claim that each element in $\boldsymbol{\delta}_{r_1,r_2}$ can be obtained by $\lambda^{\alpha,\beta}_{r_1,r_2}$. By definition, $\delta^{\alpha,\beta,\beta}_{r_1,r_2} = \#\{g \in V_\alpha | r_1, r_2 \in \boldsymbol{S}^{\alpha,\beta}_g\}$, which is equal to the second-order difference $\lambda^{\alpha,\beta}_{r_1,r_2}$, for all $\alpha, \beta \in \{0, 1\}$. On the other hand, $\delta^{0,0,1}_{r_1,r_2} = \#\{g \in V_\alpha | r_1 \in \boldsymbol{S}^{0,0}_g$ and $r_2 \in \boldsymbol{S}^{0,1}_g\}$ which is equal to the number of rows in $\mathscr{S}^{0,0}$ that contains r_1 but not r_2. Clearly, there are totally $\lambda^{0,0}_{r_1} - \lambda^{0,0}_{r_1,r_2}$ rows in $\mathscr{S}^{0,0}$ that contains r_1 without r_2. Thus we obtain $\delta^{0,0,1}_{r_1,r_2} = \lambda^{0,0}_{r_1} - \lambda^{0,0}_{r_1,r_2}$. Similarly, we have $\delta^{0,1,0}_{r_1,r_2} = \lambda^{0,0}_{r_2} - \lambda^{0,0}_{r_1,r_2}$, $\delta^{1,0,1}_{r_1,r_2} = \lambda^{1,1}_{r_2} - \lambda^{1,1}_{r_1,r_2}$, and $\delta^{1,1,0}_{r_1,r_2} = \lambda^{1,1}_{r_1} - \lambda^{1,1}_{r_1,r_2}$.

0.5 Proof of Theorem 2

Let $\boldsymbol{A}$ be the incidence matrix of V, which is also an $(n, k, 2, \boldsymbol{\Phi}_1)$-CDS. According to the Proposition 3.5 and the discussion of Section 4 in (Lin, Phoa, and Kao (2017b)), $\boldsymbol{A}$ is a

$CAOA(n, k, 2, 2, 1)$ if and only if

$$(\lambda_r^{0,0}, \lambda_r^{0,1}, \lambda_r^{1,0}, \lambda_r^{1,1}) = \begin{cases} (\mu_0, \mu_0, \mu_0, \mu_0) & \text{if } n \equiv 0 \pmod 4, \\ (\mu_1, \mu_1, \mu_1, \mu_2) & \text{if } n \equiv 1 \pmod 4, \\ (\mu_1, \mu_2, \mu_2, \mu_2) & \text{if } n \equiv 3 \pmod 4, \\ (\mu_1, \mu_2, \mu_2, \mu_1) \text{ or } (\mu_2, \mu_1, \mu_1, \mu_2) & \text{if } n \equiv 2 \pmod 4, \end{cases}$$

where $\mu_0 = n/4$, $\mu_1 = \lfloor n/4 \rfloor$, and $\mu_2 = \lceil n/4 \rceil$. The above conditions describe the pattern of $\mathbf{\Phi}_1$.

Now we discuss the pattern of $\mathbf{\Phi}_2$. By Lemmas 1 and 2, $\boldsymbol{A}$ is a uniform $CAOA(n, k, 2, 3, 1)$ if and only if there exists a matrix $\boldsymbol{\delta}$ such that $B(\boldsymbol{\delta}) = 1$ and $\boldsymbol{\delta}_{r_1,r_2} = \boldsymbol{\delta}$ for all $1 \leq r_1 < r_2 \leq k-1$. Therefore, $max(\boldsymbol{\delta}) - min(\boldsymbol{\delta}) \leq 1$, where $max(\boldsymbol{\delta})$ and $min(\boldsymbol{\delta})$ are the maximal and minimal entries in $\boldsymbol{\delta}$. Another natural condition is that the sum of the frequency of all level-combination should be equal to the run size n. Thus, the conditions $\delta_{r_1,r_2}^{0,0,0} + \cdots + \delta_{r_1,r_2}^{1,1,1} = n$ and $max(\boldsymbol{\delta}) - min(\boldsymbol{\delta}) \leq 1$ hold if and only if $min(\boldsymbol{\delta}) = m$ and $m \leq max(\boldsymbol{\delta}) \leq m+1$.

Let $n = 8m + l$ where $l \in \{0, 1, \ldots, 7\}$. When $l = 1$, we have $n \equiv 1 \pmod 4$, this implies that $\lambda_r^{0,0} = \mu_1 = \lfloor n/4 \rfloor$. According to above discussion and Lemma 2, $\delta_{r_1,r_2}^{0,0,0} = \lambda_{r_1,r_2}^{0,0}$ which is equal to either $m+1$ or m. By Proposition 1, if $\lambda_{r_1,r_2}^{0,0} = m + 1 = \lceil n/8 \rceil$, then $\boldsymbol{\delta}_{r_1,r_2}^{1,0,0} = \lambda_{r_1,r_2}^{1,0} = \lambda_{r_2-r_1}^{0,0} - \lambda_{r_1,r_2}^{0,0} = m-1$ for $1 \leq r_1 < r_2 \leq k-1$. It contradicts to the assumption that $min(\boldsymbol{\delta}) = m$. On the other hand, if $\lambda_{r_1,r_2}^{0,0} = m = \lfloor n/8 \rfloor$, then $\delta_{r_1,r_2}^{0,0,1} = \cdots = \delta_{r_1,r_2}^{1,1,0} = m$ and $\delta_{r_1,r_2}^{1,1,1} = m+1$. It means that $B(\boldsymbol{\delta}_{r_1,r_2}) = 1$, so the entries in $\boldsymbol{\delta}$ are all m except the last entry equals $m+1$. Following the above discussion, we summarize all cases that satisfy the conditions as shown below.

l	$\delta^{0,0,0}$	$\delta^{0,0,1}$	$\delta^{0,1,0}$	$\delta^{0,1,1}$	$\delta^{1,0,0}$	$\delta^{1,0,1}$	$\delta^{1,1,0}$	$\delta^{1,1,1}$
0	m	m	m	m	m	m	m	m
1	m	m	m	m	m	m	m	$m+1$
2	$m+1$	m	m	m	m	m	m	$m+1$
3	m	m	$m+1$	m	m	$m+1$	$m+1$	m
4	m	$m+1$	$m+1$	m	$m+1$	m	m	$m+1$
4	$m+1$	m	m	$m+1$	m	$m+1$	$m+1$	m
5	$m+1$	m	m	$m+1$	m	$m+1$	$m+1$	$m+1$
6	m	$m+1$	$m+1$	$m+1$	$m+1$	$m+1$	$m+1$	m
7	m	$m+1$	$m+1$	$m+1$	$m+1$	$m+1$	$m+1$	$m+1$

There are two patterns of the case $l = 4$, but they are equivalent by exchanging the sets V_0 and V_1. It is obvious that the bandwidth is zero when $l = 0$, which implies that A is a uniform $CAOA(n, k, 2, 3, 0)$. This completes the proof.

0.6 Proof of Theorem 3

Following the discussion preceding Theorem 3, we firstly show the existence of a $(n/2, n/4-1; \lambda_1, \cdots, \lambda_{n/2-1})$ GDS, say D, where all λs are equal to $n/8-1$ except $\lambda_{n/4} = 0$. The required GDS has a special pattern that all differences appear equally often except one, which is related to a tough issue in combinatorial design, called *cyclic relative difference sets* (Elliott and Butson (1966)). According to the Result 1 in the literature, Pott, Reuschling, and Schmidt (1997) proved the existence of the specific cyclic relative difference sets when its order n is a prime power. Replacing the parameters n and u in the literate by $n/4-1$ and 2 respectively, the resulted cyclic relative difference set is a GDS that all λs are equal to $n/8-1$ except $\lambda_{n/4} = 0$. Therefore, if $n/4-1$ is a prime power, then there exists a $(n/2, n/4-1; \lambda_1, \cdots, \lambda_{n/2-1})$ GDS.

Let $V_0 = D \cup (D + n/2) \cup \{g, g + n/4\}$ where $g, g + n/4 \in D^c$. Now we claim that V_0 is a $(n, n/2; \lambda_1, \cdots, \lambda_{n-1})$ GDS where all λs are equal to $n/4$ except $\lambda_{n/4} = \lambda_{3n/4} = 1$ and $\lambda_{n/2} = n/2-2$. Let $D' = D \cup (D + n/2)$. By the Lemma 4.5 in (Lin, Phoa, and Kao (2017b)), D' is a $(n, n/2-2; \lambda_1, \cdots, \lambda_{n-1})$ GDS where all λs are equal to $n/4-2$ except $\lambda_{n/4} = \lambda_{3n/4} = 0$

and $\lambda_{n/2} = n/2 - 2$.

Recall that there exists $g, g' \in D^c$ such that $g' - g = n/4$. It can be confirmed by algebra that (i) the elements in $g - D'$ and $(g + n/4) - D'$ are all distinct respectively, (ii) $(g - D') \cap ((g + n/4) - D') = \emptyset$, and (iii) $n/4, n/2, 3n/4 \notin (g - D') \cup ((g + n/4) - D')$.

By (i) and (ii), we have total exactly $n-4$ distinct differences from $(g-D')\cup((g+n/4)-D')$. By (iii), all differences appear once except $n/4$, $n/2$, and $3n/4$ when we calculate the differences from $(g - D') \cup ((g + n/4) - D')$. Similarly, all differences appear once except $n/4$, $n/2$, and $3n/4$ if the differences are from $(D' - g) \cup (D' - (g + n/4))$. In addition, the differences from $\{g, g + n/4\}$ are $n/4$ and $3n/4$. Hence, $D' \cup \{g, g + n/4\} = V_0$ is a $(n, n/2; \lambda_1, \cdots, \lambda_{n-1})$ GDS where all λ's equal $(n/4 - 2) + 2 = n/4$ except $\lambda_{n/4} = \lambda_{3n/4} = 1$ and $\lambda_{n/2} = n/2 - 2$. This guarantees that the condition $\lambda_r^{0,0} = n/4$ holds for $r = 1, 2, \cdots, n/4 - 1$.

Next we claim that $\lambda^{0,0}_{r_1,r_2} = n/8$ for $1 \leq r_1 < r_2 \leq n/4 - 1$. Here, we firstly prove that $\lambda^{0,0}_{r_1,r_2} = \lambda^{1,1}_{r_1,r_2}$ for $1 \leq r_1 < r_2 \leq n/4 - 1$. Define $T^i_{r_1,r_2} = \{(x, y, z)|$ for all $x, y, z \in V_i$ such that $x - y \equiv r_1 \pmod n$ and $x - z \equiv r_2 \pmod n\}$ for $i = 0, 1$, where $V_0 = D \cup (D + n/2) \cup \{g, g + n/4\}$ and $V_1 = (D + n/4) \cup (D + 3n/4) \cup \{g + n/2, g + 3n/4\}$. Trivially, $|T^i_{r_1,r_2}| = \lambda^{i,i}_{r_1,r_2}$. Let u' be the element $u + n/4 \in (S + n/4)$ where $u \in S$. We show that there is a one-to-one correspondence between the triplets in $T^0_{r_1,r_2}$ and $T^1_{r_1,r_2}$, which implies that $\lambda^{0,0}_{r_1,r_2} = |T^0_{r_1,r_2}| = |T^1_{r_1,r_2}| = \lambda^{1,1}_{r_1,r_2}$. Without loss of generality, we assume that $\lambda^{0,0}_{r_1,r_2} \neq 0$. It implies that there exists $x, y, z \in V_0$ such that $x - y \equiv r_1 \pmod n$ and $x - z \equiv r_2 \pmod n$. There are three cases for the triplet (x, y, z), they are summarized as below.

(i) If $x, y, z \in D' \subset V_0$, then there exists $x', y', z' \in (D' + n/4) \subset V_1$ such that $x' - y' = (x + n/4) - (y + n/4) = x - y \equiv r_1 \pmod n$ and $x' - z' \equiv r_2 \pmod n$. Therefore, there is a one-to-one correspondence between the triplets $(x, y, z) \in T^0_{r_1,r_2}$ and $(x', y', z') \in T^1_{r_1,r_2}$.

(ii) If one component in $(x, y, z) \in T^0_{r_1,r_2}$ is not in D', then it must be in $\{g, g+n/4\}$. Without loss of generality, we assume that $x \in \{g, g+n/4\}$. If $x = g$ and $y, z \in D'$, then there exists $x' = g + 3n/4 \in V_1$ and $y', z' \in (D' + 3n/4) \subset V_1$ such that $x' - y' = (g + 3n/4) - (y + 3n/4) = g - y \equiv r_1 \pmod{n}$ and $x' - z' = (g + 3n/4) - (z + 3n/4) = g - y \equiv r_2 \pmod{n}$. If $x = g + n/4$ and $y, z \in D'$, then we have $x - y = (g + n/4) - y \equiv r_1 \pmod{n}$ and $x - z = (g + n/4) - z \equiv r_2 \pmod{n}$. Again, there exists $x' = g + n/2 \in V_1$ and $y', z' \in (D' + n/4)$ such that $x' - y' = (g + n/2) - (y + n/4) = (g + n/4) - y \equiv r_1 \pmod{n}$ and $x' - z' = (g + n/2) - (z + n/4) = (g + n/4) - z \equiv r_2 \pmod{n}$. So there is a one-to-one correspondence between the triplets $(x, y, z) \in T^0_{r_1,r_2}$ and $(x', y', z') \in T^1_{r_1,r_2}$.

(iii) If only one component in (x, y, z) is in D', then it can be shown that either one of $\{r_1, r_2\}$ is equal to $\lfloor n/4 \rfloor$ or $r_2 - r_1 = \lfloor n/4 \rfloor$. In this case, the equality $\lambda^{0,0}_{r_1,r_2} = \lambda^{1,1}_{r_1,r_2}$ does not always hold.

According to the above summary, we prove that $\lambda^{0,0}_{r_1,r_2} = |T^0_{r_1,r_2}| = |T^1_{r_1,r_2}| = \lambda^{1,1}_{r_1,r_2}$ when $r_1, r_2, r_2 - r_1 \neq n/4, 3n/4$. By Proposition 1 (i), $\lambda^{0,0}_{r_1,r_2} = n/8$ because of $\lambda^{0,0}_{r_1,r_2} = \lambda^{1,1}_{r_1,r_2}$ for $1 \leq r_1 < r_2 \leq n/4 - 1$ and $1 \leq r \leq n/4 - 1$. By Proposition 1 (ii), $\lambda^{\alpha,\beta}_{r_1,r_2} = n/2 - (n/4 + n/4 - \lambda^{\alpha,\alpha}_{r_1,r_2}) = \lambda^{\alpha,\alpha}_{r_1,r_2} = n/8$ for $\alpha, \beta \in \{0, 1\}$ and $1 \leq r_1 < r_2 \leq n/4 - 1$.

Let $\boldsymbol{\Phi}_1 = (n/4)\boldsymbol{J}_2$, $\boldsymbol{\Phi}_2 = (n/8)\boldsymbol{J}_2$, $V = \{V_0, V_1\}$ is an $(n, n/4, 2; \boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2)$-HCDS. By Theorem 2, the incidence matrix of V is a uniform $CAOA(n, n/4, 2, 3, 0)$.

Tables for Generating Vectors

According to the Corollary 1, we list the generating vectors of $CAOA(s^k, k, s, t, 0)$ when $8 \leq s^k \leq 1000$ and $2 \leq s \leq 10$. All the designs in Table 1 are constructed via De Bruijn frequency graphs. In addition, Table 2 shows the generating vectors of $CAOA(n, k, 2, 3, 0)$, which are constructed by Theorem 3, when $n/4 - 1$ is a prime power and $8 \leq n \leq 392$.

Table 1: $CAOA(s^k, k, s, t, 0)$ for $8 \leq s^k \leq 1000$ and $2 \leq s \leq 10$, where $2 \leq t \leq k$.

n	s	k	Generating Vector
8	2	3	00011101
16	2	4	0000111101011001
32	2	5	00000111110101101110010100110001
64	2	6	0000001111110101011101101111001001011001101001110001010001100001
128	2	7	00000001111111010101101011110110111011111001001100101010010111001101 1001110100111100010010001011000110100011100001010000011000001
256	2	8	0000000011111111010101010111010110110101111101101111011101111110010010 1001001110010101100101101001011110011001101010011011100111011001110 1001111100010001001100010101000101110001100100011011000111010001110 0001001000010110000110100001110000010100000110000001

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Pott, A., Reuschling, D., and Schmidt, B. (1997). A multiplier theorem for projections of affine difference sets. *Journal of statistical planning and inference* **62(1)**, 63–67.

n	s	k	Generating Vector
512	2	9	000000000111111110101010110101011110101101110101110110101111101101 10111110111011110111111100100100101100100110100100111100101001100101 01010010101110010110110010111010010111110011001110011010110011011010 01101111001110101001110111001111011001111101001111110001000110001001 01000100111000101001000101011000101101000101111000110011000110101000 110111000111001000111011000111101000111110000010001000010011000010101 00001011100000110010000110110000111010000111100000010010000001011000001 101000000111000000010100000001100000001
9	3	2	200102112
27	3	3	0001112220222121021101201002
81	3	4	000011112222020212022221212200220122102211210021012110211101011201020 1100120010002
243	3	5	00000111112222202022021220221202222121221222002020021200222012020121 20122210202102121022211202112121122000220012201022011221002210122110 22111210002100121010210112110021101211102111101011011120100201012011 020111001010011200102001100012000100002
729	3	6	00000011111122222202020212020222021212021222022022122022212022222121 21222122122220020020120020220021020021120021220022020022120022220120 12022012102012112012122012202012212012222102022102102112102122102202 10221210222211202211211212211220211221211222000202000212000222001202 00121200122201020201021201022201120201121201122210020210021210022210 12021012121012221102021102121102221112021112121112200002200012200102 20011220100220101220110220111221000221001221010221011221100221101221 11022111121000021000121001021001121010021010121011021011121100021100 1211010211011211100211101211110211110101011101101111201000201001201 01020101120110020110120111020111100100101100110100111200100200101200 1102001110001010001120001020001100001200001000002
16	4	2	3001020311213223
64	4	3	0001112223330331332320321322022123023121021131031101201301002003
256	4	4	00001111222233330303130323033313132313332323300330133023310331133123 32033213322320032013202321032113212322032213222020212022212122302030 21302231203121312200220122102211230023012310231121002101211021113100 3101311031110101120102011301030110012001300100020003
25	5	2	4001020304112131422324334
125	5	3	00011122233344404414424434304314324330331332340341342320321322420421 422022123023124024121021131031141041101201301401002003004

n	s	k	Generating Vector
625	5	4	0000111122223333444404041404240434044441414241434144424243424443434400440144024403441044114412441344204421442244234430443144324433430043014302430343104311431243134320432143224323433043314332433303031303230333131323133323233403040314032403341304131413241334230423142324233003301330233103311331233203321332233400340134023410341134123420342134223200320132023210321132123220322132224200420142024210421142124220422142220202120222121223020302130223120312131224020402140224120412141220022012210221123002301231023112400240124102411210021012110211131003101311031114100410141104111010112010201130103011401040110012001300140010002000300004
36	6	2	500102030405112131415223242533435445
216	6	3	000111222333444555055155255355454054154254354404414424434504514524534304314324335305315325330331332340341342350351352320321322420421422520521522022123023124024125025121021131031141041151051101201301401501002003004005
49	7	2	6001020304050611213141516223242526334353644546556
343	7	3	0001112223334445556660661662663664665650651652653654655055155255355456056156256356454054154254354464064164264364404414424434504514524534604614624634304314324335305315325336306316326330331332340341342350351352360361362320321322420421422520521522620621622022123023124024125025126026121021131031141041151051161061101201301401501601002003004005006
64	8	2	7001020304050607112131415161722324252627334353637445464755657667
512	8	3	0001112223334445556667770771772773774775776760761762763764765766066166266366466566706716726736746756506516526536546557507517527537547550551552553554560561562563564570571572573574540541542543544640641642643644740741742743744044144244345045145245346046146246347047147247343043143243353053153253363063163263373073173273303313323403413423503513523603613623703713723203213224204214225205215226206216227207217220221230231240241250251260261270271210211310311410411510511610611710711012013014015016017010020030040050060 07
81	9	2	800102030405060708112131415161718223242526272833435363738445464748556575866768778

n	s	k	Generating Vector
729	9	3	000111222333444555666777888088188288388488588688788708718728738748758 768770771772773774775776780781782783784785786760761762763764765766686 086186286386486586660661662663664665670671672673674675680681682683684 685650651652653654655750751752753754755850851852853854855505515525535 545605615625635645705715725735745805815825835845405415425435446406641 642643644740741742743744840841842843844044144244345045145245346046614 624634704714724734804814824834304314324335305315325336306316326333730 731732733830831832833033133234034134235035135236036136237037137233803 813823203213224204214225205215226206216227207217228208218220221230 23 124024125025126026127027128028121021131031141041151051161061171071118 1081101201301401501601701801002003004005006007008
100	10	2	9001020304050607080911213141516171819223242526272829334353637383944 5 4647484955657585966768697787988 9
1000	10	3	000111222333444555666777888999099199299399499599699799898098198298 39 849859869879880881882883884885886887890891892893894895896897870871 87 287387487587687797097197297397497597697707717727737747757767807817 82 783784785786790791792793794795796760761762763764765766860861862863 86 486586696096196296396496596606616626636646656706716726736746756806 81 682683684685690691692693694695650651652653654655750751752753754755 85 085185285385485595095195295395495505515525535545605615625635645705 71 572573574580581582583584590591592593594540541542543544640641642643 64 474074174274374484084184284384494094194294394404414424434504514524 53 460461462463470471472473480481482483490491492493430431432433530531 53 253363063163263373073173273383083183283393093193293303313323403413 42 350351352360361362370371372380381382390391392320321322420421422520 52 152262062162227207217228208218229209219220221230231240241250251260 61 270271280281290291210211310311410411510511610611710711810811910911 01 201301401501601701801901002003004005006007008009

Table 2: $CAOA(n, k, 2, 3, 0)$ for $8 \leq n \leq 392$.

n	k	Generating Vector
8	3	00010111
16	4	0010110000111101
24	6	000100111010000101111011
32	8	00100010110111000010001111011101
40	10	1000010110011110100010000101110111101001
48	12	000011001010111100110100000011001011111100110101
56	14	00000110010100111110011010100000011001010111111001101011
72	18	00010000110100010011101111001011101000010000110100010111101111001011 1011
80	20	00101000110010000010110101110011011111000010100011001000001111010111 001101111101
96	24	00110101000110111111001011001010111001000000110000110101000110111111 0011110010101110010000001101
104	26	10101111001110110010111110010100001100010011010000001010111100111011 001011111101010000110001001101000001
112	28	10001111101110010001010010000111000000100011011101011011010001111011 10010001010010010111000001000110111010110111
120	30	00011100101011101111110010010011100011010100010000001101101000011100 1010111011111100100101111000110101000100000011011011
128	32	00010110100010011000100001111010111010010111011001110111100001000001 011010001001100010000111101111101001011101100111011110000101
152	38	00100100111111101001111010101100011100110110110000000101100001010100 11100010001001001111111010011110101011000111011101101100000001011000 0101010011100011
168	42	01100011111010001010011111011110101101100010011100000101110101100000 10000101001001100110001111101000101001111101111010110110011001110000 0101110101100000100001010010011
176	44	00000001100011110001001110110100100110101010111111100111000011101100 01001011011001010100000000011000111100010011101101001001101010111111 1110011100001110110001001011011001010101

n	k	Generating Vector
192	48	0011011001000000010111100101111010101110011100101100100110111111010 0001101000010101000110001100001101100100000001011110010111101010110 01110011110010011011111110100001101000010101000110001101
200	50	1011110111111100100101100110000111001010100010111001000010000001101 10100110011110001101010111010000101111011111110010010110011000011100 1010100010111101000010000000110110100110011110001101010111010001
216	54	001111000111110111011000000101011000100010100100101100110000011100000 10001001111110101001110111010110110100100011110001111101110110000001 01011000100010100100101101110000111000001000100111111010100111011101 011011010011
240	60	00111001101010010000001000011101101000101011100000011001001011000110 01010110111111011110001001011101010001111110011011000011100110101001 00000010000111011010001010111000000110010011110001100101011011111101 111000100101110101000111111001101101
248	62	00000010111001110111100011101001111011011010001001101111010100111111 01000110001000011100010110000100100101110110010000101010000000101110 01110111100011101001111011011010001001101111010101111111010001100010 00011100010110000100100101110110010000101011
272	68	00011000110101101110100001100000001010100110100001000111110010011010 11100111001010010001011110011111110101011001011110111000001101100100 00011000110101101110100001100000001010100110100001000111110010011011 11100111001010010001011110011111110101011001011110111000001101100101
288	72	00111111111001110001111010001110111001000100100001011011011001010101 00101100000000011000111000010111000100011011101101111010010010011010 10101100001111111110011100011110100011101110010001001000010110110110 010101010011110000000000110001110000101110001000110111011011110100100 1001101010101101
296	74	00001000111000011011000101110010101011111111001000010011100101101101 11010011110111000111100100111010001101010100000000110111101100011010 01001000101000001000111000011011000101110010101011111111001000010011 10010110110111010111110111000111100100111010001101010100000000110111 101100011010010010001011
320	80	00110000000011101110100001101000001101100111001010000110100001000100 10101011001011001111111100010001011110010111110010011000110101111001 01111011101101010100110000110000000011101110100001101000001101100111 00101000011010000100010010101011001111001111111100010001011110010111 110010011000110101111001011110111011010101001101

n	k	Generating Vector
328	82	1000110011110101000111111000101101101111101011100011110110101001 0000 0010110011011001110011000010101110000001110100100100000101000111 0000 1001010110111111010011001000100011001111010100011111100010110110 1111 1010111000111101101010010000001011001101110111001100001010111000 0001 110100100100000101000111000010010101101111111010011001001
336	84	0011110111011000110001111111100111101011110100000101100101010110 1100 1001110111010010110000100010011100111000000001100001010000101111 1010 0110101010010011011000100010110000111101110110001100011111111001 1110 1011110100000101100101010110110010011101110100111100001000100111 0011 10000000011000010100001011111010011010101001001101100010001011 01
360	90	0110001111010001000111101101100101111101010110000000101000011000 1110 1001000100001011011000100111000010111011100001001001101000000101 01001 1111110101111001110001011011101111010010011001100011110100010001 1110 1101100101111101010110000000101000011000111010010001000010110110 0110 0111000010111011100001001001101000000101010011111110101111001110 00101 1011101111010010 0111
392	98	0000010010010110011100101011010001011111000100010001000100101000 0100 0011111100100110000111000101001111101101101001100011010100101110 1000 0011101110111011101101011110111100000011011001111000111010100000 0100 1001011001110010101101000101111100010001000100010010100001000011 1111 0010011000011100010101111110110110100110001101010010111010000011 1011 101110111011010111101111000000110110011110001110 1011