CONSTRUCTION OF STRONG GROUP-ORTHOGONAL ARRAYS

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Supplementary Material

In this supplemental file, we provide proofs of the theorems and two large tables.

S1. Proof of Theorem 1

For the convenience of proof, define $A = (A_1, \ldots, A_g)$ and $B = (B_1, \ldots, B_g)$. Let a_{ij}, b_{ij} and t_{ij} be the jth columns of A_i, B_i and T_i, respectively.

(i) From Lemma 6.27 of Hedayat, Sloane and Stufken (1999) and the definition of A, we can see that A is an $\mathrm{OA}(sn_0, gs, s, 2)$. It is easy to see that after collapsing the factors of $\tilde{T}$ into s levels, $\tilde{T}$ becomes A. Thus any two distinct columns of $\tilde{T}$ can be collapsed into an $\mathrm{OA}(sn_0, 2, s, 2)$.

(ii) First we consider the space-filling property in two dimensions. Let $D = (d_{ij})_{s\times s}$ and d_j be the jth column of D. Without loss of generality, we assume $d_{11} = \cdots = d_{s1} = 0$. For other cases, the proof is similar. Here A and B are $\mathrm{OA}(sn_0, gs, s, 2)$'s. If $(a_{i_1j_1}, a_{i_2j_2}, b_{i_2j_2})$ and $(a_{i_2j_2}, a_{i_1j_1}, b_{i_1j_1})$ are OAs of strength 3 for $i_1 \neq i_2$, then $(t_{i_1j_1}, t_{i_2j_2})$ can be collapsed into an $\mathrm{OA}(sn_0, 2, s \times s^2, 2)$ and an $\mathrm{OA}(sn_0, 2, s^2 \times s, 2)$. Therefore, we only need to prove that $(a_{i_1j_1}, a_{i_2j_2}, b_{i_2j_2})$ is an OA of strength 3 in the following two cases.

(a) Here we consider the case: $j_1 = j_2 = 1$. Then $(a_{i_11}, a_{i_21}, b_{i_21}) = (d_1 \oplus c_{i_1}, d_1 \oplus$

$c_{i_2}, d_s \oplus c_{i_2})$ is an OA$(sn_0, 3, s, 3)$ due to its special structure

$$\begin{pmatrix} c_{i_1} & c_{i_2} & d_{1s} \dot{+} c_{i_2} \\ \vdots & \vdots & \vdots \\ c_{i_1} & c_{i_2} & d_{ss} \dot{+} c_{i_2} \end{pmatrix}.$$

(b) Next, we consider the case: $j_1 = 1$ and $j_2 > 1$. According to the definitions of A and B, $(a_{i_1 1}, a_{i_2 j_2}, b_{i_2 j_2})$ can be written as

$$\begin{pmatrix} c_{i_1} & d_{1j_2} \dot{+} c_{i_2} & d_{1(j_2-1)} \dot{+} c_{i_2} \\ \vdots & \vdots & \vdots \\ c_{i_1} & d_{sj_2} \dot{+} c_{i_2} & d_{s(j_2-1)} \dot{+} c_{i_2} \end{pmatrix}.$$

As (c_{i_1}, c_{i_2}) is an OA$(n_0, 2, s, 2)$, then $(c_{i_1}, d_{1j_2} \dot{+} c_{i_2})$ forms an OA$(n_0, 2, s, 2)$. And for any level combination $(\alpha, \beta, \gamma_1)$ in $(c_{i_1}, d_{1j_2} \dot{+} c_{i_2}, d_{1(j_2-1)} \dot{+} c_{i_2})$, there is a corresponding level combination $(\alpha, \beta, \gamma_k)$ in $(c_{i_1}, d_{kj_2} \dot{+} c_{i_2}, d_{k(j_2-1)} \dot{+} c_{i_2})$ for $k = 2, \ldots, s$, where $\gamma_1, \ldots, \gamma_s$ are distinct with each other and they are a permutation on $\{0, 1, \ldots, s-1\}$. Thus $(a_{i_1 1}, a_{i_2 j_2}, b_{i_2 j_2})$ is an OA$(sn_0, 3, s, 3)$.

For other cases $j_1, j_2 > 1$, the structures of $(a_{i_1 j_1}, a_{i_2 j_2}, b_{i_2 j_2})$'s can be considered the same as the one in case (b) because $(d_{j_1} \oplus c_{i_1}, d_{j_2} \oplus c_{i_2}, d_{j_2-1} \oplus c_{i_2})$ is equivalent to $(d_1 \oplus c_{i_1}, (d_{j_2} - d_{j_1}) \oplus c_{i_2}, (d_{j_2-1} - d_{j_1}) \oplus c_{i_2})$. Thus for $\tilde{T}$, any two columns from different T_i's can be collapsed into an OA$(sn_0, 2, s \times s^2, 2)$ and an OA$(sn_0, 2, s^2 \times s, 2)$.

Now we consider the column orthogonality of $\tilde{T}$. For all the cases considered above, we know that $(a_{i_1 j_1}, a_{i_2 j_2}, b_{i_2 j_2})$ for $i_1 \neq i_2$ is an OA$(sn_0, 3, s, 3)$. Without loss of generality, now assume the s levels in A and B are centered. Correspondingly we have $t_{i_1 j_1}^T t_{i_2 j_2} = (s a_{i_1 j_1} + b_{i_1 j_1})^T (s a_{i_2 j_2} + b_{i_2 j_2}) = s^2 a_{i_1 j_1}^T a_{i_2 j_2} + s a_{i_1 j_1}^T b_{i_2 j_2} + s b_{i_1 j_1}^T a_{i_2 j_2} + b_{i_1 j_1}^T b_{i_2 j_2} = 0$, implying that any two columns from different groups of $\tilde{T}$ are column orthogonal.

(iii) Similarly to the proof in (ii), we can get that $(a_{i_1j_1}, a_{i_2j_2}, a_{i_2j_3})$ with $i_1 \neq i_2$ and $j_2 \neq j_3$ must be an OA of strength 3. Thus for the resulting design, any three distinct columns from two different groups T_i and T_j with $i \neq j$ can be collapsed into an OA$(sn_0, 3, s, 3)$.

In summary, $\tilde{T}$ is an SGOA$(sn_0, gs, s^2, 2)$.

S2. Proof of Theorem 2

From Algorithm 1, we can see that $\tilde{T}$ becomes A after collapsing the factors into s levels. Therefore, we only need to prove that $A = (A_1, \ldots, A_g)$ achieves stratifications on $s \times s \times s$ grids with a proportion π. There are $\binom{gs}{3}$ possible combinations of choosing 3 distinct columns out of gs columns. Now we count the number of the combinations $(a_{i_1j_1}, a_{i_2j_2}, a_{i_3j_3})$'s which are OA$(sn_0, 3, s, 3)$'s, for $1 \leq i_1, i_2, i_3 \leq g$ and $1 \leq j_1, j_2, j_3 \leq s$. There are only two cases:

(a) Any two of i_k's, $k = 1, 2, 3$, are equal, for example $i_1 = i_2$ and $i_3 \neq i_1$. There are $gs(gs - s)(s - 1)/2$ possible choices for $(a_{i_1j_1}, a_{i_2j_2}, a_{i_3j_3})$. And from the proof of Theorem 1, under all these choices, $(a_{i_1j_1}, a_{i_2j_2}, a_{i_3j_3})$'s are OA$(sn_0, 3, s, 3)$'s.

(b) i_1, i_2 and i_3 are not equal to each other. As C is regular, any $(a_{i_1j_1}, a_{i_2j_2}, a_{i_3j_3})$ is an OA$(sn_0, 3, s, 3)$ if and only if $a_{i_3j_3}$ cannot be represented by the linear combination of $a_{i_1j_1}$ and $a_{i_2j_2}$ according to Theorem 3.29 of Hedayat, Sloane and Stufken (1999). For any given $a_{i_1j_1}$ and $a_{i_2j_2}$ with $i_1 \neq i_2$, there are $s - 1$ columns among all other $gs - 2s$ columns which can be represented by the linear combination of $a_{i_1j_1}$ and $a_{i_2j_2}$ since C is saturated. Thus there are $gs(gs - s)(gs - 2s - s + 1)/6$ choices in this case that can make each $(a_{i_1j_1}, a_{i_2j_2}, a_{i_3j_3})$ being an array of strength 3. Therefore for design

A, there are M three-tuples that are OA$(sn_0, 3, s, 3)$'s, where

$$M = \frac{gs(gs-s)(s-1)}{2} + \frac{gs(gs-s)(gs-3s+1)}{6} = \frac{gs(gs-s)(gs-2)}{6}.$$

Thus the proportion is equal to $\pi = (gs-s)/(gs-1)$.

S3. Proof of Theorem 3

For the convenience of proof, we denote $E = (E_1, \dots, E_g)$, $F = (F_1, \dots, F_g)$ and $G = (G_1, \dots, G_g)$, and let e_{ij}, f_{ij} and g_{ij} be the jth columns of E_i, F_i and G_i respectively, for $i = 1, \dots, g$ and $j = 1, \dots, s$. From Lemma 6.27 of Hedayat, Sloane and Stufken (1999), it is easy to see that E, F and G are OA$(s^2n_0, gs, s, 2)$'s. We first verify that the level size of T is s^3. From the construction, we only need to prove that (e_{ij}, f_{ij}, g_{ij}) is an OA$(s^2n_0, 3, s, 3)$, for any $i = 1, \dots, g$ and $j = 1, \dots, s$. Let d_j denote the jth column of D, for $j = 1, \dots, s$. Then according to the definitions of E, F and G, for $j = 1, j = 2, j \geq 3$, (e_{ij}, f_{ij}, g_{ij}) can be written as

$$\begin{pmatrix} d_1 & d_s & d_{s-1} \\ d_1 \dot{+} 1 & d_s & d_{s-1} \\ \vdots & \vdots & \vdots \\ d_1 \dot{+} (s-1) & d_s & d_{s-1} \end{pmatrix} \oplus c_i, \begin{pmatrix} d_2 & d_1 & d_s \\ d_2 \dot{+} 1 & d_1 & d_s \\ \vdots & \vdots & \vdots \\ d_2 \dot{+} (s-1) & d_1 & d_s \end{pmatrix} \oplus c_i, \text{and} \begin{pmatrix} d_j & d_{j-1} & d_{j-2} \\ d_j \dot{+} 1 & d_{j-1} & d_{j-2} \\ \vdots & \vdots & \vdots \\ d_j \dot{+} (s-1) & d_{j-1} & d_{j-2} \end{pmatrix} \oplus c_i,$$

respectively. For $j \geq 3$, from Lemma 6.12 of Hedayat, Sloane and Stufken (1999), we know $(d_j, d_{j-1}, d_{j-2}) \oplus c_i$ is an OA$(sn_0, 3, s, 2)$. And for any level combination $(\alpha_0, \beta, \gamma)$ in $(d_j, d_{j-1}, d_{j-2}) \oplus c_i$, there is a corresponding level combination $(\alpha_k, \beta, \gamma)$ in $(d_j \dot{+} k, d_{j-1}, d_{j-2}) \oplus c_i$ for $k = 1, \dots, s-1$, where $\alpha_0, \dots, \alpha_{s-1}$ are distinct with each other and they are a permutation on $\{0, 1, \dots, s-1\}$. Thus (e_{ij}, f_{ij}, g_{ij}) is an OA$(s^2n_0, 3, s, 3)$ for $j \geq 3$. And we can show that (e_{ij}, f_{ij}, g_{ij}) is an OA$(s^2n_0, 3, s, 3)$ for $j = 1$ or $j = 2$ by the similar argument.

(i) Now we prove the space-filling property in any two dimensions. It is easy to see that T becomes E after collapsing the factors into s levels, and it becomes $sE+F$ after collapsing the factors into s^2 levels. Let t_{ij} denote the jth column of T_i for $i = 1, \ldots, g$ and $j = 1, \ldots, s$. Here we want to verify that any two distinct columns $(t_{i_1 j_1}, t_{i_2 j_2})$ can be collapsed into an $\mathrm{OA}(s^2 n_0, 2, s \times s^2, 2)$ and an $\mathrm{OA}(s^2 n_0, 2, s^2 \times s, 2)$. To prove this, we only need to verify that $(e_{i_1 j_1}, e_{i_2 j_2}, f_{i_2 j_2})$ and $(e_{i_1 j_1}, f_{i_1 j_1}, e_{i_2 j_2})$ are $\mathrm{OA}(s^2 n_0, 3, s, 3)$'s. We now give the proof for $(e_{i_1 j_1}, e_{i_2 j_2}, f_{i_2 j_2})$ in three cases: (a) $i_1 = i_2$ and $j_1 \neq j_2$; (b) $i_1 \neq i_2$ and $j_1 = j_2$; and (c) $i_1 \neq i_2$ and $j_1 \neq j_2$. The proof for $(e_{i_1 j_1}, f_{i_1 j_1}, e_{i_2 j_2})$ is similar. For case (a), here we assume that $j_1 < j_2$, then $(e_{i_1 j_1}, e_{i_1 j_2}, f_{i_1 j_2})$ can be written as

$$\begin{pmatrix} d_{j_1} & d_{j_2} & d_{j_2-1} \\ d_{j_1} \dot{+} 1 & d_{j_2} \dot{+} 1 & d_{j_2-1} \\ \vdots & \vdots & \vdots \\ d_{j_1} \dot{+} (s-1) & d_{j_2} \dot{+} (s-1) & d_{j_2-1} \end{pmatrix} \oplus c_{i_1}.$$

It is easy to verify that $((d_{j_1} \dot{+} k, d_{j_2} \dot{+} k) \oplus c_{i_1})$ is an $\mathrm{OA}(sn_0, 2, s, 2)$, for $k = 0, \ldots, s-1$. And based on the properties of the difference scheme, for any level combination $(\alpha, \beta, \gamma_0)$ in $((d_{j_1}, d_{j_2}, d_{j_2-1}) \oplus c_{i_1})$, there is a corresponding level combination $(\alpha, \beta, \gamma_k)$ in $((d_{j_1} \dot{+} k, d_{j_2} \dot{+} k, d_{j_2-1}) \oplus c_{i_1})$ for $k = 1, \ldots, s-1$, where $\gamma_0, \ldots, \gamma_{s-1}$ are distinct with each other and they are a permutation on $\{0, 1, \ldots, s-1\}$. Thus $(e_{i_1 j_1}, e_{i_1 j_2}, f_{i_1 j_2})$ is an $\mathrm{OA}(s^2 n_0, 3, s, 3)$. The proofs of cases (b) and (c) are similar, so we omit them here.

(ii) First we prove the space-filling property of two columns selected from different groups. We consider the two columns $(t_{i_1 j_1}, t_{i_2 j_2})$ with $i_1 \neq i_2$. Here we need to verify that it can be collapsed into an OA$(s^2n_0, 2, s \times s^3, 2)$ and an OA$(s^2n_0, 2, s^3 \times s, 2)$. To prove this, we only need to verify that $(e_{i_1j_1}, t_{i_2j_2})$ with $i_1 \neq i_2$ is an OA$(s^2n_0, 2, s \times s^3, 2)$. Write it as

$$\left(\begin{pmatrix} d_{j_1} \\ d_{j_1} \dot{+} 1 \\ \vdots \\ d_{j_1} \dot{+} (s-1) \end{pmatrix} \oplus c_{i_1}, s^2 \begin{pmatrix} d_{j_2} \\ d_{j_2} \dot{+} 1 \\ \vdots \\ d_{j_2} \dot{+} (s-1) \end{pmatrix} \oplus c_{i_2} + s \begin{pmatrix} d_{j_2-1} \\ d_{j_2-1} \\ \vdots \\ d_{j_2-1} \end{pmatrix} \oplus c_{i_2} + \begin{pmatrix} d_{j_2-2} \\ d_{j_2-2} \\ \vdots \\ d_{j_2-2} \end{pmatrix} \oplus c_{i_2} \right),$$

where (d_{j_2-1}, d_{j_2-2}) takes (d_s, d_{s-1}) and (d_1, d_s) respectively when j_2 is 1 and 2. It is easy to see that $((d_{j_2} \dot{+} k, d_{j_2-1}, d_{j_2-2}) \oplus c_{i_2})$ contains s^2 distinct three-tuples based on $\{0, 1, \ldots, s-1\}$ for $k = 0, 1, \ldots, s-1$. Then for each k, $s^2(d_{j_2} \dot{+} k) \oplus c_{i_2} + sd_{j_2-1} \oplus c_{i_2} + d_{j_2-2} \oplus c_{i_2}$ consists of s^2 distinct levels in $\{0, 1, \ldots, s^3-1\}$, each of which occurs λs times, where λ is the index of C. Here we denote the s^2 different levels as $V_k = \{v_{k,0}, \ldots, v_{k,s^2-1}\}$ for $k = 0, 1, \ldots, s-1$. Since (c_{i_1}, c_{i_2}) is an OA of strength 2, $((d_{j_1} \dot{+} k) \oplus c_{i_1}, s^2(d_{j_2} \dot{+} k) \oplus c_{i_2} + sd_{j_2-1} \oplus c_{i_2} + d_{j_2-2} \oplus c_{i_2})$ must be an OA of strength of 2 based on $\{0, 1, \ldots, s-1\} \times \{v_{k,0}, v_{k,1}, \ldots, v_{k,s^2-1}\}$, of which each two-tuple occurs λ times. $V_0, V_1, \ldots, V_{s-1}$ are disjoint with each other, and $\bigcup_{k=0}^{s-1} V_k$ is a permutation on $\{0, 1, \ldots, s^3-1\}$. Thus, each two-tuple based on $\{0, 1, \ldots, s-1\} \times \{0, 1, \ldots, s^3-1\}$ occurs λ times in $(e_{i_1j_1}, t_{i_2j_2})$, implying that it is an OA$(s^2n_0, 2, s \times s^3, 2)$.

Then let us verify the column orthogonality. Now assume the s levels in E, F and G are centered. As discussed in (ii), $(e_{i_1j_1}, e_{i_2j_2}, f_{i_2j_2})$ is an OA of strength 3 for any $i_1 \neq i_2$ or $j_1 \neq j_2$, which is also true for $(e_{i_1j_1}, e_{i_2j_2}, g_{i_2j_2})$. Correspondingly we have

$$t_{i_1j_1}^T t_{i_2j_2} = (s^2 e_{i_1j_1} + sf_{i_1j_1} + g_{i_1j_1})^T (s^2 e_{i_2j_2} + sf_{i_2j_2} + g_{i_2j_2}) = s(f_{i_1j_1}^T g_{i_2j_2} + g_{i_1j_1}^T f_{i_2j_2}).$$

When $i_1 \neq i_2$, both $(f_{i_1j_1}, g_{i_2j_2})$ and $(f_{i_2j_2}, g_{i_1j_1})$ are OA$(s^2n_0, 2, s, 2)$'s, thus $f_{i_1j_1}^T g_{i_2j_2} = g_{i_1j_1}^T f_{i_2j_2} = 0$. Hence any two columns $t_{i_1j_1}$ and $t_{i_2j_2}$ with $i_1 \neq i_2$ are column orthogonal, implying that any two columns from different groups in $\overline{T}$ are column orthogonal.

(iii) Now we consider the three-dimensional space-filling property. It is easy to see that T becomes E after collapsing the factors into s levels. Thus it is equal to consider the three-dimensional property of E. Write it as

$$E = (E_1, \dots, E_g) = \begin{pmatrix} D \oplus c_1 & D \oplus c_2 & \dots & D \oplus c_g \\ D_1 \oplus c_1 & D_1 \oplus c_2 & \dots & D_1 \oplus c_g \\ \vdots & \vdots & & \vdots \\ D_{s-1} \oplus c_1 & D_{s-1} \oplus c_2 & \dots & D_{s-1} \oplus c_g \end{pmatrix}.$$

In the proof of Theorem 1, we have shown that any three distinct columns from two different groups of $(D \oplus c_1, \dots, D \oplus c_g)$ can be collapsed into an OA$(sn_0, 3, s, 3)$, which also holds for $(D_k \oplus c_1, \dots, D_k \oplus c_g)$ with $k = 1, \dots, s-1$. Thus, any three distinct columns from two different groups of E can be collapsed into an OA$(s^2n_0, 3, s, 3)$.

Now the proof is complete, $\overline{T}$ is an SGOA$(s^2n_0, gs, s^3, 3)$.

S4. Proof of Theorem 4

It is easy to see that $\overline{T}$ becomes E after collapsing the factors into s levels. Thus it is equal to consider the three-dimensional space-filling property of E. Here we only need to verify that $(e_{i_1j_1}, e_{i_2j_2}, e_{i_3j_3})$ is an OA$(s^{p+2}, 3, s, 3)$ when i_1, i_2 and i_3 are all different. It can be written as

$$\left(\begin{pmatrix} d_{j_1} \\ d_{j_1} \dot{+} 1 \\ \vdots \\ d_{j_1} \dot{+} (s-1) \end{pmatrix} \oplus c_{i_1}, \begin{pmatrix} d_{j_2} \\ d_{j_2} \dot{+} 1 \\ \vdots \\ d_{j_2} \dot{+} (s-1) \end{pmatrix} \oplus c_{i_2}, \begin{pmatrix} d_{j_3} \\ d_{j_3} \dot{+} 1 \\ \vdots \\ d_{j_3} \dot{+} (s-1) \end{pmatrix} \oplus c_{i_3} \right).$$

We can see that it is an $\mathrm{OA}(s^{p+2}, 3, s, 3)$ if $(c_{i_1}, c_{i_2}, c_{i_3})$ is an OA of strength 3. Thus we only consider the situation that $(c_{i_1}, c_{i_2}, c_{i_3})$ cannot form an OA of strength 3, which is equivalent to the situation that the three columns $(c_{i_1}, c_{i_2}, c_{i_3})$ are not linearly independent according to Theorem 3.29 of Hedayat, Sloane and Stufken (1999). Since C is a regular $\mathrm{OA}(s^p, g_1, s, 2)$ whose each generated column can be represented as the sum of q independent columns, where $2 \le q \le p$, then for any three columns that are not linearly independent, one of them can be expressed as the sum of the remaining two columns. Without of generality, we assume $c_{i_3} = c_{i_1} \dot{+} c_{i_2}$. Based on the properties of the difference scheme, for any row (x_1, x_2, x_3) with the relation $x_3 = x_1 \dot{+} x_2 \dot{+} \gamma_0$ in $(d_{j_1} \oplus c_{i_1}, d_{j_2} \oplus c_{i_2}, d_{j_3} \oplus c_{i_3})$, there is a corresponding row (x_1, x_2, x_3') in $((d_{j_1} \dot{+} k) \oplus c_{i_1}, (d_{j_2} \dot{+} k) \oplus c_{i_2}, (d_{j_3} \dot{+} k) \oplus c_{i_3})$ which has the relation $x_3' = x_1 \dot{+} x_2 \dot{+} \gamma_k$ for $k = 1, \ldots, s-1$, where $\gamma_0, \ldots, \gamma_{s-1}$ are distinct with each other and they are a permutation on $\{0, 1, \ldots, s-1\}$.

S5. Large Tables

Table S.1: The SGOA$(64, 20, 16, 2)$ in Example 2.

Pre-collapsing																				Post-collapsing																			
T_1				T_2				T_3				T_4				T_5				A_1				A_2				A_3				A_4				A_5			
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	0	0	0	0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	0	0	0	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
5	5	5	5	0	0	0	0	5	5	5	5	10	10	10	10	15	15	15	15	1	1	1	1	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
5	5	5	5	5	5	5	5	0	0	0	0	15	15	15	15	10	10	10	10	1	1	1	1	1	1	1	1	0	0	0	0	3	3	3	3	2	2	2	2
5	5	5	5	10	10	10	10	15	15	15	15	0	0	0	0	5	5	5	5	1	1	1	1	2	2	2	2	3	3	3	3	0	0	0	0	1	1	1	1
5	5	5	5	15	15	15	15	10	10	10	10	5	5	5	5	0	0	0	0	1	1	1	1	3	3	3	3	2	2	2	2	1	1	1	1	0	0	0	0
10	10	10	10	0	0	0	0	10	10	10	10	15	15	15	15	5	5	5	5	2	2	2	2	0	0	0	0	2	2	2	2	3	3	3	3	1	1	1	1
10	10	10	10	5	5	5	5	15	15	15	15	10	10	10	10	0	0	0	0	2	2	2	2	1	1	1	1	3	3	3	3	2	2	2	2	0	0	0	0
10	10	10	10	10	10	10	10	0	0	0	0	5	5	5	5	15	15	15	15	2	2	2	2	2	2	2	2	0	0	0	0	1	1	1	1	3	3	3	3
10	10	10	10	15	15	15	15	5	5	5	5	0	0	0	0	10	10	10	10	2	2	2	2	3	3	3	3	1	1	1	1	0	0	0	0	2	2	2	2
15	15	15	15	0	0	0	0	15	15	15	15	5	5	5	5	10	10	10	10	3	3	3	3	0	0	0	0	3	3	3	3	1	1	1	1	2	2	2	2
15	15	15	15	5	5	5	5	10	10	10	10	0	0	0	0	15	15	15	15	3	3	3	3	1	1	1	1	2	2	2	2	0	0	0	0	3	3	3	3
15	15	15	15	10	10	10	10	5	5	5	5	15	15	15	15	0	0	0	0	3	3	3	3	2	2	2	2	1	1	1	1	3	3	3	3	0	0	0	0
15	15	15	15	15	15	15	15	0	0	0	0	10	10	10	10	5	5	5	5	3	3	3	3	3	3	3	3	0	0	0	0	2	2	2	2	1	1	1	1
3	4	9	14	3	4	9	14	3	4	9	14	3	4	9	14	3	4	9	14	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
3	4	9	14	6	1	12	11	6	1	12	11	6	1	12	11	6	1	12	11	0	1	2	3	1	0	3	2	1	0	3	2	1	0	3	2	1	0	3	2
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3	4	9	14	12	11	6	1	12	11	6	1	12	11	6	1	12	11	6	1	0	1	2	3	3	2	1	0	3	2	1	0	3	2	1	0	3	2	1	0
6	1	12	11	3	4	9	14	6	1	12	11	9	14	3	4	12	11	6	1	1	0	3	2	0	1	2	3	1	0	3	2	2	3	0	1	3	2	1	0
6	1	12	11	6	1	12	11	3	4	9	14	12	11	6	1	9	14	3	4	1	0	3	2	1	0	3	2	0	1	2	3	3	2	1	0	2	3	0	1
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9	14	3	4	9	14	3	4	3	4	9	14	6	1	12	11	12	11	6	1	2	3	0	1	2	3	0	1	0	1	2	3	1	0	3	2	3	2	1	0
9	14	3	4	12	11	6	1	6	1	12	11	3	4	9	14	9	14	3	4	2	3	0	1	3	2	1	0	1	0	3	2	0	1	2	3	2	3	0	1
12	11	6	1	3	4	9	14	12	11	6	1	6	1	12	11	9	14	3	4	3	2	1	0	0	1	2	3	3	2	1	0	1	0	3	2	2	3	0	1
12	11	6	1	6	1	12	11	9	14	3	4	3	4	9	14	12	11	6	1	3	2	1	0	1	0	3	2	2	3	0	1	0	1	2	3	3	2	1	0
12	11	6	1	9	14	3	4	6	1	12	11	12	11	6	1	3	4	9	14	3	2	1	0	2	3	0	1	1	0	3	2	3	2	1	0	0	1	2	3
12	11	6	1	12	11	6	1	3	4	9	14	9	14	3	4	6	1	12	11	3	2	1	0	3	2	1	0	0	1	2	3	2	3	0	1	1	0	3	2
1	8	14	7	1	8	14	7	1	8	14	7	1	8	14	7	1	8	14	7	0	2	3	1	0	2	3	1	0	2	3	1	0	2	3	1	0	2	3	1
1	8	14	7	4	13	11	2	4	13	11	2	4	13	11	2	4	13	11	2	0	2	3	1	1	3	2	0	1	3	2	0	1	3	2	0	1	3	2	0
1	8	14	7	11	2	4	13	11	2	4	13	11	2	4	13	11	2	4	13	0	2	3	1	2	0	1	3	2	0	1	3	2	0	1	3	2	0	1	3
1	8	14	7	14	7	1	8	14	7	1	8	14	7	1	8	14	7	1	8	0	2	3	1	3	1	0	2	3	1	0	2	3	1	0	2	3	1	0	2
4	13	11	2	1	8	14	7	4	13	11	2	11	2	4	13	14	7	1	8	1	3	2	0	0	2	3	1	1	3	2	0	2	0	1	3	3	1	0	2
4	13	11	2	4	13	11	2	1	8	14	7	14	7	1	8	11	2	4	13	1	3	2	0	1	3	2	0	0	2	3	1	3	1	0	2	2	0	1	3
4	13	11	2	11	2	4	13	14	7	1	8	1	8	14	7	4	13	11	2	1	3	2	0	2	0	1	3	3	1	0	2	0	2	3	1	1	3	2	0
4	13	11	2	14	7	1	8	11	2	4	13	4	13	11	2	1	8	14	7	1	3	2	0	3	1	0	2	2	0	1	3	1	3	2	0	0	2	3	1
11	2	4	13	1	8	14	7	11	2	4	13	14	7	1	8	4	13	11	2	2	0	1	3	0	2	3	1	2	0	1	3	3	1	0	2	1	3	2	0
11	2	4	13	4	13	11	2	14	7	1	8	11	2	4	13	1	8	14	7	2	0	1	3	1	3	2	0	3	1	0	2	2	0	1	3	0	2	3	1
11	2	4	13	11	2	4	13	1	8	14	7	4	13	11	2	14	7	1	8	2	0	1	3	2	0	1	3	0	2	3	1	1	3	2	0	3	1	0	2
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14	7	1	8	1	8	14	7	14	7	1	8	4	13	11	2	11	2	4	13	3	1	0	2	0	2	3	1	3	1	0	2	1	3	2	0	2	0	1	3
14	7	1	8	4	13	11	2	11	2	4	13	1	8	14	7	14	7	1	8	3	1	0	2	1	3	2	0	2	0	1	3	0	2	3	1	3	1	0	2
14	7	1	8	11	2	4	13	4	13	11	2	14	7	1	8	1	8	14	7	3	1	0	2	2	0	1	3	1	3	2	0	3	1	0	2	0	2	3	1
14	7	1	8	14	7	1	8	1	8	14	7	11	2	4	13	4	13	11	2	3	1	0	2	3	1	0	2	0	2	3	1	2	0	1	3	1	3	2	0
2	12	7	9	2	12	7	9	2	12	7	9	2	12	7	9	2	12	7	9	0	3	1	2	0	3	1	2	0	3	1	2	0	3	1	2	0	3	1	2
2	12	7	9	7	9	2	12	7	9	2	12	7	9	2	12	7	9	2	12	0	3	1	2	1	2	0	3	1	2	0	3	1	2	0	3	1	2	0	3
2	12	7	9	8	6	13	3	8	6	13	3	8	6	13	3	8	6	13	3	0	3	1	2	2	1	3	0	2	1	3	0	2	1	3	0	2	1	3	0
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7	9	2	12	2	12	7	9	7	9	2	12	8	6	13	3	13	3	8	6	1	2	0	3	0	3	1	2	1	2	0	3	2	1	3	0	3	0	2	1
7	9	2	12	7	9	2	12	2	12	7	9	13	3	8	6	8	6	13	3	1	2	0	3	1	2	0	3	0	3	1	2	3	0	2	1	2	1	3	0
7	9	2	12	8	6	13	3	13	3	8	6	2	12	7	9	7	9	2	12	1	2	0	3	2	1	3	0	3	0	2	1	0	3	1	2	1	2	0	3
7	9	2	12	13	3	8	6	8	6	13	3	7	9	2	12	2	12	7	9	1	2	0	3	3	0	2	1	2	1	3	0	1	2	0	3	0	3	1	2
8	6	13	3	2	12	7	9	8	6	13	3	13	3	8	6	7	9	2	12	2	1	3	0	0	3	1	2	2	1	3	0	3	0	2	1	1	2	0	3
8	6	13	3	7	9	2	12	13	3	8	6	8	6	13	3	2	12	7	9	2	1	3	0	1	2	0	3	3	0	2	1	2	1	3	0	0	3	1	2
8	6	13	3	8	6	13	3	2	12	7	9	7	9	2	12	13	3	8	6	2	1	3	0	2	1	3	0	0	3	1	2	1	2	0	3	3	0	2	1
8	6	13	3	13	3	8	6	7	9	2	12	2	12	7	9	8	6	13	3	2	1	3	0	3	0	2	1	1	2	0	3	0	3	1	2	2	1	3	0
13	3	8	6	2	12	7	9	13	3	8	6	7	9	2	12	8	6	13	3	3	0	2	1	0	3	1	2	3	0	2	1	1	2	0	3	2	1	3	0
13	3	8	6	7	9	2	12	8	6	13	3	2	12	7	9	13	3	8	6	3	0	2	1	1	2	0	3	2	1	3	0	0	3	1	2	3	0	2	1
13	3	8	6	8	6	13	3	7	9	2	12	13	3	8	6	2	12	7	9	3	0	2	1	2	1	3	0	1	2	0	3	3	0	2	1	0	3	1	2
13	3	8	6	13	3	8	6	2	12	7	9	8	6	13	3	7	9	2	12	3	0	2	1	3	0	2	1	0	3	1	2	2	1	3	0	1	2	0	3

Table S.2: An SGOA$(81, 12, 27, 3)$.

Pre-collapsing												Post-collapsing											
T_1			T_2			T_3			T_4			E_1			E_2			E_3			E_4		
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	13	13	13	13	13	13	26	26	26	0	0	0	1	1	1	1	1	1	2	2	2
0	0	0	26	26	26	26	26	26	13	13	13	0	0	0	2	2	2	2	2	2	1	1	1
13	13	13	0	0	0	13	13	13	13	13	13	1	1	1	0	0	0	1	1	1	1	1	1
13	13	13	13	13	13	26	26	26	0	0	0	1	1	1	1	1	1	2	2	2	0	0	0
13	13	13	26	26	26	0	0	0	26	26	26	1	1	1	2	2	2	0	0	0	2	2	2
26	26	26	0	0	0	26	26	26	26	26	26	2	2	2	0	0	0	2	2	2	2	2	2
26	26	26	13	13	13	0	0	0	13	13	13	2	2	2	1	1	1	0	0	0	1	1	1
26	26	26	26	26	26	13	13	13	0	0	0	2	2	2	2	2	2	1	1	1	0	0	0
7	11	21	7	11	21	7	11	21	7	11	21	0	1	2	0	1	2	0	1	2	0	1	2
7	11	21	11	21	7	11	21	7	21	7	11	0	1	2	1	2	0	1	2	0	2	0	1
7	11	21	21	7	11	21	7	11	11	21	7	0	1	2	2	0	1	2	0	1	1	2	0
11	21	7	7	11	21	11	21	7	11	21	7	1	2	0	0	1	2	1	2	0	1	2	0
11	21	7	11	21	7	21	7	11	7	11	21	1	2	0	1	2	0	2	0	1	0	1	2
11	21	7	21	7	11	7	11	21	21	7	11	1	2	0	2	0	1	0	1	2	2	0	1
21	7	11	7	11	21	21	7	11	21	7	11	2	0	1	0	1	2	2	0	1	2	0	1
21	7	11	11	21	7	7	11	21	11	21	7	2	0	1	1	2	0	0	1	2	1	2	0
21	7	11	21	7	11	11	21	7	7	11	21	2	0	1	2	0	1	1	2	0	0	1	2
5	19	15	5	19	15	5	19	15	5	19	15	0	2	1	0	2	1	0	2	1	0	2	1
5	19	15	15	5	19	15	5	19	19	15	5	0	2	1	1	0	2	1	0	2	2	1	0
5	19	15	19	15	5	19	15	5	15	5	19	0	2	1	2	1	0	2	1	0	1	0	2
15	5	19	5	19	15	15	5	19	15	5	19	1	0	2	0	2	1	1	0	2	1	0	2
15	5	19	15	5	19	19	15	5	5	19	15	1	0	2	1	0	2	2	1	0	0	2	1
15	5	19	19	15	5	5	19	15	19	15	5	1	0	2	2	1	0	0	2	1	2	1	0
19	15	5	5	19	15	19	15	5	19	15	5	2	1	0	0	2	1	2	1	0	2	1	0
19	15	5	15	5	19	5	19	15	15	5	19	2	1	0	1	0	2	0	2	1	1	0	2
19	15	5	19	15	5	15	5	19	5	19	15	2	1	0	2	1	0	1	0	2	0	2	1
9	9	9	9	9	9	9	9	9	9	9	9	1	1	1	1	1	1	1	1	1	1	1	1
9	9	9	22	22	22	22	22	22	8	8	8	1	1	1	2	2	2	2	2	2	0	0	0
9	9	9	8	8	8	8	8	8	22	22	22	1	1	1	0	0	0	0	0	0	2	2	2
22	22	22	9	9	9	22	22	22	22	22	22	2	2	2	1	1	1	2	2	2	2	2	2
22	22	22	22	22	22	8	8	8	9	9	9	2	2	2	2	2	2	0	0	0	1	1	1
22	22	22	8	8	8	9	9	9	8	8	8	2	2	2	0	0	0	1	1	1	0	0	0
8	8	8	9	9	9	8	8	8	8	8	8	0	0	0	1	1	1	0	0	0	0	0	0
8	8	8	22	22	22	9	9	9	22	22	22	0	0	0	2	2	2	1	1	1	2	2	2
8	8	8	8	8	8	22	22	22	9	9	9	0	0	0	0	0	0	2	2	2	1	1	1
16	20	3	16	20	3	16	20	3	16	20	3	1	2	0	1	2	0	1	2	0	1	2	0
16	20	3	20	3	16	20	3	16	3	16	20	1	2	0	2	0	1	2	0	1	0	1	2
16	20	3	3	16	20	3	16	20	20	3	16	1	2	0	0	1	2	0	1	2	2	0	1
20	3	16	16	20	3	20	3	16	20	3	16	2	0	1	1	2	0	2	0	1	2	0	1
20	3	16	20	3	16	3	16	20	16	20	3	2	0	1	2	0	1	0	1	2	1	2	0
20	3	16	3	16	20	16	20	3	3	16	20	2	0	1	0	1	2	1	2	0	0	1	2
3	16	20	16	20	3	3	16	20	3	16	20	0	1	2	1	2	0	0	1	2	0	1	2
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3	16	20	3	16	20	20	3	16	16	20	3	0	1	2	0	1	2	2	0	1	1	2	0
14	1	24	14	1	24	14	1	24	14	1	24	1	0	2	1	0	2	1	0	2	1	0	2
14	1	24	24	14	1	24	14	1	1	24	14	1	0	2	2	1	0	2	1	0	0	2	1
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24	14	1	14	1	24	24	14	1	24	14	1	2	1	0	1	0	2	2	1	0	2	1	0
24	14	1	24	14	1	1	24	14	14	1	24	2	1	0	2	1	0	0	2	1	1	0	2
24	14	1	1	24	14	14	1	24	1	24	14	2	1	0	0	2	1	1	0	2	0	2	1
1	24	14	14	1	24	1	24	14	1	24	14	0	2	1	1	0	2	0	2	1	0	2	1
1	24	14	24	14	1	14	1	24	24	14	1	0	2	1	2	1	0	1	0	2	2	1	0
1	24	14	1	24	14	24	14	1	14	1	24	0	2	1	0	2	1	2	1	0	1	0	2
18	18	18	18	18	18	18	18	18	18	18	18	2	2	2	2	2	2	2	2	2	2	2	2
18	18	18	4	4	4	4	4	4	17	17	17	2	2	2	0	0	0	0	0	0	1	1	1
18	18	18	17	17	17	17	17	17	4	4	4	2	2	2	1	1	1	1	1	1	0	0	0
4	4	4	18	18	18	4	4	4	4	4	4	0	0	0	2	2	2	0	0	0	0	0	0
4	4	4	4	4	4	17	17	17	18	18	18	0	0	0	0	0	0	1	1	1	2	2	2
4	4	4	17	17	17	18	18	18	17	17	17	0	0	0	1	1	1	2	2	2	1	1	1
17	17	17	18	18	18	17	17	17	17	17	17	1	1	1	2	2	2	1	1	1	1	1	1
17	17	17	4	4	4	18	18	18	4	4	4	1	1	1	0	0	0	2	2	2	0	0	0
17	17	17	17	17	17	4	4	4	18	18	18	1	1	1	1	1	1	0	0	0	2	2	2
25	2	12	25	2	12	25	2	12	25	2	12	2	0	1	2	0	1	2	0	1	2	0	1
25	2	12	2	12	25	2	12	25	12	25	2	2	0	1	0	1	2	0	1	2	1	2	0
25	2	12	12	25	2	12	25	2	2	12	25	2	0	1	1	2	0	1	2	0	0	1	2
2	12	25	25	2	12	2	12	25	2	12	25	0	1	2	2	0	1	0	1	2	0	1	2
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12	25	2	25	2	12	12	25	2	12	25	2	1	2	0	2	0	1	1	2	0	1	2	0
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12	25	2	12	25	2	2	12	25	25	2	12	1	2	0	1	2	0	0	1	2	2	0	1
23	10	6	23	10	6	23	10	6	23	10	6	2	1	0	2	1	0	2	1	0	2	1	0
23	10	6	6	23	10	6	23	10	10	6	23	2	1	0	0	2	1	0	2	1	1	0	2
23	10	6	10	6	23	10	6	23	6	23	10	2	1	0	1	0	2	1	0	2	0	2	1
6	23	10	23	10	6	6	23	10	6	23	10	0	2	1	2	1	0	0	2	1	0	2	1
6	23	10	6	23	10	10	6	23	23	10	6	0	2	1	0	2	1	1	0	2	2	1	0
6	23	10	10	6	23	23	10	6	10	6	23	0	2	1	1	0	2	2	1	0	1	0	2
10	6	23	23	10	6	10	6	23	10	6	23	1	0	2	2	1	0	1	0	2	1	0	2
10	6	23	6	23	10	23	10	6	6	23	10	1	0	2	0	2	1	2	1	0	0	2	1
10	6	23	10	6	23	6	23	10	23	10	6	1	0	2	1	0	2	0	2	1	2	1	0