SUPPLEMENT TO "TESTING HETEROSCEDASTICITY FOR REGRESSION MODELS BASED ON PROJECTIONS"

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This Supplement Material contains proofs of the theorems in the main context and the additional simulation results.

1. Proofs of the theorems

Proof of Theorem 1. Recall that $\hat{\eta}_i = \hat{\varepsilon}_i^2 - \hat{\sigma}^2$ and $\hat{\varepsilon}_i = Y_i - m(Z_i, \hat{\beta}_n)$. Then it follows that $\hat{\varepsilon}_i = \varepsilon_i - [m(Z_i, \hat{\beta}_n) - m(Z_i, \beta)]$. Consequently,

$$
\begin{aligned}
\hat{\eta}_i &= \varepsilon_i^2 + [m(Z_i, \hat{\beta}_n) - m(Z_i, \beta)]^2 - 2\varepsilon_i[m(Z_i, \hat{\beta}_n) - m(Z_i, \beta)] \\
&\quad -\frac{1}{n}\sum_{j=1}^n \varepsilon_j^2 - \frac{1}{n}\sum_{j=1}^n [m(Z_j, \hat{\beta}_n) - m(Z_j, \beta)]^2 + \frac{2}{n}\sum_{j=1}^n \varepsilon_j[m(Z_j, \hat{\beta}_n) - m(Z_j, \beta)] \\
&= \varepsilon_i^2 - \sigma^2 + \sigma^2 - \frac{1}{n}\sum_{j=1}^n \varepsilon_j^2 \\
&\quad +[m(Z_i, \hat{\beta}_n) - m(Z_i, \beta)]^2 - \frac{1}{n}\sum_{j=1}^n [m(Z_j, \hat{\beta}_n) - m(Z_j, \beta)]^2 \\
&\quad -\{2\varepsilon_i[m(Z_i, \hat{\beta}_n) - m(Z_i, \beta)] - \frac{2}{n}\sum_{j=1}^n \varepsilon_j[m(Z_j, \hat{\beta}_n) - m(Z_j, \beta)]\} \\
&=: T_{1n} + T_{2n} - T_{3n}
\end{aligned}
$$

Let $V_{jn}(\alpha, t) = (1/\sqrt{n})\sum_{i=1}^n T_{jn} I(\alpha^\top Z_i \le t)$. Then it follows that

$$V_n(\alpha, t) = V_{1n}(\alpha, t) + V_{2n}(\alpha, t) - V_{3n}(\alpha, t).$$

For $V_{1n}(\alpha, t)$, it is easy to see that

$$\begin{aligned} V_{1n}(\alpha, t) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\varepsilon_i^2 - \sigma^2 + (\sigma^2 - \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j^2)] I(\alpha^\top Z_i \le t) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_i I(\alpha^\top Z_i \le t) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} I(\alpha^\top Z_i \le t) \frac{1}{n} \sum_{i=1}^{n} \eta_i. \end{aligned}$$

By Theorem 24 of Chapter 2 in Pollard (1984), we obtain that

$$\sup_{\alpha, t} | \frac{1}{n} \sum_{i=1}^{n} I(\alpha^\top Z_i \le t) - F_\alpha(t) | = o_p(1).$$

Since $E(\varepsilon^4) < \infty$, it follows that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_i = O_p(1).$$

Consequently,

$$V_{1n}(\alpha, t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_i [I(\alpha^\top Z_i \le t) - F_\alpha(t)] + o_p(1),$$

uniformly in (α, t). To prove this theorem, it suffices to show that

$$V_{2n}(\alpha, t) = o_p(1) \quad \text{and} \quad V_{3n}(\alpha, t) = o_p(1) \text{ uniformly in } (\alpha, \mathrm{t}).$$

For $V_{2n}(\alpha, t)$, we have

$$\begin{aligned} V_{2n}(\alpha, t) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{ [m(Z_i, \hat{\beta}_n) - m(Z_i, \beta)]^2 - \frac{1}{n} \sum_{j=1}^{n} [m(Z_j, \hat{\beta}_n) - m(Z_j, \beta)]^2 \} I(\alpha^\top Z_i \le t) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [m(Z_i, \hat{\beta}_n) - m(Z_i, \beta)]^2 I(\alpha^\top Z_i \le t) - \\ &\quad \{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} I(\alpha^\top Z_i \le t) \} \{ \frac{1}{n} \sum_{j=1}^{n} [m(Z_j, \hat{\beta}_n) - m(Z_j, \beta)]^2 \} \\ &=: V_{21n}(\alpha, t) - V_{22n}(\alpha, t) \end{aligned}$$

By Taylor's expansion, we obtain

$$m(Z_i,\hat\beta_n)-m(Z_i,\beta)=(\hat\beta_n-\beta)^\top m'(Z_i,\beta)+\frac{1}{2}(\hat\beta_n-\beta)^\top m''(Z_i,\beta_1)(\hat\beta_n-\beta),$$

where β_1 lies between $\hat\beta_n$ and β. Then it follows that

$$\begin{aligned}V_{21n}(\alpha,t) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n(\hat\beta_n-\beta)^\top m'(Z_i,\beta)m'(Z_i,\beta)^\top(\hat\beta_n-\beta)I(\alpha^\top Z_i\le t)+\\ &\quad \frac{1}{4\sqrt{n}}\sum_{i=1}^n[(\hat\beta_n-\beta)^\top m''(Z_i,\beta_1)(\hat\beta_n-\beta)]^2 I(\alpha^\top Z_i\le t)+\\ &\quad \frac{1}{\sqrt{n}}\sum_{i=1}^n(\hat\beta_n-\beta)^\top m'(Z_i,\beta)(\hat\beta_n-\beta)^\top m''(Z_i,\beta_1)(\hat\beta_n-\beta)I(\alpha^\top Z_i\le t)\end{aligned}$$

Since $E\|m'(Z,\beta)\|^2<\infty$ and $\|m''(Z,\beta_1)\|\le M(Z)$ with $E|M(Z)|^2<\infty$, it is easy to see that

$$V_{21n}(\alpha,t)=O_p(\frac{1}{\sqrt{n}})\quad \text{uniformly in } (\alpha,\mathrm{t}).$$

Similarly, we obtain that $V_{22n}(\alpha,t)=O_p(1/\sqrt{n})$ uniformly in (α,t).

Next we consider the third term $V_{3n}(\alpha,t)$ in $V_n(\alpha,t)$. Note that

$$\begin{aligned}V_{3n}(\alpha,t) &= \frac{2}{\sqrt{n}}\sum_{i=1}^n\varepsilon_i[m(Z_i,\hat\beta_n)-m(Z_i,\beta)]I(\alpha^\top Z_i\le t)-\\ &\quad \{\frac{2}{n}\sum_{i=1}^n\varepsilon_i[m(Z_i,\hat\beta_n)-m(Z_i,\beta)]\}\{\frac{1}{\sqrt{n}}\sum_{i=1}^n I(\alpha^\top Z_i\le t)\}\\ &=: V_{31n}(\alpha,t)-V_{32n}(\alpha,t).\end{aligned}$$

For $V_{31n}(\alpha,t)$, we have

$$\begin{aligned}V_{31n}(\alpha,t) &= \frac{2}{\sqrt{n}}\sum_{i=1}^n\varepsilon_i(\hat\beta_n-\beta)^\top m'(Z_i,\beta)I(\alpha^\top Z_i\le t)-\\ &\quad \frac{1}{\sqrt{n}}\sum_{i=1}^n\varepsilon_i(\hat\beta_n-\beta)^\top m''(Z_i,\beta_1)(\hat\beta_n-\beta)I(\alpha^\top Z_i\le t)\\ &= o_p(1)\end{aligned}$$

By a similar argument, we obtain $V_{32n}(\alpha, t) = o_p(1)$ uniformly in (α, t). Altogether we complete the proof. $\square$

Proof of Theorem 2. Recall that $V_n(\alpha, t) = (1/\sqrt{n}) \sum_{i=1}^n \hat{\eta}_i I(\alpha^\top Z_i \leq t)$, where $\hat{\eta}_i = \hat{\varepsilon}_i^2 - \hat{\sigma}^2$, $\hat{\varepsilon}_i = Y_i - \hat{\beta}_n^\top X_i - \hat{g}(T_i)$, $\hat{\sigma}^2 = (1/n) \sum_{i=1}^n \hat{\varepsilon}_i^2$, and

$$\hat{g}(T_i) = \frac{1}{n} \sum_{j \neq i}^{n} [Y_j - \hat{\beta}_n^\top X_j] K_h(T_i - T_j)/\hat{f}_i(T_i).$$

Decompose $\hat{g}(T_i)$ as following,

$$\hat{g}(T_i) = \tilde{g}(T_i) - (\hat{\beta}_n - \beta)^\top \frac{1}{n} \sum_{j \neq i}^{n} X_j K_h(T_i - T_j)/\hat{f}_i(T_i),$$

where $\tilde{g}(T_i) = (1/n) \sum_{j \neq i}^n [Y_j - \beta^\top X_j] K_h(T_i - T_j)/\hat{f}_i(T_i)$. Consequently, we obtain that

$$\hat{\varepsilon}_i = \varepsilon_i - (\hat{\beta}_n - \beta)^\top \{X_i - \hat{E}(X|T_i)\} - \{\tilde{g}(T_i) - g(T_i)\}.$$

Then it follows that

$$\begin{aligned}
\hat{\eta}_i &= \varepsilon_i^2 - \sigma^2 + (\sigma^2 - \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i^2) \\
&\quad + \{(\hat{\beta}_n - \beta)^\top [X_i - \hat{E}(X|T_i)]\}^2 - \frac{1}{n} \sum_{i=1}^{n} \{(\hat{\beta}_n - \beta)^\top [X_i - \hat{E}(X|T_i)]\}^2 \\
&\quad + [\tilde{g}(T_i) - g(T_i)]^2 - \frac{1}{n} \sum_{i=1}^{n} [\tilde{g}(T_i) - g(T_i)]^2 \\
&\quad - 2\varepsilon_i (\hat{\beta}_n - \beta)^\top [X_i - \hat{E}(X|T_i)] + (\hat{\beta}_n - \beta)^\top \frac{2}{n} \sum_{i=1}^{n} \varepsilon_i [X_i - \hat{E}(X|T_i)] \\
&\quad - 2\varepsilon_i [\tilde{g}(T_i) - g(T_i)] + \frac{2}{n} \sum_{i=1}^{n} \varepsilon_i [\tilde{g}(T_i) - g(T_i)] \\
&\quad + 2(\hat{\beta}_n - \beta)^\top [X_i - \hat{E}(X|T_i)][\tilde{g}(T_i) - g(T_i)] - (\hat{\beta}_n - \beta)^\top \frac{2}{n} \sum_{i=1}^{n} [X_i - \hat{E}(X|T_i)][\tilde{g}(T_i) - g(T_i)] \\
&=: T_{1n} + T_{2n} + T_{3n} - T_{4n} - T_{5n} + T_{6n}.
\end{aligned}$$

Let $V_{jn}(\alpha,t) = (1/\sqrt{n})\sum_{i=1}^n T_{jn} I(\alpha^\top Z_i \le t)$. First we consider $V_{1n}(\alpha,t)$. Note that

$$\begin{aligned} V_{1n}(\alpha,t) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n[\varepsilon_i^2-\sigma^2+(\sigma^2-\frac{1}{n}\sum_{i=1}^n\varepsilon_i^2)]I(\alpha^\top Z_i\le t) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n\eta_i I(\alpha^\top Z_i\le t)-\frac{1}{\sqrt{n}}\sum_{i=1}^n I(\alpha^\top Z_i\le t)\frac{1}{n}\sum_{i=1}^n\eta_i. \end{aligned}$$

By the standard empirical process theory, see, e.g. Pollard (1984, Chapter II), we have

$$\sup_{\alpha,t}|\frac{1}{n}\sum_{i=1}^n I(\alpha^\top Z_i\le t)-F_\alpha(t)| = o_p(1).$$

Then it follows that

$$V_{1n}(\alpha,t) = \frac{1}{\sqrt{n}}\sum_{i=1}^n\eta_i[I(\alpha^\top Z_i\le t)-F_\alpha(t)]+o_p(1),$$

uniformly in (α,t). Next we show that the rest terms $V_{jn}(\alpha,t) = o_p(1)$ uniformly in (α,t) for $2\le j\le 6$.

Recall that

$$\begin{aligned} V_{2n}(\alpha,t) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n\{(\hat\beta_n-\beta)^\top[X_i-\hat E(X|T_i)]\}^2 I(\alpha^\top Z_i\le t) - \\ & \frac{1}{n}\sum_{i=1}^n\{(\hat\beta_n-\beta)^\top[X_i-\hat E(X|T_i)]\}^2\frac{1}{\sqrt{n}}\sum_{i=1}^n I(\alpha^\top Z_i\le t). \end{aligned}$$

Then it follows that

$$
\begin{aligned}
V_{2n}(\alpha,t) \;=\; & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{(\hat\beta_n-\beta)^\top[X_i-E(X|T_i)]\}^2 I(\alpha^\top Z_i\le t) - \\
& \frac{1}{n}\sum_{i=1}^{n}\{(\hat\beta_n-\beta)^\top[X_i-E(X|T_i)]\}^2 \frac{1}{\sqrt{n}}\sum_{i=1}^{n} I(\alpha^\top Z_i\le t) + \\
& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{(\hat\beta_n-\beta)^\top[\hat E(X|T_i)-E(X|T_i)]\}^2 I(\alpha^\top Z_i\le t) - \\
& \frac{1}{n}\sum_{i=1}^{n}\{(\hat\beta_n-\beta)^\top[\hat E(X|T_i)-E(X|T_i)]\}^2 \frac{1}{\sqrt{n}}\sum_{i=1}^{n} I(\alpha^\top Z_i\le t) - \\
& \frac{2}{\sqrt{n}}\sum_{i=1}^{n}(\hat\beta_n-\beta)^\top[X_i-E(X|T_i)](\hat\beta_n-\beta)^\top[\hat E(X|T_i)-E(X|T_i)]I(\alpha^\top Z_i\le t) + \\
& \frac{2}{n}\sum_{i=1}^{n}(\hat\beta_n-\beta)^\top[X_i-E(X|T_i)](\hat\beta_n-\beta)^\top[\hat E(X|T_i)-E(X|T_i)]\frac{1}{\sqrt{n}}\sum_{i=1}^{n} I(\alpha^\top Z_i\le t).
\end{aligned}
$$

Since $\sup_t |\hat E(X|T=t)-E(X|T=t)| = O_p(\log n/\sqrt{nh}+h)$, we obtain that

$$
V_{2n}(\alpha,t) = O_p(\frac{1}{\sqrt{n}}) + O_p(\frac{1}{\sqrt{n}})O_p(\frac{\log n}{\sqrt{nh}}+h)^2 + O_p(\frac{1}{\sqrt{n}})O_p(\frac{\log n}{\sqrt{nh}}+h).
$$

Thus we have $V_{2n}(\alpha,t)=o_p(1)$ uniformly in (α,t). By similar arguments, we can show that $V_{jn}(\alpha,t)=o_p(1)$ uniformly in (α,t) for $j=3,4,6$.

Now we consider the term $V_{5n}(\alpha,t)$. Note that

$$
\begin{aligned}
V_{5n}(\alpha,t) \;=\; & \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i[\tilde g(T_i)-g(T_i)]I(\alpha^\top Z_i\le t) - \frac{2}{\sqrt{n}}\sum_{i=1}^{n} I(\alpha^\top Z_i\le t)\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i[\tilde g(T_i)-g(T_i)] \\
\;=\; & \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i[\tilde g(T_i)-g(T_i)][I(\alpha^\top Z_i\le t)-F_\alpha(t)] - \\
& \frac{2}{\sqrt{n}}\sum_{i=1}^{n}[I(\alpha^\top Z_i\le t)-F_\alpha(t)]\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i[\tilde g(T_i)-g(T_i)]
\end{aligned}
$$

Then it follows that

$$V_{5n}(\alpha,t)=\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i[\tilde{g}(T_i)-g(T_i)][I(\alpha^\top Z_i\le t)-F_\alpha(t)]+O_p(\log n/\sqrt{nh}+h),$$

uniformly in (α,t). Let $r(t)=g(t)f(t)$ and $\hat{r}(T_i)=(1/n)\sum_{j\neq i}^n[Y_j-\beta^\top X_j]K_h(T_i-T_j)$. Then $\hat{r}(T_i)=\tilde{g}(T_i)\hat{f}(T_i)$. Consequently,

$$\begin{aligned}\tilde{g}(T_i)-g(T_i) &= \frac{\hat{r}(T_i)}{\hat{f}(T_i)}-\frac{r(T_i)}{f(T_i)}=\frac{\hat{r}(T_i)-r(T_i)}{f(T_i)}-g(T_i)\frac{\hat{f}(T_i)-f(T_i)}{f(T_i)}\\ &\quad -\frac{[\hat{r}(T_i)-r(T_i)][\hat{f}(T_i)-f(T_i)]}{\hat{f}(T_i)f(T_i)}+\frac{g(T_i)[\hat{f}(T_i)-f(T_i)]^2}{\hat{f}(T_i)f(T_i)}.\end{aligned}$$

Then we obtain that

$$\begin{aligned}V_{5n}(\alpha,t) &= \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i\frac{\hat{r}(T_i)-r(T_i)}{f(T_i)}[I(\alpha^\top Z_i\le t)-F_\alpha(t)]-\\ &\quad \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i g(T_i)\frac{\hat{f}(T_i)-f(T_i)}{f(T_i)}[I(\alpha^\top Z_i\le t)-F_\alpha(t)]+\\ &\quad O_p(\log n/\sqrt{nh}+h)+O_p(\sqrt{n})O_p(\log n/\sqrt{nh}+h)^2\\ &=: J_{1n}+J_{2n}+O_p(\log n/\sqrt{nh}+h)+O_p((\log n)^2/\sqrt{n}h+\sqrt{n}h^2)\end{aligned}$$

It will be shown that J_{1n} and J_{2n} converge to zero in probability uniformly in (α,t). We only give the detailed arguments for J_{1n}. The arguments for J_{2n} are similar. Note that

$$\begin{aligned}J_{1n} &= \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\frac{\varepsilon_i}{f(T_i)}\left(\frac{1}{nh}\sum_{j\neq i}^{n}(Y_j-\beta^\top X_j)K(\frac{T_i-T_j}{h})-r(T_i)\right)[I(\alpha^\top Z_i\le t)-F_\alpha(t)]\\ &= \frac{2}{hn^{3/2}}\sum_{i\neq j}^{n}\frac{\varepsilon_i}{f(T_i)}\left\{[g(T_j)+\varepsilon_j]K(\frac{T_i-T_j}{h})-hr(T_i)\right\}[I(\alpha^\top Z_i\le t)-F_\alpha(t)].\end{aligned}$$

Define $\tau_i=(X_i,T_i,\varepsilon_i)$ and

$$f_{\alpha,t}(\tau_i,\tau_j)=\frac{\varepsilon_i}{f(T_i)}\left\{[g(T_j)+\varepsilon_j]K(\frac{T_i-T_j}{h})-hr(T_i)\right\}[I(\alpha^\top Z_i\le t)-F_\alpha(t)].$$

Then it follows that

$$J_{1n} = \frac{2}{hn^{3/2}} \sum_{i \neq j}^{n} f_{\alpha,t}(\tau_i, \tau_j) = \frac{1}{hn^{3/2}} \sum_{i \neq j}^{n} [f_{\alpha,t}(\tau_i, \tau_j) + f_{\alpha,t}(\tau_j, \tau_i)]$$

Let $w_{\alpha,t}(\tau_i, \tau_j) = f_{\alpha,t}(\tau_i, \tau_j) + f_{\alpha,t}(\tau_j, \tau_i)$ and define

$$\tilde{J}_{1n} = \sum_{i \neq j}^{n} \{w_{\alpha,t}(\tau_i, \tau_j) - E(w_{\alpha,t}(\tau_i, \tau_j)|\tau_i) - E(w_{\alpha,t}(\tau_i, \tau_j)|\tau_j)\}.$$

Then $\tilde{J}_{1n}$ is a $\mathbb{P}$-degenerate U-process (see, Nolan and Pollard (1987)). Here $\mathbb{P}$ is the probability measure of (X, T, ε). Consider the class of functions

$$\mathcal{F}_n = \{w_{\alpha,t}(\tau_1, \tau_2) - E(w_{\alpha,t}(\tau_1, \tau_2)|\tau_1) - E(w_{\alpha,t}(\tau_1, \tau_2)|\tau_2) : \alpha \in \mathbb{S}^{p+1},\ t \in \mathbb{R}\}.$$

Then $\mathcal{F}_n$ is a $\mathbb{P}$-degenerate class of functions with an envelope

$$\begin{aligned} G_n(\tau_1, \tau_2) &= |\frac{\varepsilon_1}{f(T_1)} \left\{ [g(T_2) + \varepsilon_2] K(\frac{T_1 - T_2}{h}) - \int g(t) f(t) K(\frac{T_1 - t}{h}) dt \right\} | + \\ &\quad |\frac{\varepsilon_2}{f(T_2)} \left\{ [g(T_1) + \varepsilon_1] K(\frac{T_2 - T_1}{h}) - \int g(t) f(t) K(\frac{T_2 - t}{h}) dt \right\} | \end{aligned}$$

It is well known that the class of indictor functions is a VC-class. Then it follows that

$$N_2\{u(T_n G_n^2)^{1/2}, L_2(T_n), \mathcal{F}_n\} \leq C u^{-w},$$

where the constants C and w do not depend on n,

$$T_n g^2 = \sum_{i \neq j} [g^2(\tau_{2i}, \tau_{2j}) + g^2(\tau_{2i}, \tau_{2j-1}) + g^2(\tau_{2i-1}, \tau_{2j}) + g^2(\tau_{2i-1}, \tau_{2j-1})],$$

and $N_2\{u, L_2(T_n), \mathcal{F}_n\}$ is the covering number of $\mathcal{F}_n$ under the semi-metric $L_2(T_n)$. By Theorem 6 of Nolan and Pollard (1987), we obtain that

$$E(\sup_{\alpha,t} |\tilde{J}_{1n}|) \leq CE[\theta_n + \gamma_n J_n(\theta_n/\gamma_n)],$$

---

where C is a universal constant, $\gamma_n = (T_n G_n^2)^{1/2}$, $\theta_n = (1/4)\sup_{\mathcal{F}_n}(T_n g^2)^{1/2}$, and

$$J_n(s) = \int_0^s \log N_2\{u(T_n G_n^2)^{1/2}, L_2(T_n), \mathcal{F}_n\} du.$$

Therefore, we have

$$E(\sup_{\alpha,t} |\tilde{J}_{1n}|) \leq CE[\gamma_n/4 + J_n(1/4)\gamma_n] \leq CE(T_n G_n^2)^{1/2}.$$

It is easy to see that $E(T_n G_n^2) = O(n^2 h)$. Thus we obtain that $\sup_{\alpha,t} |\tilde{J}_{1n}| = O_p(nh^{1/2})$.

Recall that

$$J_{1n} = \frac{1}{hn^{3/2}}\tilde{J}_{1n} + \frac{1}{hn^{3/2}}\sum_{i\neq j}^n \{E(w_{\alpha,t}(\tau_i,\tau_j)|\tau_i) + E(w_{\alpha,t}(\tau_i,\tau_j)|\tau_j)\}.$$

To prove $J_{1n} = o_p(1)$ uniformly in (α, t), it remains to show that

$$\frac{1}{hn^{3/2}}\sum_{i\neq j}^n \{E(w_{\alpha,t}(\tau_i,\tau_j)|\tau_i) + E(w_{\alpha,t}(\tau_i,\tau_j)|\tau_j)\} = o_p(1),$$

uniformly in (α, t). Note that

$$\begin{aligned}
& \frac{1}{hn^{3/2}}\sum_{i\neq j}^n E(w_{\alpha,t}(\tau_i,\tau_j)|\tau_i) \\
= \; & \frac{n-1}{hn^{3/2}}\sum_{i=1}^n \frac{\varepsilon_i}{f(T_i)}[\int r(t)K(\frac{T_i - t}{h})dt - hr(T_i)][I(\alpha^\top Z_i \leq t) - F_\alpha(t)] \\
= \; & \frac{n-1}{n}\frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{\varepsilon_i}{f(T_i)}[\int r(T_i - hu)K(u)du - r(T_i)][I(\alpha^\top Z_i \leq t) - F_\alpha(t)] \\
= \; & \frac{n-1}{n}\frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{\varepsilon_i}{f(T_i)}[\int (-hu)r'(\zeta_i)K(u)du][I(\alpha^\top Z_i \leq t) - F_\alpha(t)] \\
= \; & O_p(h),
\end{aligned}$$

where ζ_i lies between T_i and $T_i - hu$. Similarly, we have $(1/hn^{3/2}) \sum_{i\neq j}^n E(w_{\alpha,t}(\tau_i,\tau_j)|\tau_j) = O_p(h)$ uniformly in (α, t). Thus $J_{1n} = o_p(1)$ uniformly in (α, t). Altogether, we obtain

$$V_n(\alpha,t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \eta_i [I(\alpha^\top Z_i \le t) - F_\alpha(t)] + o_p(1),$$

Hence we complete the proof. □

Proof of Theorem 3. Similar to the arguments in Theorem 1, we have

$$V_n(\alpha,t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (\varepsilon_i^2 - \sigma^2)[I(\alpha^\top Z_i \le t) - F_\alpha(t)] + o_p(1).$$

Then under the alternative H_{1n}, we obtain that

$$\begin{aligned} V_n(\alpha,t) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n [\varepsilon_i^2 - \sigma^2 - c_n s(Z_i)][I(\alpha^\top Z_i \le t) - F_\alpha(t)] + \\ &\quad c_n \frac{1}{\sqrt{n}} \sum_{i=1}^n s(Z_i)[I(\alpha^\top Z_i \le t) - F_\alpha(t)] + o_p(1). \end{aligned}$$

If $c_n = 1/\sqrt{n}$, then the first sum in $V_n(\alpha,t)$ converges to $V_\infty(\alpha,t)$ and the seconds tends to $E\{s(Z)[I(\alpha^\top Z \le t) - F_\alpha(t)]\}$. If $\sqrt{n}c_n \to \infty$, then $V_n(\alpha,t)$ tends to infinity. Altogether we complete the proof. □

Proof of Theorem 4. The proof of Theorem 4 follows the same line as in the proof of Theorem 3.2 in Zhu, Fujikoshi and Naito (2001) with some extra complications that arise from the indicator functions $I(\beta^\top Z \le t)$ involving the projections. Since the class of indicator functions

$$\mathcal{F} = \{f(z) = I(\beta^\top z \le t) : \beta \in \mathbb{S}^p, t \in \mathbb{R}\}$$

is also a VC-class, the proof can be very similar to that of Theorem 3.2 in Zhu, Fujikoshi and Naito (2001). Thus we omit it here. □

References

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Table 1: Empirical sizes and powers of HCM_n, T_n^G, T_n^{ZH}, and T_n^{ZFN} for H_{13} and H_{14} in Example 1.

	a	HCM_n		T_n^G		T_n^{ZH}		T_n^{ZFN}	
		n=100	n=200	n=100	n=200	n=100	n=200	n=100	n=200
$H_{13}, p = 2$	0.0	0.049	0.050	0.042	0.054	0.026	0.043	0.067	0.048
	0.1	0.102	0.169	0.110	0.147	0.036	0.060	0.081	0.178
	0.2	0.307	0.555	0.256	0.488	0.077	0.187	0.238	0.576
	0.3	0.566	0.902	0.467	0.836	0.184	0.412	0.516	0.890
	0.4	0.782	0.993	0.712	0.977	0.310	0.687	0.772	0.986
	0.5	0.922	0.999	0.892	0.998	0.497	0.880	0.910	1.000
$H_{13}, p = 4$	0.0	0.052	0.057	0.070	0.060	0.020	0.030	0.063	0.055
	0.1	0.089	0.114	0.107	0.110	0.038	0.037	0.022	0.056
	0.2	0.191	0.406	0.169	0.288	0.057	0.067	0.054	0.219
	0.3	0.389	0.758	0.295	0.589	0.060	0.125	0.125	0.528
	0.4	0.596	0.931	0.471	0.834	0.076	0.215	0.240	0.816
	0.5	0.756	0.989	0.635	0.958	0.138	0.362	0.344	0.932
$H_{13}, p = 8$	0.0	0.056	0.043	0.077	0.071	0.055	0.044	0.061	0.052
	0.1	0.075	0.081	0.079	0.077	0.054	0.053	0.022	0.030
	0.2	0.138	0.266	0.096	0.144	0.045	0.064	0.020	0.011
	0.3	0.261	0.556	0.149	0.300	0.053	0.062	0.008	0.010
	0.4	0.427	0.833	0.226	0.445	0.068	0.072	0.013	0.011
	0.5	0.602	0.945	0.308	0.626	0.075	0.105	0.010	0.019
$H_{14}, p = 2$	0.0	0.046	0.048	0.074	0.064	0.033	0.036	0.051	0.052
	0.1	0.582	0.907	0.421	0.756	0.139	0.312	0.596	0.931
	0.2	0.956	0.999	0.926	1.000	0.473	0.883	0.954	1.000
	0.3	0.991	1.000	0.997	1.000	0.783	0.994	0.989	1.000
	0.4	0.993	1.000	1.000	1.000	0.882	0.999	0.984	1.000
	0.5	0.992	1.000	1.000	1.000	0.927	0.999	0.975	1.000
$H_{14}, p = 4$	0.0	0.033	0.053	0.072	0.055	0.027	0.029	0.041	0.048
	0.1	0.448	0.805	0.281	0.520	0.060	0.097	0.319	0.762
	0.2	0.886	0.998	0.706	0.974	0.199	0.366	0.690	0.983
	0.3	0.945	1.000	0.885	1.000	0.292	0.652	0.718	0.979
	0.4	0.966	0.998	0.964	1.000	0.446	0.805	0.678	0.961
	0.5	0.966	1.000	0.963	1.000	0.504	0.863	0.560	0.916
$H_{14}, p = 8$	0.0	0.041	0.042	0.079	0.059	0.045	0.045	0.031	0.047
	0.1	0.332	0.655	0.164	0.253	0.072	0.065	0.010	0.029
	0.2	0.683	0.972	0.346	0.698	0.133	0.139	0.012	0.054
	0.3	0.838	0.988	0.538	0.918	0.166	0.211	0.003	0.041
	0.4	0.877	0.992	0.631	0.959	0.231	0.273	0.003	0.035
	0.5	0.882	0.986	0.677	0.975	0.278	0.324	0.003	0.011

Table 2: Empirical sizes and powers of HCM_n, T_n^G, T_n^{ZH}, and T_n^{ZFN} for H_{21} and H_{22} in Example 2.

	a	HCM_n		T_n^G		T_n^{ZH}		T_n^{ZFN}	
		n=100	n=200	n=100	n=200	n=100	n=200	n=100	n=200
$H_{21}, q=2$	0.0	0.044	0.045	0.047	0.056	0.044	0.051	0.035	0.055
	0.1	0.325	0.671	0.286	0.513	0.088	0.210	0.253	0.747
	0.2	0.771	0.988	0.685	0.977	0.283	0.670	0.636	0.988
	0.3	0.941	1.000	0.884	0.998	0.504	0.892	0.797	0.994
	0.4	0.970	1.000	0.956	1.000	0.664	0.980	0.840	0.993
	0.5	0.985	1.000	0.990	1.000	0.763	0.995	0.816	0.990
$H_{21}, q=4$	0.0	0.040	0.037	0.059	0.047	0.027	0.032	0.025	0.034
	0.1	0.210	0.527	0.170	0.290	0.041	0.093	0.023	0.274
	0.2	0.583	0.942	0.434	0.787	0.100	0.250	0.086	0.679
	0.3	0.819	0.991	0.617	0.965	0.182	0.425	0.110	0.732
	0.4	0.895	0.997	0.750	0.985	0.254	0.575	0.144	0.729
	0.5	0.901	1.000	0.815	0.996	0.322	0.681	0.106	0.664
$H_{21}, q=8$	0.0	0.042	0.039	0.065	0.056	0.046	0.038	0.024	0.026
	0.1	0.133	0.330	0.106	0.160	0.057	0.048	0.002	0.003
	0.2	0.409	0.854	0.190	0.408	0.066	0.073	0.000	0.001
	0.3	0.594	0.966	0.311	0.660	0.100	0.116	0.000	0.003
	0.4	0.736	0.978	0.388	0.792	0.130	0.173	0.001	0.001
	0.5	0.780	0.979	0.447	0.858	0.155	0.200	0.002	0.006
$H_{22}, q=2$	0.0	0.046	0.039	0.052	0.052	0.024	0.048	0.039	0.047
	0.1	0.155	0.327	0.128	0.207	0.046	0.087	0.067	0.235
	0.2	0.477	0.847	0.375	0.700	0.138	0.329	0.248	0.767
	0.3	0.798	0.995	0.695	0.973	0.289	0.693	0.503	0.955
	0.4	0.937	1.000	0.893	0.998	0.521	0.911	0.640	0.959
	0.5	0.966	0.999	0.978	1.000	0.692	0.978	0.719	0.941
$H_{22}, q=4$	0.0	0.040	0.041	0.059	0.064	0.041	0.038	0.033	0.049
	0.1	0.097	0.220	0.116	0.142	0.028	0.056	0.011	0.051
	0.2	0.328	0.703	0.230	0.496	0.065	0.105	0.023	0.236
	0.3	0.624	0.977	0.436	0.809	0.100	0.232	0.043	0.437
	0.4	0.807	0.991	0.686	0.971	0.203	0.416	0.055	0.499
	0.5	0.893	0.988	0.812	0.997	0.254	0.593	0.045	0.430
$H_{22}, q=8$	0.0	0.044	0.040	0.068	0.057	0.051	0.039	0.033	0.033
	0.1	0.081	0.120	0.072	0.078	0.042	0.053	0.008	0.002
	0.2	0.229	0.526	0.132	0.267	0.081	0.054	0.001	0.001
	0.3	0.472	0.877	0.236	0.478	0.071	0.104	0.001	0.001
	0.4	0.662	0.968	0.329	0.665	0.115	0.151	0.000	0.000
	0.5	0.720	0.944	0.445	0.843	0.138	0.206	0.000	0.000

Table 3: Empirical sizes and powers of HCM_n, T_n^G, T_n^{ZH}, and T_n^{ZFN} for H_{23} and H_{24} in Example 2.

	a	HCM_n		T_n^G		T_n^{ZH}		T_n^{ZFN}	
		n=100	n=200	n=100	n=200	n=100	n=200	n=100	n=200
$H_{23}, q=2$	0.0	0.044	0.049	0.058	0.057	0.017	0.034	0.035	0.067
	0.1	0.101	0.279	0.143	0.248	0.061	0.110	0.126	0.412
	0.2	0.342	0.739	0.339	0.688	0.145	0.350	0.331	0.853
	0.3	0.606	0.943	0.615	0.926	0.276	0.636	0.642	0.986
	0.4	0.724	0.992	0.782	0.992	0.417	0.855	0.752	0.999
	0.5	0.844	0.999	0.867	0.998	0.555	0.929	0.831	0.998
$H_{23}, q=4$	0.0	0.046	0.041	0.053	0.048	0.026	0.024	0.024	0.015
	0.1	0.096	0.161	0.106	0.151	0.039	0.041	0.015	0.125
	0.2	0.215	0.528	0.223	0.442	0.054	0.110	0.037	0.439
	0.3	0.368	0.834	0.342	0.727	0.089	0.195	0.118	0.744
	0.4	0.527	0.935	0.508	0.873	0.131	0.316	0.210	0.846
	0.5	0.652	0.977	0.598	0.950	0.187	0.413	0.233	0.916
$H_{23}, q=8$	0.0	0.042	0.048	0.073	0.050	0.046	0.046	0.024	0.028
	0.1	0.059	0.111	0.085	0.098	0.048	0.051	0.008	0.010
	0.2	0.138	0.331	0.117	0.192	0.052	0.048	0.006	0.007
	0.3	0.244	0.594	0.185	0.347	0.066	0.067	0.001	0.009
	0.4	0.329	0.741	0.256	0.479	0.067	0.072	0.001	0.009
	0.5	0.403	0.852	0.270	0.572	0.081	0.098	0.003	0.014
$H_{24}, q=2$	0.0	0.040	0.062	0.058	0.041	0.034	0.044	0.047	0.059
	0.1	0.199	0.476	0.210	0.453	0.099	0.223	0.232	0.699
	0.2	0.596	0.951	0.563	0.916	0.260	0.671	0.584	0.983
	0.3	0.847	0.995	0.827	0.995	0.460	0.923	0.766	0.989
	0.4	0.945	0.999	0.927	1.000	0.617	0.976	0.791	0.990
	0.5	0.956	1.000	0.968	1.000	0.735	0.991	0.805	0.990
$H_{24}, q=4$	0.0	0.041	0.041	0.047	0.048	0.030	0.038	0.022	0.026
	0.1	0.106	0.314	0.128	0.253	0.058	0.082	0.034	0.227
	0.2	0.412	0.836	0.334	0.719	0.088	0.195	0.065	0.610
	0.3	0.656	0.969	0.535	0.923	0.159	0.397	0.121	0.739
	0.4	0.804	0.991	0.679	0.978	0.237	0.548	0.133	0.717
	0.5	0.873	0.994	0.810	0.992	0.294	0.648	0.094	0.651
$H_{24}, q=8$	0.0	0.047	0.031	0.057	0.059	0.053	0.047	0.021	0.020
	0.1	0.092	0.208	0.086	0.129	0.065	0.066	0.003	0.001
	0.2	0.269	0.654	0.159	0.337	0.061	0.090	0.001	0.002
	0.3	0.470	0.877	0.262	0.564	0.111	0.112	0.001	0.004
	0.4	0.604	0.958	0.359	0.740	0.115	0.165	0.000	0.002
	0.5	0.706	0.967	0.411	0.796	0.137	0.182	0.001	0.001

Table 4: Empirical sizes and powers of HCM_n, T_n^G, T_n^{ZH}, and T_n^{ZFN} for H_{25} and H_{26} in Example 2.

	a	HCM_n		T_n^G		T_n^{ZH}		T_n^{ZFN}	
		n=100	n=200	n=100	n=200	n=100	n=200	n=100	n=200
$H_{25}, q = 2$	0.0	0.039	0.057	0.053	0.059	0.027	0.041	0.033	0.072
	0.1	0.071	0.170	0.105	0.225	0.057	0.098	0.133	0.392
	0.2	0.227	0.515	0.281	0.597	0.142	0.338	0.290	0.835
	0.3	0.419	0.850	0.500	0.865	0.253	0.622	0.514	0.979
	0.4	0.589	0.960	0.680	0.973	0.375	0.835	0.676	0.996
	0.5	0.725	0.985	0.805	0.996	0.491	0.922	0.740	0.997
$H_{25}, q = 4$	0.0	0.036	0.047	0.062	0.046	0.036	0.040	0.028	0.022
	0.1	0.051	0.101	0.087	0.144	0.040	0.043	0.020	0.101
	0.2	0.132	0.336	0.168	0.355	0.056	0.089	0.032	0.381
	0.3	0.242	0.609	0.277	0.581	0.071	0.180	0.086	0.660
	0.4	0.353	0.810	0.386	0.791	0.107	0.312	0.138	0.817
	0.5	0.496	0.913	0.514	0.902	0.137	0.388	0.182	0.878
$H_{25}, q = 8$	0.0	0.041	0.051	0.048	0.043	0.053	0.045	0.023	0.015
	0.1	0.052	0.059	0.061	0.078	0.045	0.048	0.005	0.007
	0.2	0.095	0.180	0.093	0.157	0.055	0.050	0.003	0.000
	0.3	0.136	0.364	0.143	0.274	0.059	0.082	0.002	0.008
	0.4	0.209	0.574	0.195	0.407	0.057	0.079	0.002	0.009
	0.5	0.342	0.698	0.254	0.505	0.076	0.078	0.002	0.012
$H_{26}, q = 2$	0.0	0.042	0.038	0.050	0.051	0.034	0.038	0.026	0.045
	0.1	0.035	0.082	0.074	0.100	0.052	0.105	0.055	0.135
	0.2	0.099	0.211	0.123	0.258	0.099	0.286	0.140	0.546
	0.3	0.169	0.554	0.230	0.551	0.182	0.626	0.327	0.895
	0.4	0.238	0.719	0.329	0.769	0.309	0.813	0.492	0.966
	0.5	0.277	0.815	0.392	0.875	0.343	0.915	0.580	0.981
$H_{26}, q = 4$	0.0	0.045	0.037	0.062	0.064	0.026	0.032	0.042	0.044
	0.1	0.041	0.041	0.069	0.086	0.035	0.041	0.022	0.020
	0.2	0.051	0.093	0.093	0.153	0.044	0.059	0.013	0.073
	0.3	0.077	0.130	0.136	0.218	0.047	0.128	0.020	0.156
	0.4	0.084	0.225	0.161	0.352	0.057	0.159	0.034	0.242
	0.5	0.095	0.248	0.180	0.414	0.084	0.209	0.040	0.290
$H_{26}, q = 8$	0.0	0.043	0.041	0.056	0.052	0.055	0.040	0.040	0.037
	0.1	0.046	0.036	0.077	0.072	0.047	0.041	0.021	0.022
	0.2	0.062	0.051	0.095	0.099	0.059	0.045	0.019	0.007
	0.3	0.054	0.080	0.110	0.115	0.043	0.055	0.008	0.005
	0.4	0.062	0.115	0.113	0.147	0.060	0.076	0.005	0.008
	0.5	0.080	0.104	0.123	0.144	0.071	0.072	0.003	0.007