A Unified Theory for Robust Bayesian Prediction Under a General Class of Regret Loss Functions

Nader Nematollahi[a,*], Mohammad Jafari Jozani[b] and Razieh Jafaraghaie[c]

[a] *Department of Statistics, Allameh Tabataba'i University, Tehran, Iran.*

[b] *Department of Statistics, University of Manitoba, Winnipeg, MB, CANADA, R3T 2N2*

[c] *Department of Mathematics and Statistics, Shoushtar Branch, Islamic Azad University, Shoushtar, Iran.*

Supplementary Material

Summary: This supplementary material provides the details for deriving some of the results in the paper. It also contains a comprehensive account for real data applications associated with our second and third studies in Section 5 of the paper. We provide complimentary numerical results to show how PRGM predictors of the finite population mean perform compared with their corresponding Bayes predictors.

S1 Proof of Lemma 1

Suppose that $L(\gamma(\mathbf{y}),\delta) \leq M$ for all $(\gamma(\mathbf{y}),\delta) \in \mathbb{R}^2$. Since $L(\gamma(\mathbf{y}),\delta)$ is a strictly BS loss function in both $\gamma(\mathbf{y})$ and δ, then $\lim_{\gamma(\mathbf{y})\to\infty} L(\gamma(\mathbf{y}),\delta) = \lim_{\delta\to\infty} L(\gamma(\mathbf{y}),\delta) = M$. Denote the unique (finite) minimizer of $\int L(\gamma(\mathbf{y}),\delta(\mathbf{y}_s))h(\gamma(\mathbf{y})|\mathbf{y}_s)d\gamma(\mathbf{y})$ in δ to be δ^π. Since $L(\cdot,\cdot)$ is bounded,

$$\begin{aligned}&\lim_{\delta(\mathbf{y}_s)\to\infty}\int L(\gamma(\mathbf{y}),\delta(\mathbf{y}_s))h(\gamma(\mathbf{y})|\mathbf{y}_s)d\gamma\\&\quad=\int \lim_{\delta(\mathbf{y}_s)\to\infty} L(\gamma(\mathbf{y}),\delta(\mathbf{y}_s))h(\gamma(\mathbf{y})|\mathbf{y}_s)d\gamma(\mathbf{y}) = M.\end{aligned}$$

*Corresponding author: nematollahi@atu.ac.ir

Also, $0 < \int L(\gamma(\mathbf{y}), \delta^\pi(\mathbf{y}_s))h(\gamma(\mathbf{y})|\mathbf{y}_s)d\gamma(\mathbf{y}) = m < M$. Hence,

$$\begin{aligned}
&\lim_{\delta(\mathbf{y}_s)\to\infty}\left\{\int L(\gamma(\mathbf{y}), \delta(\mathbf{y}_s))h(\gamma(\mathbf{y})|\mathbf{y}_s)d\gamma(\mathbf{y}) - \inf_{\delta(\mathbf{y}_s)}\int L(\gamma(\mathbf{y}), \delta(\mathbf{y}_s))h(\gamma(\mathbf{y})|\mathbf{y}_s)d\gamma(\mathbf{y})\right\}\\
&= \lim_{\delta(\mathbf{y}_s)\to\infty}\left\{\int L(\gamma(\mathbf{y}), \delta(\mathbf{y}_s))h(\gamma(\mathbf{y})|\mathbf{y}_s)d\gamma(\mathbf{y}) - \int L(\gamma(\mathbf{y}), \delta^\pi(\mathbf{y}_s))h(\gamma(\mathbf{y})|\mathbf{y}_s)d\gamma(\mathbf{y})\right\}\\
&= M - m\\
&< M = \lim_{\delta(\mathbf{y}_s)\to\infty} L(\delta^\pi(\mathbf{y}_s), \delta(\mathbf{y}_s)).
\end{aligned}$$

Thus, for sufficiently large $\delta(\mathbf{y}_s)$ and bounded $L(\cdot,\cdot)$, the equality (2.3) in the paper can not be true and this completes the proof.

S2 Proof of Theorem 1

First note that $L(\gamma(\mathbf{y}), \delta)$ is a regret loss function and it is a strictly BS in both $\gamma(\mathbf{y})$ and δ. Since

$$\underline{\delta}(\mathbf{y}_s) = \inf_{\pi\in\Gamma}\delta^\pi(\mathbf{y}_s) \le \delta^\pi(\mathbf{y}_s) \le \sup_{\pi\in\Gamma}\delta^\pi(\mathbf{y}_s) = \overline{\delta}(\mathbf{y}_s),$$

one can easily see that, for all $\mathbf{y}_s \in \mathbb{R}^n$,

$$\begin{aligned}
\sup_{\pi\in\Gamma} R(\delta(\mathbf{y}_s), \delta^\pi(\mathbf{y}_s)) &= \sup_{\pi\in\Gamma} L(\delta^\pi(\mathbf{y}_s), \delta(\mathbf{y}_s))\\
&= \max\left\{L(\underline{\delta}(\mathbf{y}_s), \delta(\mathbf{y}_s)), L(\overline{\delta}(\mathbf{y}_s)), \delta(\mathbf{y}_s))\right\}.
\end{aligned}$$

By the BS property of $L(\overline{\delta}, \delta)$ ($L(\underline{\delta}, \delta)$) in δ, we conclude that $L(\overline{\delta}, \delta)$ ($L(\underline{\delta}, \delta)$) is a strictly decreasing (increasing) function of δ for $\underline{\delta} < \delta < \overline{\delta}$. Let $D(\delta) = L(\overline{\delta}, \delta) - L(\underline{\delta}, \delta)$, then $D(\delta)$ is a decreasing function of δ for $\underline{\delta} < \delta < \overline{\delta}$ and $D(\overline{\delta})D(\underline{\delta}) < 0$. So, there exists a $\delta^p(\mathbf{y}_s)$ such that $D(\delta^p(\mathbf{y}_s)) = 0$, or equivalently $L(\overline{\delta}(\mathbf{y}_s), \delta^p(\mathbf{y}_s)) = L(\underline{\delta}(\mathbf{y}_s), \delta^p(\mathbf{y}_s))$ for all $\mathbf{y}_s$. Also, for any $\delta \in \mathcal{D}$, $\max\{L(\underline{\delta}(\mathbf{y}_s), \delta(\mathbf{y}_s)), L(\overline{\delta}(\mathbf{y}_s)),$ $\delta(\mathbf{y}_s))\}$ is increasing in either direction about $\delta^p(\mathbf{y}_s)$. By the BS property of

$L(\underline{\delta}(\mathbf{y}_s), \delta(\mathbf{y}_s))$ and $L(\overline{\delta}(\mathbf{y}_s)), \delta(\mathbf{y}_s))$, we have

$$\max\left\{L(\underline{\delta}(\mathbf{y}_s), \delta(\mathbf{y}_s)), L(\overline{\delta}(\mathbf{y}_s)), \delta(\mathbf{y}_s))\right\}$$
$$= \begin{cases} L(\overline{\delta}(\mathbf{y}_s), \delta(\mathbf{y}_s)) & \text{if } \underline{\delta}(\mathbf{y}_s) \leq \delta(\mathbf{y}_s) < \delta^p(\mathbf{y}_s), \\ L(\underline{\delta}(\mathbf{y}_s), \delta(\mathbf{y}_s)) & \text{if } \delta^p(\mathbf{y}_s) < \delta(\mathbf{y}_s) \leq \overline{\delta}(\mathbf{y}_s). \end{cases}$$

This shows that the minimum of $\sup_{\pi \in \Gamma} R(\delta(\mathbf{y}_s), \delta^\pi(\mathbf{y}_s))$ over all $\delta \in \mathcal{D}$ happens at $\delta^p(\mathbf{y}_s)$ and $L(\overline{\delta}(\mathbf{y}_s), \delta^p(\mathbf{y}_s)) = L(\underline{\delta}(\mathbf{y}_s), \delta^p(\mathbf{y}_s))$ for all $\mathbf{y}_s$.

S3 Derivation of PRGM predictors in Example 1

Note that all equation numbers are the same as those in the paper.

S3.1 SE loss function

Under Γ_μ, using (4.3), the Bayes predictor of $\gamma_1(\mathbf{y})$ is given by

$$\delta^\pi(\mathbf{y}_s) = \overline{y}_s - (1-f)(\overline{y}_s - \mu)B_0,$$

where $B_0 = \frac{\sigma^2}{\sigma^2 + n\tau_0^2}$. Since $\mu_1 \leq \mu \leq \mu_2$, using Table 1, the PRGM predictor of $\gamma_1(\mathbf{y})$ is

$$\delta_\mu^{PRGM}(\mathbf{y}_s) = \overline{y}_s - (1-f)B_0(\overline{y}_s - \frac{\mu_1 + \mu_2}{2}) = \overline{y}_s - (1-f)(\overline{y}_s - \mu^*)B_0 \tag{4.4}$$

Note that δ^{PRGM} is a Bayes predictor of $\gamma_1(\mathbf{y})$ w.r.t. $\pi_{\mu^*} \in \Gamma_\mu$ with $\mu^* = \frac{\mu_1+\mu_2}{2} \in [\mu_1, \mu_2]$. Under the class Γ_{τ^2} of priors, the Bayes predictor of $\gamma_1(\mathbf{y})$ is given by

$$\delta^\pi(\mathbf{y}_s) = \overline{y}_s - (1-f)(\overline{y}_s - \mu_0)B,$$

where $B_2 \leq B \leq B_1$ and $B_i = \frac{\sigma^2}{\sigma^2 + n\tau_i^2}$, $i = 1, 2$. For $\overline{y}_s \neq \mu_0$, and using Table 1, one can easily show that the PRGM predictor of $\gamma_1(\mathbf{y})$ is

$$\delta_\tau^{PRGM}(\mathbf{y}_s) = \overline{y}_s - (1-f)(\overline{y}_s - \mu_0)B^*, \tag{4.5}$$

where $B^* = \frac{B_1+B_2}{2} = \frac{\sigma^2}{\sigma^2+n\tau_*^2}$ and $\tau_*^2 = \frac{2n\tau_1^2\tau_2^2+\sigma^2(\tau_1^2+\tau_2^2)}{2\sigma^2+n\tau_1^2+n\tau_2^2}$. Here again, δ_τ^{PRGM} is a Bayes predictor of $\gamma_1(\mathbf{y})$ w.r.t. $\pi_{\tau_*} \in \Gamma_{\tau^2}$, $(\tau_1^2 \leq \tau_*^2 \leq \tau_2^2)$. Under the class Γ_ϵ of priors, the Bayes predictor of $\gamma_1(\mathbf{y})$ is obtained as

$$\delta^\pi(\mathbf{y}_s) = \lambda\delta^{\pi_0}(\mathbf{y}_s) + (1-\lambda)\delta^q(\mathbf{y}_s),$$

with $\delta^{\pi_0}(\mathbf{y}_s) = E_{\pi_0}(\gamma_1(\mathbf{y})|\mathbf{y}_s)$, $\delta^q(\mathbf{y}_s) = E_q(\gamma_1(\mathbf{y})|\mathbf{y}_s)$ and $\lambda = (1-\epsilon)\frac{m(\mathbf{y}_s|\pi_0)}{m(\mathbf{y}_s|\pi)}$. Note that $m(\mathbf{y}_s|\pi) = (1-\epsilon)m(\mathbf{y}_s|\pi_0) + \epsilon m(\mathbf{y}_s|q)$ with $m(\mathbf{y}_s|\pi_0)$ and $m(\mathbf{y}_s|q)$ denoting the marginal (predictive) densities of $\mathbf{y}_s$ under the prior distributions π_0 and q, respectively. Using Theorem 3 of Ghosh and Kim (1993), we have

$$\underline{\delta}(\mathbf{y}_s) = f\,\overline{y}_s + (1-f)\frac{a\delta^0(\mathbf{y}_s) + \theta_l f(\mathbf{y}_s|\theta_l)}{a + f(\mathbf{y}_s|\theta_l)}$$

and

$$\overline{\delta}(\mathbf{y}_s) = f\,\overline{y}_s + (1-f)\frac{a\delta^0(\mathbf{y}_s) + \theta_u f(\mathbf{y}_s|\theta_u)}{a + f(\mathbf{y}_s|\theta_u)},$$

where $a = \frac{1-\epsilon}{\epsilon}m(\mathbf{y}_s|\pi_0)$, $f(\mathbf{y}_s|\theta_.)$ is the conditional density of $\mathbf{y}_s$ given $\theta_.$, $\delta^0(\mathbf{y}_s) = E_{\pi_0}(\theta|\mathbf{y}_s)$, $\theta_l = \frac{\sigma}{\sqrt{n}}\nu_l + \overline{y}_s$, $\theta_u = \frac{\sigma}{\sqrt{n}}\nu_u + \overline{y}_s$, and the values of ν_l and ν_u are obtained as the solutions of the following equation

$$e^{-\nu^2/2} - c\nu^2 - b\nu + c = 0,$$

by noting that $\nu = \frac{\sqrt{n}(\theta-\overline{y}_s)}{\sigma}$, $c = a(2\pi\sigma^2)^{\frac{n}{2}}\exp\{\frac{\sum_{i\in s}(y_i-\overline{y}_s)^2}{2\sigma^2}\}$ and $b = c\frac{\sqrt{n}(\overline{y}_s-\delta^0)}{\sigma}$. Now, using Table 1, the PRGM predictor of $\gamma_1(\mathbf{y})$ under the SEL function and the class Γ_ϵ of prior distributions is given by

$$\delta_\epsilon^{PRGM}(\mathbf{y}_s) = f\overline{y}_s + \frac{(1-f)}{2}\left(\frac{a\delta^0(\mathbf{y}_s) + \theta_l f(\mathbf{y}_s|\theta_l)}{a + f(\mathbf{y}_s|\theta_l)} + \frac{a\delta^0(\mathbf{y}_s) + \theta_u f(\mathbf{y}_s|\theta_u)}{a + f(\mathbf{y}_s|\theta_u)}\right). \quad (4.6)$$

Note that if $\pi_0(\cdot)$ is the density of $N(\mu_0, \tau_0^2)$-distribution then $\delta^0(\mathbf{y}_s) = \overline{y}_s - B_0(\overline{y}_s - \mu_0)$, and if $\epsilon = 0$ then $\Gamma_\epsilon = \{\pi_0\}$, and accordingly

$$\delta_0^{PRGM}(\mathbf{y}_s) = \delta^{\pi_0}(\mathbf{y}_s) = \overline{y}_s - (1-f)(\overline{y}_s - \mu_0)B_0.$$

One can easily observe that $\delta_{\epsilon}^{PRGM}(\mathbf{y}_s)$ can be considered as a compromise between the Bayes predictor under the prior distribution π_0 (associated with $\epsilon = 0$ corresponding to the case where one is very confident in π_0) and the predictor obtained as the mid-range of the class of Bayes predictors under the ϵ-contaminated class of priors when ϵ is close to one.

S3.2 LINEX loss function

Under the class Γ_{μ} of priors, and using the conditional distribution of $\gamma_1(\mathbf{y})$ given $\mathbf{y}_s$, the Bayes predictor of $\gamma_1(\mathbf{y})$ under the LINEX loss function is given by

$$\begin{aligned}\delta^{\pi}(\mathbf{y}_s) &= \frac{-1}{c} \ln E(e^{-c\gamma(\mathbf{y})}|\mathbf{y}_s) \\ &= \overline{y}_s - (1-f)B_0(\overline{y}_s - \mu) - \frac{c(1-f)\sigma^2}{2n}(1-(1-f)B_0),\end{aligned}$$

where $\mu_1 \leq \mu \leq \mu_2$. Hence, from Table 1, the PRGM predictor of $\gamma_1(\mathbf{y})$ is obtained as follows

$$\begin{aligned}\delta_{\tau}^{PRGM}(\mathbf{y}_s) = \overline{y}_s &- \frac{c(1-f)\sigma^2}{2n}(1-(1-f)B_0) \\ &- \frac{1}{c} \ln \frac{e^{cB_0(1-f)(\overline{y}_s-\mu_1)} - e^{cB_0[(1-f)(\overline{y}_s-\mu_2)}}{c(\mu_2-\mu_1)(1-f)B_0}. \qquad (4.8)\end{aligned}$$

Under the class Γ_{τ^2} of priors, the Bayes predictor of $\gamma_1(\mathbf{y})$ is given by

$$\delta^{\pi}(\mathbf{y}_s) = \overline{y}_s - (1-f)B(\overline{y}_s - \mu_0) - \frac{c(1-f)\sigma^2}{2n}(1-(1-f)B),$$

where $B_2 \leq B \leq B_1$ with $B_i = \frac{\sigma^2}{\sigma^2+n\tau_i^2}$, $i = 1, 2$. Given $\overline{y}_s \neq \mu_0$ and $c \neq 0$, using Table 1, the PRGM predictor of $\gamma_1(\mathbf{y})$ under the LINEX loss function with respect to class Γ_{τ^2} of priors has the following form

$$\begin{aligned}\delta_{\tau}^{PRGM}(\mathbf{y}_s) = \overline{y}_s &- \frac{c(1-f)\sigma^2}{2n} \\ &- \frac{1}{c} \ln \frac{e^{cB_2[(1-f)(\overline{y}_s-\mu_0)-\frac{c(1-f)^2\sigma^2}{2n}]} - e^{cB_1[(1-f)(\overline{y}_s-\mu_0)-\frac{c(1-f)^2\sigma^2}{2n}]}}{c(B_2-B_1)[(1-f)(\overline{y}_s-\mu_0)-\frac{c(1-f)^2\sigma^2}{2n}]}. \qquad (4.9)\end{aligned}$$

Under the class Γ_ϵ of priors, we expand Theorem 3 of Ghosh and Kim (1993) and find

$$\underline{\delta}(\mathbf{y}_s) = f\overline{y}_s - \frac{c\sigma^2(1-f)^2}{2(N-n)} - \frac{1}{c}\ln\frac{a_0 + e^{-c(1-f)\theta_l} f(\mathbf{y}_s|\theta_l)}{a + f(\mathbf{y}_s|\theta_l)},$$

and

$$\overline{\delta}(\mathbf{y}_s) = f\overline{y}_s - \frac{c\sigma^2(1-f)^2}{2(N-n)} - \frac{1}{c}\ln\frac{a_0 + e^{-c(1-f)\theta_u} f(\mathbf{y}_s|\theta_u)}{a + f(\mathbf{y}_s|\theta_u)},$$

where $t = E(e^{-c(1-f)\theta}|\mathbf{y}_s)$, $a_0 = at$, and a, θ_l and θ_u are defined in Example 1. Also, the values of ν_l and ν_u are the solutions of the following equation

$$e^{-c(1-f)(\nu\frac{\sigma}{\sqrt{n}}+\overline{y}_s)}[-c(1-f)(k + e^{-\frac{\nu^2}{2}}) - k\nu\frac{\sqrt{n}}{\sigma}] + kt\nu\frac{\sqrt{n}}{\sigma} = 0,$$

by noting that $\nu = \frac{\sqrt{n}(\theta-\overline{y}_s)}{\sigma}$, and $k = a(2\pi\sigma^2)^{\frac{n}{2}}\exp\{\frac{\sum_{i\in s}(y_i-\overline{y}_s)^2}{2\sigma^2}\}$. Now, using Table 1, the PRGM predictor of $\gamma(\mathbf{y})$ under the LINEX loss function and class Γ_ϵ of prior distributions is easy to obtain as follows

$$\begin{aligned}\delta_\epsilon^{PRGM}(\mathbf{y}_s) = f\overline{y}_s &- \frac{c\sigma^2(1-f)^2}{2(N-n)} \\ &- \frac{1}{c}\ln\frac{\frac{a_0+e^{-c(1-f)\theta_u}f(\mathbf{y}_s|\theta_u)}{a+f(\mathbf{y}_s|\theta_u)} - \frac{a_0+e^{-c(1-f)\theta_l}f(\mathbf{y}_s|\theta_l)}{a+f(\mathbf{y}_s|\theta_l)}}{\ln\frac{a_0+e^{-c(1-f)\theta_u}f(\mathbf{y}_s|\theta_u)}{a+f(\mathbf{y}_s|\theta_u)} - \ln\frac{a_0+e^{-c(1-f)\theta_l}f(\mathbf{y}_s|\theta_l)}{a+f(\mathbf{y}_s|\theta_l)}}. \qquad (4.10)\end{aligned}$$

S4 Derivation of PRGM predictors in Example 2

Note that all equation numbers are the same as those in the paper.

S4.1 SE loss function

Under the class Γ_μ of priors, the Bayes predictor of $\gamma_1(\mathbf{y})$ using (4.16) is the posterior mean, that is

$$\delta^\pi(\mathbf{y}_s) = f\overline{y}_s + ((1-B_{s0})\mu + \frac{b_s}{d_s}B_{s0})\frac{a_s}{N},$$

where $B_{s0} = \frac{\tau_0^2 d_s}{\sigma^2+\tau_0^2 d_s}$. Since $\mu_1 \le \mu \le \mu_2$, using Table 1, the PRGM predictor of $\gamma_1(\mathbf{y})$ is given by

$$\delta_{\mu}^{PRGM}(\mathbf{y}_s) = f\overline{y}_s + ((1-B_{s0})\mu^* + \frac{b_s}{d_s}B_{s0})\frac{a_s}{N}. \tag{4.19}$$

Note that δ^{PRGM} is itself a Bayes predictor of $\gamma_1(\mathbf{y})$ w.r.t. $\pi_{\mu^*} \in \Gamma_\mu$ with $\mu^* = \frac{\mu_1+\mu_2}{2} \in [\mu_1, \mu_2]$.

Under the class Γ_{τ^2} of priors, the Bayes predictor of $\gamma_1(\mathbf{y})$ is

$$\delta^\pi = f\bar{y}_s + \frac{a_s}{N}\mu_0 + \frac{a_s}{N}\left(\frac{b_s}{d_s} - \mu_0\right)B_s$$

where $B_{s1} \le B_s \le B_{s2}$ and $B_{si} = \frac{\tau_i^2 d_s}{\sigma^2+\tau_i^2 d_s}$, $i = 1, 2$. For $\dfrac{b_s}{d_s} \neq \mu_0$, and using Table 1, one can easily show that the PRGM predictor of $\gamma_1(\mathbf{y})$ is given by

$$\delta_{\tau}^{PRGM}(\mathbf{y}_s) = f\bar{y}_s + \frac{a_s}{N}\mu_0 + \frac{a_s}{N}\left(\frac{b_s}{d_s} - \mu_0\right)B_s^*, \tag{4.20}$$

where $B_s^* = \frac{B_{s1}+B_{s2}}{2} = \frac{\tau_s^{2*} d_s}{\sigma^2+\tau_s^{2*} d_s}$ and $\tau_s^{2*} = \dfrac{2d_s\tau_1^2\tau_2^2 + \sigma^2(\tau_1^2+\tau_2^2)}{2\sigma^2 + d_s(\tau_1^2+\tau_2^2)}$. Here again, δ_{τ}^{PRGM} is a Bayes predictor of $\gamma_1(\mathbf{y})$ w.r.t. $\pi_{\tau_s^*} \in \Gamma_{\tau^2}$, $(\tau_1^2 \le \tau_s^{2*} \le \tau_2^2)$. Under the class Γ_ϵ of priors, we extend Theorem 3 of Ghosh and Kim (1993) and find $\underline{\delta}(\mathbf{y}_s)$ and $\overline{\delta}(\mathbf{y}_s)$ as follow

$$\underline{\delta}(\mathbf{y}_s) = f\,\overline{y}_s + (1-f)\bar{x}_{\bar{s}}\frac{a\delta^0(\mathbf{y}_s) + \beta_l f(\mathbf{y}_s|\beta_l)}{a + f(\mathbf{y}_s|\beta_l)},$$

and

$$\overline{\delta}(\mathbf{y}_s) = f\,\overline{y}_s + (1-f)\bar{x}_{\bar{s}}\frac{a\delta^0(\mathbf{y}_s) + \beta_u f(\mathbf{y}_s|\beta_u)}{a + f(\mathbf{y}_s|\beta_u)},$$

where $a = \frac{1-\epsilon}{\epsilon}m(\mathbf{y}_s|\pi_0)$, $f(\mathbf{y}_s|\beta_.)$ is the conditional density of $\mathbf{y}_s$ given $\beta_.$, $\delta^0(\mathbf{y}_s) = E_{\pi_0}(\beta|\mathbf{y}_s)$,

$\beta_l = \frac{\sigma}{\sqrt{d_s}}\nu_l + \frac{b_s}{d_s}$, $\beta_u = \frac{\sigma}{\sqrt{d_s}}\nu_u + \frac{b_s}{d_s}$, and the values of ν_l and ν_u are the solutions of the following equation

$$e^{-\nu^2/2} - c\nu^2 - b\nu + c = 0,$$

by noting that $\nu = \sqrt{d_s}(\beta - \frac{b_s}{d_s})/\sigma$, and $c = a(2\pi\sigma^2)^{\frac{n}{2}} exp\{\frac{1}{2\sigma^2}\sum_{i\in s}(y_i^2 - \frac{b_s^2}{d_s})\}$ and $b = c\frac{\sqrt{d_s}}{\sigma}(\frac{b_s}{d_s} - \delta^0(\mathbf{y}_s))$. So, using Table 1, the PRGM predictor of $\gamma_1(\mathbf{y})$ under the SEL function and the class Γ_ϵ of prior distributions is given by

$$\delta_\epsilon^{PRGM}(\mathbf{y}_s) = f\overline{y}_s + \frac{(1-f)\bar{x}_{\bar{s}}}{2}\left(\frac{a\delta^0(\mathbf{y}_s) + \beta_l f(\mathbf{y}_s|\beta_l)}{a + f(\mathbf{y}_s|\beta_l)} + \frac{a\delta^0(\mathbf{y}_s) + \beta_u f(\mathbf{y}_s|\beta_u)}{a + f(\mathbf{y}_s|\beta_u)}\right). \tag{4.21}$$

S4.2 LINEX loss function

Considering the class Γ_μ of priors for β in model (4.14), first the Bayes predictor of $\gamma(\mathbf{y})$ using (4.18) is

$$\delta^\pi(\mathbf{y}_s) = \frac{1}{N}a_s(1-B_{s0})\mu + f\bar{y}_s + a_s b_s\frac{B_{s0}}{Nd_s} - \frac{c\sigma^2}{2N^2}\left((N-n) + \frac{B_{s0}}{d_s}a_s^2\right).$$

For $a_s > 0$,

$$\underline{\delta}^\pi(\mathbf{y}_s) = \frac{1}{N}a_s(1-B_{s0})\mu_1 + f\bar{y}_s + a_s b_s\frac{B_{s0}}{Nd_s} - \frac{c\sigma^2}{2N^2}\left((N-n) + \frac{B_{s0}}{d_s}a_s^2\right),$$

and

$$\overline{\delta}^\pi(\mathbf{y}_s) = \frac{1}{N}a_s(1-B_{s0})\mu_2 + f\bar{y}_s + a_s b_s\frac{B_{s0}}{Nd_s} - \frac{c\sigma^2}{2N^2}\left((N-n) + \frac{B_{s0}}{d_s}a_s^2\right).$$

For $a_s < 0$,

$$\underline{\delta}^\pi(\mathbf{y}_s) = \frac{1}{N}a_s(1-B_{s0})\mu_2 + f\bar{y}_s + a_s b_s\frac{B_{s0}}{Nd_s} - \frac{c\sigma^2}{2N^2}\left((N-n) + \frac{B_{s0}}{d_s}a_s^2\right),$$

and

$$\overline{\delta}^\pi(\mathbf{y}_s) = \frac{1}{N}a_s(1-B_{s0})\mu_1 + f\bar{y}_s + a_s b_s\frac{B_{s0}}{Nd_s} - \frac{c\sigma^2}{2N^2}\left((N-n) + \frac{B_{s0}}{d_s}a_s^2\right).$$

So, we have

$$\begin{aligned}\delta_\mu^{PRGM}(\mathbf{y}_s) = f\bar{y}_s + a_s b_s\frac{B_{s0}}{Nd_s} - \frac{c\sigma^2}{2N^2}\left((N-n) + \frac{B_{s0}}{d_s}a_s^2\right)\\ - \frac{1}{c}\ln\frac{e^{-\frac{ca_s}{N}(1-B_{s0})\mu_2} - e^{-\frac{ca_s}{N}(1-B_{s0})\mu_1}}{-c\frac{1}{N}a_s(1-B_{s0})(\mu_2-\mu_1)}.\end{aligned} \tag{4.22}$$

Under the class Γ_{τ^2} of priors, the Bayes predictor of $\gamma_1(\mathbf{y})$ is given by

$$\delta^\pi(\mathbf{y}_s) = (\frac{a_s}{N}(\frac{b_s}{d_s} - \mu_0) - \frac{c\sigma^2}{2N^2}\frac{a_s^2}{d_s})B_s + f\bar{y}_s + \frac{a_s}{N}\mu_0 - \frac{c\sigma^2}{2N}(1-f),$$

where $B_{s1} \leq B_s \leq B_{s2}$ and $B_{si} = \frac{\tau_i^2 d_s}{\sigma^2+\tau_i^2 d_s}$, $i = 1, 2$. Given $\frac{\overline{y}_s}{\bar{x}_s} \neq \mu_0$ and $c \neq 0$, the PRGM predictor of $\gamma_1(\mathbf{y})$ under the LINEX loss function with respect to class Γ_{τ^2} of priors is obtained as follow

$$\begin{aligned}\delta_\tau^{PRGM}(\mathbf{y}_s) &= f\bar{y}_s + \frac{a_s}{N}\mu_0 - \frac{c\sigma^2}{2N}(1-f)\\ &\quad - \frac{1}{c}\ln\frac{e^{-cB_{s1}(\frac{a_s}{N}(\frac{b_s}{d_s}-\mu_0)-\frac{c\sigma^2}{2N^2}\frac{a_s^2}{d_s})} - e^{-cB_{s2}(\frac{a_s}{N}(\frac{b_s}{d_s}-\mu_0)-\frac{c\sigma^2}{2N^2}\frac{a_s^2}{d_s})}}{-c(B_{s1}-B_{s2})(\frac{a_s}{N}(\frac{b_s}{d_s}-\mu_0)-\frac{c\sigma^2}{2N^2}\frac{a_s^2}{d_s})}. \quad (4.23)\end{aligned}$$

Under the class Γ_ϵ of priors, we expand Theorem 3 of Ghosh and Kim (1993) and find $\underline{\delta}(\mathbf{y}_s)$ and $\overline{\delta}(\mathbf{y}_s)$ as follow

$$\underline{\delta}(\mathbf{y}_s) = f\overline{y}_s - \frac{c\sigma^2(1-f)^2}{2(N-n)} - \frac{1}{c}\ln\frac{b_0 + e^{-c(1-f)\beta_l\bar{x}_{\bar{s}}}f(\mathbf{y}_s|\beta_l)}{a + f(\mathbf{y}_s|\beta_l)},$$

and

$$\overline{\delta}(\mathbf{y}_s) = f\overline{y}_s - \frac{c\sigma^2(1-f)^2}{2(N-n)} - \frac{1}{c}\ln\frac{b_0 + e^{-c(1-f)\beta_u\bar{x}_{\bar{s}}}f(\mathbf{y}_s|\beta_u)}{a + f(\mathbf{y}_s|\beta_u)},$$

where $a = \frac{1-\epsilon}{\epsilon}m(\mathbf{y}_s|\pi_0)$, $t = E(e^{-c(1-f)\beta\bar{x}_{\bar{s}}}|\mathbf{y}_s)$, $b_0 = at$, $\beta_l = \frac{\sigma}{\sqrt{d_s}}\nu_l + \frac{b_s}{d_s}$, $\beta_u = \frac{\sigma}{\sqrt{d_s}}\nu_u + \frac{b_s}{d_s}$ and the values of ν_l and ν_u are the solutions of the following equation

$$e^{-c(1-f)\bar{x}_{\bar{s}}(\frac{\sigma}{\sqrt{d_s}}\nu+\frac{b_s}{d_s})}[(k + e^{-\frac{\nu^2}{2}}) - k\frac{\sqrt{d_s}}{\sigma}\nu] + kt\frac{\sqrt{d_s}}{\sigma}\nu = 0,$$

by noting that $\nu = \sqrt{d_s}(\beta - \frac{b_s}{d_s})/\sigma$, and $k = a(2\pi\sigma^2)^{\frac{n}{2}}exp\{\frac{1}{2\sigma^2}(\sum_{i\in s} y_i^2 - \frac{b_s^2}{d_s})\}$. So, from Table 1,

$$\begin{aligned}\delta_\epsilon^{PRGM}(\mathbf{y}_s) &= f\overline{y}_s - \frac{c\sigma^2(1-f)^2}{2(N-n)}\\ &\quad - \frac{1}{c}\ln\frac{\frac{b_0+e^{-c(1-f)\beta_u\bar{x}_{\bar{s}}}f(\mathbf{y}_s|\beta_u)}{a+f(\mathbf{y}_s|\beta_u)} - \frac{b_0+e^{-c(1-f)\beta_l\bar{x}_{\bar{s}}}f(\mathbf{y}_s|\beta_l)}{a+f(\mathbf{y}_s|\beta_l)}}{\ln\frac{b_0+e^{-c(1-f)\beta_u\bar{x}_{\bar{s}}}f(\mathbf{y}_s|\beta_u)}{a+f(\mathbf{y}_s|\beta_u)} - \ln\frac{b_0+e^{-c(1-f)\beta_l\bar{x}_{\bar{s}}}f(\mathbf{y}_s|\beta_l)}{a+f(\mathbf{y}_s|\beta_l)}}. \quad (4.24)\end{aligned}$$

S5 Applications and simulation studies

In this section, we consider models (4.14) and (4.25) and present two more real world applications of our results and perform further simulation studies. To this end, we first consider the problem of predicting the average seventh-month weight of sheep in a finite population consisting of 224 sheep at the Research Farm of Ataturk University, Erzurum, Turkey, which were used by Ozturk et al. (2005) and Jafaraghaie and Nematollahi (2018). We compute and compare the performance of the Bayes and PRGM predictors of the underlying population mean. Also, we adopt the non-normal model (4.25) and use a cancer dataset corresponding to remission time (in months) of 128 bladder cancer patients from a study conducted by the American Cancer Society. We predict the average remission time using the Bayes and PRGM approach and compare their performance over different classes of priors using a simulation study.

S5.1 Application to a Sheep data

This data set contains the values of the seventh-month weights, mother's weight at mating and birth weight of 224 sheep which are measured in kilograms (see Ozturk et al. (2005) for details). Jafaraghaie and Nematollahi (2018) considered the seventh-month weight and mother's weight at mating as response and auxiliary variables, respectively, and showed that the regression model (4.14) provides a good fit to this data. They showed that the response variable has approximately normal distribution with mean 28.11 and variance 15.21, and the maximum likelihood estimates of β and σ^2 were obtained as in Table 1. Also, they argued the need for robust Bayes prediction in this problem. To this end, we consider a single prior $N(0.2, 0.1)$ as well as two classes $\Gamma_\mu = \{N(\mu, \tau_0^2) : \mu \in [0.2, 0.8]\}$ and $\Gamma_{\tau^2} = \{N(\mu_0, \tau^2) : \tau^2 \in [0.1, 0.7] \subseteq \Re^+\}$ of priors for β. We obtain the Bayes predictor

Table 1: The maximum likelihood estimates of the parameters, β and σ^2.

Parameters	Estimate	Std. Error
β	0.53785	0.00422
σ^2	10.9767	1.10149

($\delta^{\pi_{\mu_0,\tau_0^2}}$ with $\mu_0 = 0.2$ and $\tau_0^2 = 0.1$), the PRGM predictor over the class Γ_μ ($\delta_{\Gamma_\mu}^{PRGM}$) and the PRGM predictor over the class Γ_{τ^2} ($\delta_{\Gamma_{\tau^2}}^{PRGM}$). Table 2 summarizes the predicted values under the SE loss function for fixed sample size $n = 50$. As we observe the PRGM predicted values are closer to the mean of the seventh-month weight of sheep, i.e., 28.11 than the corresponding Bayes predcition.

Table 2: The Bayes and PRGM predicted values of the finite population mean over Γ_μ and Γ_{τ^2} under the SE loss function.

$\delta^{\pi_{\mu_0,\tau_0^2}}$	$\delta_{\Gamma_\mu}^R$	$\delta_{\Gamma_{\tau^2}}^{PRGM}$
28.03038	28.03991	28.03496

To evaluate the performance of each predictor, we perform a simulation study similar to Subsection 5.1 and calculate the EMSE and EAB of each predictor for different sample sizes ($n = 20, 30,$ and 50). We consider two classes of prior distributions. The first class Γ_μ of priors is chosen to be $\Gamma_\mu = \{N(\mu, \tau_0^2) : \mu \in [0.2, 0.8]\}$. In this setting we obtain the PRGM predictor of the population mean under Γ_μ as well as its Bayes predictor under some specific normal distributions with $\mu_0 = 0.2, 0.4, 0.6, 0.8$ and $\tau_0^2 = 0.1, 0.3, 0.5, 0.7$ as prior distributions for θ. We also consider another class of priors $\Gamma_{\tau^2} = \{N(\mu_0, \tau^2) : \tau^2 \in [0.1, 0.7] \subseteq \Re^+\}$ and study the performance of the PRGM predictors of the population mean compared with their Bayes predictors with respect to normal prior distributions with $\mu_0 = 0.2, 0.4, 0.6, 0.8$ and $\tau_0^2 = 0.1, 0.3, 0.5, 0.7$. For simulation studies, we consider a superpopulation model $y_i = 0.53785\ x_i + \epsilon_i$ with $\epsilon_i \sim N(0, 10.9767)$. The parameters of the superpopulation model are obtained using our real-data set. We use the same steps as in Jafaraghaie and Nematollahi (2018), to generate samples from this superpopulation model and calculate the EMSE and EAB of the predictors. The estimated MSE and bias of each predictor are presented in Tables 3 and 4. From Table 3 we observe that for all values of τ_0^2 and small values of μ_0 ($\mu_0 = 0.2, 0.4$), PRGM predictors perform reasonably well compared with Bayes predictors and for large values of μ_0 ($\mu_0 = 0.6, 0.8$), Bayes predictors perform well compared with PRGM predictors. Table 4 presents that for all values of τ_0^2 and small values of μ_0 ($\mu_0 = 0.2$), and also, for moderate to large values of τ_0^2 ($\tau_0^2 = 0.3, 0.5, 0.7$) and moderate to large values of μ_0 ($\mu_0 = 0.4, 0.6, 0.8$), PRGM predictors perform reasonably well compared with Bayes predictors. We have opposite results for small values of τ_0^2 ($\tau_0^2 = 0.1$) and moderate to large values of μ_0 ($\mu_0 = 0.4, 0.6, 0.8$).

S5.2 Application to a bladder cancer data

In this subsection, we consider a data set corresponding to remission times (in months) of 128 bladder cancer patients with average remission time 9.366. This dataset was previously studied by Lee and Wang (2003) and Lemonte and Cordeiro (2013). Bladder cancer is a disease in which abnormal cells multiply without control in the bladder. The most common type of bladder cancer recapitulates the normal histology of the urothelium and is known as transitional cell carcinoma (Zea et al., 2012). These remission times are a subset of the data from a bladder cancer study conducted by the American Cancer Society. By using this dataset, we would like to predict the average remission time of the population under the

Table 3: Simulated MSE and absolute bias for the Bayes and PRGM predictors for values $\tau_0^2 = 0.1, 0.3, 0.5, 0.7$, $\mu_0 = 0.2, 0.4, 0.6, 0.8$ and $\mu \in [0.2, 0.8]$ over Γ_μ (sheep data).

		δ^{π}				δ^{PRGM}
τ_0^2	n	$\mu_0 = 0.2$	0.4	0.6	0.8	
0.1	20	0.46015	0.45908	0.45873	0.45910	0.45881
	30	0.28276	0.28222	0.28197	0.28200	0.28206
EMSE	50	0.15236	0.15224	0.15221	0.15227	0.15222
	20	0.54154	0.54097	0.54084	0.54108	0.54086
EAB	30	0.42669	0.42626	0.42602	0.42601	0.42612
	50	0.31309	0.31291	0.31281	0.31280	0.31284
0.3	20	0.46011	0.45994	0.45985	0.45985	0.45989
	30	0.28260	0.28250	0.28243	0.28239	0.28246
EMSE	50	0.15236	0.15234	0.15234	0.15234	0.15234
	20	0.54159	0.54152	0.54149	0.54150	0.54150
EAB	30	0.42655	0.42645	0.42638	0.42633	0.42641
	50	0.31302	0.31298	0.31295	0.31293	0.31296
0.5	20	0.46021	0.46013	0.46008	0.46006	0.46010
	30	0.28262	0.28256	0.28252	0.28249	0.28254
EMSE	50	0.15237	0.15237	0.15236	0.15236	0.15236
	20	0.54167	0.54164	0.54163	0.54162	0.54163
EAB	30	0.42655	0.42650	0.42645	0.42642	0.42647
	50	0.31302	0.31300	0.31298	0.31297	0.31299
0.7	20	0.46026	0.46022	0.46018	0.46016	0.46020
	30	0.28263	0.28259	0.28256	0.28254	0.28258
EMSE	50	0.15238	0.15238	0.15237	0.15237	0.15237
	20	0.54171	0.54169	0.54168	0.54168	0.54169
EAB	30	0.42655	0.42652	0.42649	0.42646	0.42650
	50	0.31302	0.31300	0.31299	0.31298	0.31300

SE loss function. The Kolmogorov-Smirnov test shows that we may assume the original data is a random sample from a gamma-distribution with shape parameter $\alpha = 1.1726$ and rate parameter $\theta = 0.125$ with a p-value of 0.5085. So, the data can be considered as a realization of a gamma superpopulation model. Then, for this data, the model (4.25) holds and the maximum likelihood estimates of the parameters is given in Table 5.

Table 4: Simulated MSE and absolute bias for the Bayes and PRGM predictors for values $\tau_0^2 = 0.1, 0.3, 0.5, 0.7$, $\mu_0 = 0.2, 0.4, 0.6, 0.8$ and $\tau^2 \in [0.1, 0.7]$ over Γ_{τ^2} (sheep data).

		δ^{π}				δ^{PRGM}
μ_0	n	$\tau_0^2 = 0.1$	0.3	0.5	0.7	
0.2	20	0.46015	0.46011	0.46021	0.46026	0.46002
	30	0.28276	0.28260	0.28262	0.28263	0.28262
EMSE	50	0.15236	0.15236	0.15237	0.15238	0.15235
	20	0.54154	0.54159	0.54167	0.54171	0.54150
EAB	30	0.42669	0.42655	0.42655	0.42655	0.42657
	50	0.31309	0.31302	0.31302	0.31302	0.31303
0.4	20	0.45908	0.45994	0.46013	0.46022	0.45961
	30	0.28222	0.28250	0.28256	0.28259	0.28239
EMSE	50	0.15224	0.15234	0.15237	0.15238	0.15231
	20	0.54097	0.54152	0.54164	0.54169	0.54132
EAB	30	0.42626	0.42645	0.42650	0.42652	0.42638
	50	0.31291	0.31298	0.31300	0.31300	0.31295
0.6	20	0.45873	0.45985	0.46008	0.46018	0.45945
	30	0.28197	0.28243	0.28252	0.28256	0.28226
EMSE	50	0.15221	0.15234	0.15236	0.15237	0.15229
	20	0.54084	0.54149	0.54163	0.54168	0.54126
EAB	30	0.42602	0.42638	0.42645	0.42649	0.42625
	50	0.31281	0.31295	0.31298	0.31299	0.31290
0.8	20	0.45910	0.45985	0.46006	0.46016	0.45952
	30	0.28200	0.28239	0.28249	0.28254	0.28223
EMSE	50	0.15227	0.15234	0.15236	0.15237	0.15231
	20	0.54108	0.54150	0.54162	0.54168	0.54132
EAB	30	0.42601	0.42633	0.42642	0.42646	0.42620
	50	0.31280	0.31293	0.31297	0.31298	0.31288

To predict the population mean, we consider a single prior $\Gamma(7,2)$ as well as two classes

$\Gamma_a = \{\Gamma(a, b_0) : a \in [3, 9]\}$ and $\Gamma_b = \{\Gamma(a_0, b) : b \in [2, 8] \subseteq \mathbb{R}^+\}$ of priors for θ. We obtain the Bayes predictor ($\delta^{\pi_{a_0,b_0}}$ with $a_0 = 7$ and $b_0 = 2$), the PRGM predictor over the class Γ_b ($\delta^{PRGM}_{\Gamma_b}$) and the PRGM predictor over the class Γ_a ($\delta^{PRGM}_{\Gamma_a}$). Table 6 summarizes the predicted values under the SE loss function when $n = 50$.

Table 5: The maximum likelihood estimates of the parameters θ and α.

Parameters	Estimate	Std. Error
θ	0.1252	0.0173
α	1.1726	0.1308

As we observe the PRGM predicted values are closer to the population mean , i.e., 9.366 than the Bayes prediction.

Table 6: The Bayes and PRGM predicted values of the finite population mean over Γ_b and Γ_a under the SE loss function.

$\delta^{\pi_{a_0,b_0}}$	$\delta^{PRGM}_{\Gamma_b}$	$\delta^{PRGM}_{\Gamma_a}$
9.265631	9.298799	9.363406

In order to obtain the bias and precision associated with each predicted value, we perform a simulation study as follows:

1. Generate $y_1^*, y_2^*, ..., y_n^*$ from a $\Gamma(1.1726, 0.1252)$ distribution.

2. Consider y_i^*, $i = 1, ..., n$ as a sample generated from the underlying model.

3. Calculate the Bayes and PRGM predictors.

4. Repeat steps 1-3 for $b = 10^4$ times and calculate the value of EMSE and EAB of the predictors using the following formula:

$$\text{EMSE} = \frac{1}{b}\sum_{i=1}^{b}(\hat{\delta}_i^k - \bar{Y})^2, \quad \text{EAB} = |\frac{1}{b}\sum_{i=1}^{b}(\hat{\delta}_i^k - \bar{Y})|, \quad k = Bayes, PRGM,$$

where, $\hat{\delta}_i^k$ is the predictor in i-th repetition of sampling and $\bar{Y}$ is the population mean.

We consider two classes of prior distributions. The first class Γ_b of priors is chosen to be $\Gamma_b = \{\Gamma(a_0, b) : b \in [2, 8] \subseteq \mathbb{R}^+\}$. In this setting we obtain the PRGM predictor of the population mean under Γ_b as well as its Bayes predictor under some specific gamma distributions with $a_0 = 3, 5, 7, 9$, $b_0 = 2, 4, 6, 8$ as prior distributions for θ. We also consider another class of priors $\Gamma_a = \{\Gamma(a, b_0) : a \in [3, 9]\}$ and study the performance of PRGM predictors of the population mean compared with their corresponding Bayes predictors with respect to normal prior distributions with $a_0 = 3, 5, 7, 9$, $b_0 = 2, 4, 6, 8$. The estimated MSE and the bias of each predictor are presented in Tables 7 and 8. From Table 7, we observe that the performance of PRGM predictors with respect to Bayes predictor are quite satisfactory in terms of EMSE as well as the associated bias for all values of a_0 and small values of b_0 ($b_0 = 2, 4$). But we have quite the opposite results for large values of b_0 ($b_0 = 6, 8$). Note that the MSE and the bias decrease as the sample size increases. From Table 8, we observe that for all values of b_0 and large values of a_0 ($a_0 = 7, 9$), PRGM predictors are preferred to Bayes predictors in terms of EMSE and EAB. Also, EMSEs and EABs decrease as the sample size increases.

Table 7: Simulated MSE and absolute bias for the Bayes and PRGM predictors of finite population mean for $a_0 = 3, 5, 7, 9$, $b_0 = 2, 4, 6, 8$ and $b \in [2, 8]$ over Γ_b (cancer data).

		δ^{π}				δ^{PRGM}
	n	$b_0 = 2$	$b_0 = 4$	$b_0 = 6$	$b_0 = 8$	
$a_0 = 3$	20	4.357	4.281	4.216	4.164	4.247
	30	2.695	2.665	2.640	2.619	2.652
EMSE	50	1.312	1.305	1.299	1.294	1.302
	20	1.711	1.690	1.671	1.655	1.681
EAB	30	1.340	1.330	1.321	1.314	1.325
	50	0.934	0.931	0.928	0.926	0.930
$a_0 = 5$	20	4.734	4.586	4.448	4.320	4.515
	30	2.854	2.793	2.737	2.685	2.765
EMSE	50	1.351	1.336	1.323	1.310	1.329
	20	1.823	1.790	1.758	1.728	1.774
EAB	30	1.392	1.376	1.360	1.346	1.368
	50	0.948	0.943	0.938	0.933	0.940
$a_0 = 7$	20	5.549	5.348	5.156	4.973	5.251
	30	3.213	3.128	3.047	2.969	3.087
EMSE	50	1.446	1.424	1.403	1.384	1.414
	20	2.008	1.967	1.928	1.889	1.947
EAB	30	1.489	1.468	1.447	1.427	1.457
	50	0.981	0.973	0.966	0.959	0.970
$a_0 = 9$	20	6.631	6.391	6.159	5.936	6.274
	30	3.716	3.611	3.510	3.412	3.560
EMSE	50	1.586	1.558	1.531	1.505	1.545
	20	2.232	2.186	2.142	2.098	2.164
EAB	30	1.620	1.594	1.569	1.545	1.582
	50	1.029	1.020	1.010	1.002	1.015

Table 8: Simulated MSE and absolute bias for the Bayes and PRGM predictors of finite population variance for $b_0 = 2, 4, 6, 8$, $a_0 = 3, 5, 7, 9$ and $a \in [3, 9]$ over Γ_a (cancer data).

		δ^{π}				δ^{PRGM}
	n	$a_0 = 3$	$a_0 = 5$	$a_0 = 7$	$a_0 = 9$	
$b_0 = 2$	20	4.357	4.734	5.549	6.631	4.974
	30	2.695	2.854	3.213	3.716	2.973
EMSE	50	1.312	1.351	1.446	1.586	1.386
	20	1.711	1.823	2.008	2.232	1.880
EAB	30	1.340	1.392	1.489	1.620	1.425
	50	0.934	0.948	0.981	1.029	0.960
$b_0 = 4$	20	4.281	4.586	5.348	6.391	4.805
	30	2.665	2.793	3.128	3.611	2.902
EMSE	50	1.305	1.336	1.424	1.558	1.368
	20	1.690	1.790	1.967	2.186	1.844
EAB	30	1.330	1.376	1.468	1.594	1.407
	50	0.931	0.943	0.973	1.020	0.954
$b_0 = 6$	20	4.216	4.448	5.156	6.159	4.647
	30	2.640	2.737	3.047	3.510	2.836
EMSE	50	1.299	1.323	1.403	1.531	1.351
	20	1.671	1.758	1.928	2.142	1.809
EAB	30	1.321	1.360	1.447	1.569	1.389
	50	0.928	0.938	0.966	1.010	0.948
$b_0 = 8$	20	4.164	4.320	4.973	5.936	4.498
	30	2.619	2.685	2.969	3.412	2.773
EMSE	50	1.294	1.310	1.384	1.505	1.335
	20	1.655	1.728	1.889	2.098	1.775
EAB	30	1.314	1.346	1.427	1.545	1.372
	50	0.926	0.933	0.959	1.002	0.942

References

Ghosh, M. and Kim, D. H. (1993). Robust Bayes estimation of the finite population mean. *The Indian Journal of Statistics* **55**, 322–342.

Jafaraghaie, R. and Nematollahi, N. (2018). Robust Bayesian prediction under a general linear-exponential posterior risk function and its application in finite population.*Communications in Statistics: Theory and Methods* **47**, 3269–3292.

Lee, E. T. and Wang, J. (2003). *Statistical methods for survival data analysis.* Wiley, New York.

Lemonte, A. J. and Cordeiro, G. M. (2013). An extended lomax distribution. *Statistics* **47(4)**, 800–816.

Ozturk, O., Bilgin, O. C. and Wolfe, D. A. (2005). Estimation of population mean and variance in flock management: a ranked set sampling approach in a finite population setting. *Journal of Statistical Computation and Simulation* **75(11)**, 905–919.

Zea, L. M., Silva, R. B., Bourguignon, M., Santos, A. M. and Cordeiro, G. M. (2012). The beta exponentiated pareto distribution with application to bladder cancer susceptibility. *International Journal of Statistics and Probability* **1(2)**, 8.