SPECTRAL DISTRIBUTION OF THE SAMPLE COVARIANCE OF HIGH-DIMENSIONAL TIME SERIES WITH UNIT ROOTS

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Supplementary Material

This note contains proofs of Lemma 1,2,3 and convergence of $\mathcal{L}\left(H_{\mathbf{\Gamma}}, H\right)$ in Onatski and Wang (2020) (OW in what follows).

S1 Proof of Lemma 1

The definition of $\boldsymbol{C}$ yields

$$\begin{aligned}\|\mathbf{\Gamma}-\boldsymbol{C}\|_F^2 &= 2\sum_{k=1}^{T-1}(T-k)\left(c_k-\gamma_k\right)^2\\ &= 2\sum_{k=1}^{T-1}\frac{(T-k)\,k^2}{T^2}\left(\gamma_k-\gamma_{k-T}\right)^2 \leq 8\sum_{k=1}^{T-1}k\gamma_k^2.\end{aligned}$$

Recall that γ_k are the Fourier coefficients of the spectral density $f(\omega)$, and that $f(\omega)$ in our case is continuous, and thus bounded and L^2, on $[0, 2\pi]$. Hence, for any $\delta > 0$, there exists $K > 0$ such that $\sum_{k>K} \gamma_k^2 \leq \delta/16$. Therefore,

$$\|\boldsymbol{\Gamma} - \boldsymbol{C}\|_F^2 \leq 8K \sum_{k=1}^{K} \gamma_k^2 + \delta T/2 \leq \delta T$$

for all sufficiently large T. Since $\delta > 0$ is arbitrary, we obtain $\|\boldsymbol{\Gamma} - \boldsymbol{C}\|_F^2 = o(T)$.

S2 Proof of convergence of $\mathcal{L}(H_\Gamma, H)$

The rank inequality together with (A.1) of OW yield

$$\mathcal{L}\left(H_{\boldsymbol{\Gamma}}, \bar{H}_{\boldsymbol{\Gamma}}\right) \leq 1/\left(2\sqrt{u}\right). \tag{S2.1}$$

Further, inequality (A.2) and Lemma 1 of OW imply that

$$\left\|\bar{\boldsymbol{A}}^{1/2}\boldsymbol{\Gamma}\bar{\boldsymbol{A}}^{1/2} - \bar{\boldsymbol{A}}^{1/2}\boldsymbol{C}\bar{\boldsymbol{A}}^{1/2}\right\|_F^2 \leq u^2 \|\boldsymbol{\Gamma} - \boldsymbol{C}\|_F^2 = o(T) \tag{S2.2}$$

for any fixed u. By Corollary A.41 of Bai and Silverstein (2010),

$$\mathcal{L}\left(\bar{H}_{\boldsymbol{\Gamma}}, \bar{H}_{\boldsymbol{C}}\right)^3 \leq \frac{1}{T}\left\|\bar{\boldsymbol{A}}^{1/2}\boldsymbol{\Gamma}\bar{\boldsymbol{A}}^{1/2} - \bar{\boldsymbol{A}}^{1/2}\boldsymbol{C}\bar{\boldsymbol{A}}^{1/2}\right\|_F^2.$$

Hence, (S2.2) yields

$$\mathcal{L}\left(\bar{H}_{\boldsymbol{\Gamma}}, \bar{H}_{\boldsymbol{C}}\right) = o(1) \tag{S2.3}$$

for any fixed u, as $T \to \infty$.

To bound $\mathcal{L}\left(\bar{H}_{\boldsymbol{C}}, H_u\right)$, note that $\bar{H}_{\boldsymbol{C}}$ is the ESD of $\bar{\boldsymbol{A}}\boldsymbol{C}$ because the eigenvalues of $\bar{\boldsymbol{A}}\boldsymbol{C}$ and $\bar{\boldsymbol{A}}^{1/2}\boldsymbol{C}\bar{\boldsymbol{A}}^{1/2}$ coincide. On the other hand, both $\bar{\boldsymbol{A}}$ and $\boldsymbol{C}$ are circulant matrices. Therefore, they are simultaneously diagonalizable by multiplication from the right by $\boldsymbol{\mathcal{F}}^*/\sqrt{T}$ and from the left by $\boldsymbol{\mathcal{F}}/\sqrt{T}$. Consider the spectral decomposition $\boldsymbol{C} = \boldsymbol{\mathcal{F}}^*\boldsymbol{D}_{\boldsymbol{C}}\boldsymbol{\mathcal{F}}/T$ with

$$\boldsymbol{D}_{\boldsymbol{C}} = \operatorname{diag}\left(d_0, d_1, ..., d_{T-1}\right).$$

Then,

$$\bar{\boldsymbol{A}}\boldsymbol{C} = \boldsymbol{\mathcal{F}}^*\bar{\boldsymbol{D}}_{\boldsymbol{C}}\boldsymbol{\mathcal{F}}/T$$

with $\bar{\boldsymbol{D}}_{\boldsymbol{C}}$ being a diagonal matrix with the first diagonal element 0 and the $t+1$-th diagonal element $\left(1-\cos_u \omega_t\right)^{-1} d_t/2$.

Recall that $f(\omega)$ can be written as

$$f(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_k \exp\left(\mathrm{i}k\omega\right). \qquad \text{(S2.4)}$$

Denote by $\sigma_T(\omega)$ the Cesàro sum of this Fourier series

$$\sigma_T(\omega) = \frac{1}{T}\sum_{k=0}^{T-1} f_k(\omega),$$

where $f_k(\omega) \equiv \frac{1}{2\pi}\sum_{s=-k}^{k}\gamma_s \exp\left(\mathrm{i}s\omega\right)$ are the partial sums of (S2.4). As shown by Lemma 4.3 of Tyrtyshnikov (1996), $d_s = 2\pi\sigma_T(\omega_s)$ for $s = 0, ..., T-1$. On the other hand, by Fejér's theorem (e.g. p.91 of Rudin (1987)) Cesàro sums uniformly converge to $f(\omega)$ as $T \to \infty$ (because $f(\omega)$

is continuous under our assumptions). Therefore,

$$\max_{s=0,\ldots,T-1} |d_s - 2\pi f(\omega_s)| = o(1)$$

and

$$\max_{s=1,\ldots,T-1} \left| \bar{\boldsymbol{D}}_{\boldsymbol{C},ss} - \frac{\pi f(\omega_s)}{1-\cos_u \omega_s} \right| = o(1). \tag{S2.5}$$

To establish the weak convergence of $\bar{H}_{\boldsymbol{C}}$ to H_u, it is sufficient to show that, for any continuous function g with bounded support

$$\lim_{T\to\infty} \frac{1}{T} \sum_{s=0}^{T-1} g\left(\bar{\boldsymbol{D}}_{\boldsymbol{C},ss}\right) = \int g(x) \mathrm{d} H_u(x).$$

But for any such function, (S2.5) yields

$$\frac{1}{T} \sum_{s=0}^{T-1} g\left(\bar{\boldsymbol{D}}_{\boldsymbol{C},ss}\right) = \frac{1}{T} \sum_{s=0}^{T-1} g\left(\frac{\pi f(\omega_s)}{1-\cos_u \omega_s}\right) + o(1).$$

Furthermore, $g\left(\frac{\pi f(\omega_s)}{1-\cos_u \omega_s}\right)$, being a continuous function of ω, is Riemann integrable, and thus,

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \sum_{s=0}^{T-1} g\left(\bar{\boldsymbol{D}}_{\boldsymbol{C},ss}\right) &= \lim_{T\to\infty} \frac{1}{T} \sum_{s=0}^{T-1} g\left(\frac{\pi f(\omega_s)}{1-\cos_u \omega_s}\right) \\ &= \int_0^{2\pi} g\left(\frac{\pi f(\omega)}{1-\cos_u \omega}\right) \mathrm{d}\omega = \int g(x) \mathrm{d} H_u(x). \end{aligned}$$

Thus, $\bar{H}_{\boldsymbol{C}}$ is indeed weakly converging to H_u as $T \to \infty$, and hence,

$$\mathcal{L}\left(\bar{H}_{\boldsymbol{C}}, H_u\right) = o(1) \tag{S2.6}$$

for any fixed u, as $T \to \infty$.

Finally, by definition,

$$H_u(x) = \frac{1}{2\pi}\mu\left(\omega \in (0, 2\pi) : \frac{\pi f(\omega)}{1 - \cos_u \omega} \leq x\right)$$

and

$$H(x) = \frac{1}{2\pi}\mu\left(\omega \in (0, 2\pi) : \frac{\pi f(\omega)}{1 - \cos \omega} \leq x\right).$$

But $\cos_u \omega \neq \cos \omega$ may only hold for

$$\omega \leq \pi / \left(2\sqrt{u}\right) \text{ or } \omega \geq 2\pi - \pi / \left(2\sqrt{u}\right).$$

Hence,

$$\mathcal{L}(H_u, H) \leq \sup_x |H(x) - H_u(x)| \leq 1/\left(2\sqrt{u}\right). \tag{S2.7}$$

Combining (S2.1), (S2.3), (S2.6), and (S2.7), and noting that $u > 0$ can be arbitrarily large, we conclude that $\mathcal{L}(H_{\mathbf{\Gamma}}, H) \to 0$ as $T \to \infty$.

S3 Proof of Lemma 2

Let us show that Lemma 2 follows from Theorem 1.1 of Bai and Zhou (2008).

Let $\boldsymbol{W} = \bar{\boldsymbol{A}}^{1/2}\mathbf{\Gamma}^{1/2}$, and let Z_k be the k-th column of $\boldsymbol{W}\boldsymbol{\eta}'$. Then,

$$\mathbb{E}Z_{ik}Z_{lk} = Cov(Z_{ik}, Z_{lk}) = \left(\bar{\boldsymbol{A}}^{1/2}\mathbf{\Gamma}\bar{\boldsymbol{A}}^{1/2}\right)_{il} \equiv t_{il},$$

which is independent from k. Moreover,

$$\left\|\bar{\boldsymbol{A}}^{1/2}\mathbf{\Gamma}\bar{\boldsymbol{A}}^{1/2}\right\| \leq u\|\mathbf{\Gamma}\| \leq 2u\left(\sum_{j=0}^{\infty}|\theta_j|\right)^2 < \infty$$

(see, e.g. p. 434 of Bai and Zhou (2008)). By (S2.3) and (S2.6), the ESD of $\bar{\boldsymbol{A}}^{1/2}\boldsymbol{\Gamma}\bar{\boldsymbol{A}}^{1/2}$ (which is the same as that of $\boldsymbol{\Gamma}^{1/2}\bar{\boldsymbol{A}}\boldsymbol{\Gamma}^{1/2}$) converges to H_u. The only assumption of Theorem 1.1 of Bai and Zhou (2008) left to verify is that for any non-random $T \times T$ matrix $\boldsymbol{B}$ with bounded norm,

$$\mathbb{E}\left(Z_k'\boldsymbol{B}Z_k - \operatorname{tr}\left(\boldsymbol{B}\bar{\boldsymbol{A}}^{1/2}\boldsymbol{\Gamma}\bar{\boldsymbol{A}}^{1/2}\right)\right)^2 = o\left(T^2\right).$$

Let $\bar{\boldsymbol{B}} = \bar{\boldsymbol{A}}^{1/2}\boldsymbol{B}\bar{\boldsymbol{A}}^{1/2}$. Clearly, for any fixed $u > 0$, $\bar{\boldsymbol{B}}$ has a bounded norm as long as $\boldsymbol{B}$ has a bounded norm. On the other hand, $Z_k = \bar{\boldsymbol{A}}^{1/2}\varepsilon_k$, where ε_k is the transpose of the k-th row of ε. Hence, it is sufficient to show that

$$\mathbb{E}\left(\varepsilon_k'\bar{\boldsymbol{B}}\varepsilon_k - \operatorname{tr}\left(\bar{\boldsymbol{B}}\boldsymbol{\Gamma}\right)\right)^2 = o\left(T^2\right).$$

But this fact was established in Bai and Zhou (2008, p. 435). To summarize, all conditions of Theorem 1.1 Bai and Zhou (2008) are satisfied and thus, $\bar{F}_{n,T}$ a.s. weakly converges to F_u as $n, T \to_c \infty$. This completes the proof.

S4 Proof of Lemma 3

Recall that T_γ is defined as the smallest integer s.t. $n/T_\gamma \leq \gamma$. Let $T = T_\infty > T_\gamma$ and let $\boldsymbol{\xi}$ be an $n \times T$ matrix with i.i.d. $N(0,1)$ entries. Consider a partition $\boldsymbol{\xi} = [\boldsymbol{\xi}_\gamma, \boldsymbol{\xi}_\infty]$, where $\boldsymbol{\xi}_\gamma$ and $\boldsymbol{\xi}_\infty$ are $n \times T_\gamma$ and $n \times (T_\infty - T_\gamma)$ respectively. Further, let $\boldsymbol{\Delta}$ be defined similarly to $\boldsymbol{\Delta}_\gamma$ with T_γ replaced by

T and partition $\boldsymbol{\Delta} = \text{diag}[\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2]$, where $\boldsymbol{\Delta}_1$ is $T_\gamma \times T_\gamma$. Then we have

$$\boldsymbol{M}_{n,T_\gamma} = \frac{n}{(T_\gamma+1)^2}\boldsymbol{\xi}_\gamma \boldsymbol{\Delta}_\gamma \boldsymbol{\xi}_\gamma' \text{ and } \boldsymbol{M}_{n,T_\infty} = \frac{n}{(T_\infty+1)^2}(\boldsymbol{\xi}_\gamma \boldsymbol{\Delta}_1 \boldsymbol{\xi}_\gamma' + \boldsymbol{\xi}_\infty \boldsymbol{\Delta}_2 \boldsymbol{\xi}_\infty').$$

Hence,

$$\boldsymbol{M}_{n,T_\infty} - \boldsymbol{M}_{n,T_\gamma} = n\boldsymbol{\xi}_\gamma \left(\frac{\boldsymbol{\Delta}_1}{(T_\infty+1)^2} - \frac{\boldsymbol{\Delta}_\gamma}{(T_\gamma+1)^2} \right) \boldsymbol{\xi}_\gamma' + n\frac{\boldsymbol{\xi}_\infty \boldsymbol{\Delta}_2 \boldsymbol{\xi}_\infty'}{(T_\infty+1)^2}.$$

First, consider $n\boldsymbol{\xi}_\gamma \left(\frac{\boldsymbol{\Delta}_1}{(T_\infty+1)^2} - \frac{\boldsymbol{\Delta}_\gamma}{(T_\gamma+1)^2} \right) \boldsymbol{\xi}_\gamma'$. Recall that the diagonal elements of $\boldsymbol{\Delta}_1$ have form $\frac{1}{2}(1-\cos \pi j/(T_\infty+1))^{-1}$ for $j \le T_\gamma$. The diagonal elements of $\boldsymbol{\Delta}_\gamma$ have a similar form with T_∞ replaced by T_γ. Since

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 \cos t$$

for some $t \in [0, x]$, we have

$$\frac{1}{2(T_\infty+1)^2}\left(1-\cos \pi j/(T_\infty+1)\right)^{-1} = \frac{1}{(\pi j)^2}\left(1 - \frac{\cos t}{12}\frac{(\pi j)^2}{(T_\infty+1)^2}\right)^{-1}$$

for some $t \in [0, \pi]$ and hence

$$\frac{1}{2(T_\infty+1)^2}\left(1-\cos \pi j/(T_\infty+1)\right)^{-1} - \frac{1}{(\pi j)^2} = \frac{\cos t}{12(T_\infty+1)^2}\left(1 - \frac{\cos t}{12}\frac{(\pi j)^2}{(T_\infty+1)^2}\right)^{-1}.$$

Since $j \le T_\gamma < T_\infty$, we have

$$1 - \frac{\cos t}{12}\frac{(\pi j)^2}{(T_\infty+1)^2} > 1 - \frac{\pi^2}{12} > \frac{1}{12},$$

and thus

$$\left| \frac{1}{2(T_\infty+1)^2}\left(1-\cos \pi j/(T_\infty+1)\right)^{-1} - \frac{1}{(\pi j)^2} \right| < \frac{1}{T_\infty^2}.$$

The inequality holds similarly for the elements of $\mathbf{\Delta}_\gamma$,

$$\left|\frac{1}{2(T_\gamma+1)^2}\left(1-\cos\pi j/(T_\gamma+1)\right)^{-1}-\frac{1}{(\pi j)^2}\right|<\frac{1}{T_\gamma^2}.$$

Therefore, we have

$$\left|\frac{1}{2(T_\infty+1)^2}\left(1-\cos\pi j/(T_\infty+1)\right)^{-1}-\frac{1}{2(T_\gamma+1)^2}\left(1-\cos\pi j/(T_\gamma+1)\right)^{-1}\right|<\frac{2}{T_\gamma^2}.$$

To summarize,

$$\left\|n\frac{\boldsymbol{\xi}_\gamma\mathbf{\Delta}_1\boldsymbol{\xi}_\gamma'}{(1+T_\infty)^2}-n\frac{\boldsymbol{\xi}_\gamma\mathbf{\Delta}_\gamma\boldsymbol{\xi}_\gamma}{(1+T_\gamma)^2}\right\|<\left\|2\frac{n}{T_\gamma}\frac{\boldsymbol{\xi}_\gamma\boldsymbol{\xi}_\gamma'}{T_\gamma}\right\|<4\gamma.$$

with high probability for sufficiently small γ. The last inequality is due to the fact that the largest eigenvalue of $\frac{\boldsymbol{\xi}_\gamma\boldsymbol{\xi}_\gamma'}{T_\gamma}$ a.s. converges to $(1+\sqrt{\gamma})^2$.

Next consider the component $n\frac{\boldsymbol{\xi}_\infty\mathbf{\Delta}_2\boldsymbol{\xi}_\infty'}{(T_\infty+1)^2}$. Since $1-\cos x>x^2/6$ for $x\in[0,\pi]$, we have

$$2(T_\infty+1)^2(1-\cos\pi j/(T_\infty+1))>(\pi j)^2/3.$$

Partition $\mathbf{\Delta}_2$ as $\mathrm{diag}[\mathbf{\Delta}_{2,1},\cdots,\mathbf{\Delta}_{2,(T_\infty-T_\gamma)/T_\gamma}]$ where each $\mathbf{\Delta}_{2,i}$ is T_γ-dimensional. (We can choose T_∞ so that $(T_\infty-T_\gamma)/T_\gamma$ is an integer, so such a representation is possible.) Using the fact that the diagonal elements of $\mathbf{\Delta}_{2,i}/(T_\infty+1)^2$ have form

$$\frac{1}{2(T_\infty+1)^2(1-\cos\pi j/(T_\infty+1))}$$

with $j=iT_\gamma+1,\cdots,(i+1)T_\gamma-1$, we find that the upper bound on the diagonal elements of $\mathbf{\Delta}_{2,i}/(T_\infty+1)^2$ equals

$$\frac{1}{2(T_\infty+1)^2(1-\cos iT_\gamma\pi/(T_\infty+1))},$$

which is no larger than $3/(i\pi T_\gamma)^2$.

Partition $\boldsymbol{\xi}_\infty$ conformably with $\mathbf{\Delta}_2$ so that $\boldsymbol{\xi}_\infty = [\boldsymbol{\xi}_{\infty,1},\cdots,\boldsymbol{\xi}_{\infty,(T_\infty-T_\gamma)/T_\gamma}]$. Then, from the above, we have

$$\left\| n\frac{\boldsymbol{\xi}_\infty \mathbf{\Delta}_2 \boldsymbol{\xi}'_\infty}{(T_\infty+1)^2}\right\| \le \frac{3n}{\pi^2 T_\gamma}\sum_{i=1}^{(T_\infty-T_\gamma)/T_\gamma}\frac{1}{i^2}\left\|\frac{\boldsymbol{\xi}_{\infty,i}\boldsymbol{\xi}'_{\infty,i}}{T_\gamma}\right\|.$$

The Gaussian concentration inequality for the singular values of a rectangular matrix with i.i.d. Gaussian entries (see Theorem II.13 of Davidson and Szarek (2001)) implies that, for any $t>0$,

$$\mathbb{P}\left(\left\|\frac{\boldsymbol{\xi}_{\infty,i}\boldsymbol{\xi}'_{\infty,i}}{T_\gamma}\right\| \ge \left(1+\sqrt{\frac{n}{T_\gamma}}+t\right)^2\right) < \exp\left(-\frac{T_\gamma t^2}{2}\right).$$

Taking $t=i^{1/4}$, we then have

$$\sum_{i=1}^{(T_\infty-T_\gamma)/T_\gamma}\mathbb{P}\left(\left\|\frac{\boldsymbol{\xi}_{\infty,i}\boldsymbol{\xi}'_{\infty,i}}{T_\gamma}\right\| \ge \left(1+\sqrt{\frac{n}{T_\gamma}}+i^{1/4}\right)^2\right) < \sum_{i=1}^{\infty}\exp\left(-\frac{T_\gamma i^{1/2}}{2}\right).$$

Clearly, the right hand side of the above inequality can be made arbitrarily small by choosing sufficiently large T_γ. Therefore, with large probability, for sufficiently large T_γ, all $\left\|\frac{\boldsymbol{\xi}_{\infty,i}\boldsymbol{\xi}'_{\infty,i}}{T_\gamma}\right\|$ are smaller than $\left(1+\sqrt{\frac{n}{T_\gamma}}+i^{1/4}\right)^2$ and

$$\left\| n\frac{\boldsymbol{\xi}_\infty \Delta_2 \boldsymbol{\xi}'_\infty}{(T_\infty+1)^2}\right\| \le \frac{3n}{\pi^2 T_\gamma}\sum_{i=1}^{(T_\infty-T_\gamma)/T_\gamma}\frac{\left(1+\sqrt{\frac{n}{T_\gamma}}+i^{1/4}\right)^2}{i^2} \le K\gamma$$

for some constant K that does not depend on $\gamma \in (0,1)$. This completes the proof.

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