Statistica Sinica **31** (2021), 1689-1706
doi:https://doi.org/10.5705/ss.202019.0173

CONSTRUCTION OF UNIFORM DESIGNS AND COMPLEX-STRUCTURED UNIFORM DESIGNS VIA PARTITIONABLE t-DESIGNS

Hengzhen Huang[1], Huangsheng Yu[1], Min-Qian Liu[2] and Dianhua Wu[1]

[1]*Guangxi Normal University and* [2]*Nankai University*

Abstract: We propose several infinite classes of new uniform designs under the discrete discrepancy criterion. The construction is based on combinatorial configurations, that is, partitionable t-designs; hence, it dose not require a computer search. Moreover, we explore certain complex structures within the proposed uniform designs. We find that some of the designs have nested or sliced subdesigns that are also uniform designs under the discrete discrepancy. The proposed uniform designs may find many applications in statistics, including computer experiments, sequential experiments, drug combination studies, and cross-validation.

Key words and phrases: Combinatorial construction, discrete discrepancy, nested and sliced design, uniformity.

1. Introduction

Uniform designs have been widely applied in manufacturing, system engineering, pharmaceutics, computer experiments, and many other fields since they were first developed in the 1980s (see Fang (1980); Santner, Williams and Notz (2003); Tan, Fang and Ross (2012)). Such designs have a number of desirable statistical properties, such as *admissibility*, *minimaxity*, and *robustness* (Fang, Li and Sudjianto (2006)). In contrast to classical experimental designs such as orthogonal arrays, uniform designs are practically useful because they can investigate many high-level factors simultaneously using fairly economical experimental runs. In this sense, high-level designs are the main focus of uniform designs, which makes their systematic construction difficult. Over the last 20 years, two kinds of construction methods have been proposed: the algorithmic optimization approach (see Fang et al. (2018, Ch. 4), and the references therein), and combinatorial configurations (see Fang et al. (2018, Ch. 3.6), and the references therein). The second method uses the properties of various combinatorial configurations and construction techniques frequently used in design theory to obtain uniform

Corresponding author: Dianhua Wu, College of Mathematics and Statistics, Guangxi Normal University, Guilin 541004, China. E-mail: dhwugx@163.com.

designs without needing a computer search.

Note that a uniform design is an important kind *space-filling* design, and a number of design criteria are used to measure the uniformity of a design. Here, we focus on the *discrete discrepancy* criterion (DD), proposed by Hickernell and Liu (2002), as the uniformity measure. The DD is closely related to many other conventional design criteria, such as the non-orthogonal measure $E(f_{NOD})$, generalized minimum aberration, centered L_2-discrepancy, and wrap-around L_2-discrepancy (Fang, Lin and Liu (2003); Fang et al. (2004); Sun, Chen and Liu (2011)). This makes the DD popular and justifies using it as a uniformity measure. In the past decade, different combinatorial configurations have been used to construct many large, and even some infinite classes of uniform designs under the DD. However, most existing studies are limited to two classes of combinatorial configurations, namely resolvable designs and orthogonal arrays, which restricts the spectrum of existing uniform designs. A comprehensive overview of resolvable designs, orthogonal arrays, and their relationships with uniform designs can be found in Fang et al. (2018). In this study, we use an important class of combinatorial configurations called partitionable t-$(v, k, 1)$ designs to construct uniform designs under the DD. The constructed uniform designs are newly obtained because partitionability in design theory differs from resolvability and orthogonality (Colbourn and Dinita (2007)). Furthermore, we explore certain complex structures within the proposed uniform designs. We find that some of the proposed uniform designs have nested or sliced subdesigns that are also uniform designs under the DD. The proposed uniform designs can be applied to routine and complex tasks in statistics, including factorial experiments, computer experiments, sequential experiments, drug combination studies, stochastic optimization, and cross-validation (e.g., see Fang, Li and Sudjianto (2006); Qian, Tang and Wu (2009); Qian and Wu (2009); Tan, Fang and Ross (2012)).

The remainder of this paper is organized as follows. Section 2 introduces some basic concepts, including the DD, U-type design, uniform design, and partitionable t-design. This section also provides a lower bound for the DD that can be used as a benchmark when constructing uniform designs. A new construction method for uniform designs under the DD is presented in Section 3. Construction methods for nested and sliced uniform designs are given in Sections 4 and 5, respectively. Section 6 concludes the paper. All technical proofs are provided in the Appendix.

2. Basic Concepts

A uniform design aims to choose a set of points over a design region such that these points are uniformly scattered. The measure of uniformity plays a key role in the concept of a uniform design. In the literature, various discrepancies have been proposed for measuring the uniformity of a design. In most cases, the discrepancies are defined based on a type of design called a U-type design.

Definition 1. A U-type design with n runs and m factors, with levels $q_1, \ldots, q_m$, respectively, is an $n \times m$ matrix such that the q_j levels in the jth column appear equally often. This design is denoted by $U(n; q_1 \times \cdots \times q_m)$. When some q_j are equal, we denote this by $U(n; q_1^{r_1} \times \cdots \times q_s^{r_s})$, with $r_1 + \cdots + r_s = m$. If all q_j are equal, denoted by $U(n; q^m)$, the design is said to be symmetrical; otherwise, it is asymmetrical.

A U-type design is also called a *balanced design* or a *lattice design* (Fang, Li and Sudjianto (2006)). The set of all such $U(n; q_1 \times \cdots \times q_m)$ is denoted by $\mathcal{U}(n; q_1 \times \cdots \times q_m)$. By using a reproducing kernel in a Hilbert space and choosing an appropriate kernel function, the DD is defined on a U-type design, and measures how far apart the empirical distribution of the design points is from the discrete uniform distribution on the design region. For further details on the DD, see Fang, Li and Sudjianto (2006). In what follows, we use $D(\mathbf{X}; a, b)$ to denote the DD with parameters a and b ($a > b > 0$) for a U-type design $\mathbf{X}$. The condition $a > b > 0$ ensures that the chosen kernel function is positive definite. Any choice for the parameters a and b satisfying such a condition is feasible for the DD (i.e., they do not affect the ranks of the designs). If a U-type design $U(n; q_1 \times \cdots \times q_m)$ takes the minimum of the DD over $\mathcal{U}(n; q_1 \times \cdots \times q_m)$, it is called a uniform design under the DD, and is denoted by $U_n(q_1 \times \cdots \times q_m)$.

Let λ_{ij} ($i, j = 1, \ldots, n, i \neq j$) be the number of coincidences between the ith row and the jth row of $\mathbf{X}$. Then $m - \lambda_{ij}$ is the Hamming distance between these two rows. The following fact about the analytical expression and the lower bound of the DD on $\mathcal{U}(n; q_1 \times \cdots \times q_m)$ is extracted from page 76 of Fang, Li and Sudjianto (2006).

Fact 1. Let $\mathbf{X}$ be a U-type design $U(n; q_1 \times \cdots \times q_m)$, $\lambda = (\sum_{j=1}^m n/q_j - m)/(n-1)$ and $\gamma = \lfloor \lambda \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. Then,

$$D^2(\mathbf{X}; a, b) = \frac{a^m}{n} + \frac{b^m}{n^2} \sum_{i,j=1, i \neq j}^{n} \left(\frac{a}{b}\right)^{\lambda_{ij}} - \prod_{j=1}^{m} \left\lfloor \frac{a + (q_j - 1)b}{q_j} \right\rfloor,$$

$$\sum_{i,j=1,i\neq j}^{n} \left(\frac{a}{b}\right)^{\lambda_{ij}} \geq n(n-1)\left\lfloor(\gamma+1-\lambda)\left(\frac{a}{b}\right)^{\gamma}+(\lambda-\gamma)\left(\frac{a}{b}\right)^{\gamma+1}\right\rfloor, \quad (2.1)$$

and the lower bound on the right-hand side of (2.1) can be achieved if and only if all λ_{ij} take the same value γ, or take only two values γ and $\gamma+1$.

The necessary and sufficient condition stated in Fact 1 indicates that a uniform design under the DD seeks a set of points with levels that are as different as possible. Now, let us introduce the definition of a partitionable t-$(v,k,1)$ design.

Definition 2. A t-$(v,k,1)$ design is a pair $(V,\mathcal{B})$, where V is a v-element set of points, and $\mathcal{B}$ is a collection of k-element subsets of V (called blocks) with the property that every t-element subset of V is contained in exactly one block. A t-$(v,k,1)$ design is called partitionable if its block set $\mathcal{B}$ can be divided into several $(t-1)$-$(v,k,1)$ designs.

According to the above definition, it is easy to see that in a t-$(v,k,1)$ design, there are exactly $\binom{v}{t}\Big/\binom{k}{t}$, or $v(v-1)\cdots(v-t+1)/(k(k-1)\cdots(k-t+1))$ blocks. As a result, in a partitionable t-$(v,k,1)$ design, there are exactly $((v-t+1)/(k-t+1))$ $(t-1)$-$(v,k,1)$ designs, each with $\binom{v}{t-1}\Big/\binom{k}{t-1}$, or $v(v-1)\cdots(v-t+2)/(k(k-1)\cdots(k-t+2))$ blocks. For brevity, each $(t-1)$-$(v,k,1)$ design in a partitionable t-$(v,k,1)$ design is called a *parallel class*. Below is an example.

Example 1. Let $V=\{0,1,2,3,4,5,6,7,8\}$ be a nine-element set of points. Then, the 84 blocks in Table 1 comprise a 3-$(9,3,1)$ design. Furthermore, there are seven parallel classes, denoted by $\mathcal{P}_1,\ldots,\mathcal{P}_7$, respectively, in this design. That is, the 12 blocks in column $\mathcal{P}_i$ comprise a 2-$(9,3,1)$ design, for $i=1,\ldots,7$. In other words, Table 1 displays a partitionable 3-$(9,3,1)$ design with seven 2-$(9,3,1)$ designs as its parallel classes. The design in Table 1 is obtained from Lu (1983).

3. Construction of Uniform Designs

This section explores the relationship between partitionable t-designs and uniform designs under the DD. A similar investigation was undertaken by Tang (2005). As shown below, several infinite classes of symmetrical uniform designs that achieve the lower bound given in Fact 1 can be constructed from partitionable t-designs.

Let $(V,\mathcal{B})$ be a partitionable t-$(v,k,1)$ design and, without loss of generality,

Table 1. A partitionable 3-$(9,3,1)$ design in Example 1.

$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_6$	$\mathcal{P}_7$
{1,2,4}	{2,3,5}	{3,4,6}	{0,4,5}	{1,5,6}	{0,2,6}	{0,1,3}
{3,5,6}	{0,4,6}	{0,1,5}	{1,2,6}	{0,2,3}	{1,3,4}	{2,4,5}
{1,5,7}	{2,6,7}	{0,3,7}	{1,4,7}	{2,5,7}	{3,6,7}	{0,4,7}
{2,6,8}	{0,3,8}	{1,4,8}	{2,5,8}	{3,6,8}	{0,4,8}	{1,5,8}
{0,3,4}	{1,4,5}	{2,5,6}	{0,3,6}	{0,1,4}	{1,2,5}	{2,3,6}
{2,3,7}	{3,4,7}	{4,5,7}	{5,6,7}	{0,6,7}	{0,1,7}	{1,2,7}
{4,5,8}	{5,6,8}	{0,6,8}	{0,1,8}	{1,2,8}	{2,3,8}	{3,4,8}
{0,1,6}	{0,1,2}	{1,2,3}	{2,3,4}	{3,4,5}	{4,5,6}	{0,5,6}
{4,6,7}	{0,5,7}	{1,6,7}	{0,2,7}	{1,3,7}	{2,4,7}	{3,5,7}
{1,3,8}	{2,4,8}	{3,5,8}	{4,6,8}	{0,5,8}	{1,6,8}	{0,2,8}
{0,2,5}	{1,3,6}	{0,2,4}	{1,3,5}	{2,4,6}	{0,3,5}	{1,4,6}
{0,7,8}	{1,7,8}	{2,7,8}	{3,7,8}	{4,7,8}	{5,7,8}	{6,7,8}

set $V=\{0,1,\ldots,v-1\}$. Denote

$$n=\binom{v}{t-1},\quad m=\frac{v-t+1}{k-t+1}\quad\text{and}\quad q=\frac{v(v-1)\cdots(v-t+2)}{k(k-1)\cdots(k-t+2)}. \tag{3.1}$$

Suppose $\mathcal{B}=\mathcal{P}_1\cup\mathcal{P}_2\cup\cdots\cup\mathcal{P}_m$, where each parallel class $\mathcal{P}_i=\cup_j\{P_{ij}\}$, for $i=1,2,\ldots,m$, $j=1,2,\ldots,q$, represents the blocks of a $(t-1)$-$(v,k,1)$ design. Then, a U-type design $U(n;q^m)$ can be constructed as follows.

Construction 1.

Step 1. Order the q blocks P_{ij} in each $\mathcal{P}_i$ randomly, for $j=1,2,\ldots,q$, and $i=1,2,\ldots,m$.

Step 2. Order the n $(t-1)$-subsets of V lexicographically; they will be used as the row labels of the design.

Step 3. For each $\mathcal{P}_i$, for $i=1,2,\ldots,m$, construct a corresponding n-vector $\mathbf{x}_i$, with coordinate labelled by $(l_1,l_2,\ldots,l_{t-1})$ taking the value j, if the $(t-1)$-subset $\{l_1,l_2,\ldots,l_{t-1}\}$ occurs together in the jth block of $\mathcal{P}_i$.

Step 4. Column-juxtapose the $\mathbf{x}_i$ to form an $n\times m$ matrix $\mathbf{X}=(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m)$.

Example 2. The U-type design $U(36;12^7)$ listed in Table 2 is derived from the partitionable 3-$(9,3,1)$ design in Example 1 by using Construction 1.

We have the following theorem; the proof can be found in the Appendix.

Theorem 1. *The U-type design $U(n;q^m)$ derived from a partitionable t-$(v,k,1)$ design using Construction 1 is a uniform design under the DD. Moreover, any of the possible level-combinations between any two columns appears at most once.*

Table 2. A U-type design $U(36; 12^7)$.

Run number	Row label	I	II	III	IV	V	VI	VII	Run number	Row label	I	II	III	IV	V	VI	VII
1	(0,1)	8	8	2	7	5	6	1	19	(2,6)	4	3	5	2	11	1	5
2	(0,2)	11	8	11	9	2	1	10	20	(2,7)	6	3	12	9	3	9	6
3	(0,3)	5	4	3	5	2	11	1	21	(2,8)	4	10	12	4	7	7	10
4	(0,4)	5	2	11	1	5	4	3	22	(3,4)	5	6	1	8	8	2	7
5	(0,5)	11	9	2	1	10	11	8	23	(3,5)	2	1	10	11	8	11	9
6	(0,6)	8	2	7	5	6	1	8	24	(3,6)	2	11	1	5	4	3	5
7	(0,7)	12	9	3	9	6	6	3	25	(3,7)	6	6	3	12	9	3	9
8	(0,8)	12	4	7	7	10	4	10	26	(3,8)	10	4	10	12	4	7	7
9	(1,2)	1	8	8	2	7	5	6	27	(4,5)	7	5	6	1	8	8	2
10	(1,3)	10	11	8	11	9	2	1	28	(4,6)	9	2	1	10	11	8	11
11	(1,4)	1	5	4	3	5	2	11	29	(4,7)	9	6	6	3	12	9	3
12	(1,5)	3	5	2	11	1	5	4	30	(4,8)	7	10	4	10	12	4	7
13	(1,6)	8	11	9	2	1	10	11	31	(5,6)	2	7	5	6	1	8	8
14	(1,7)	3	12	9	3	9	6	6	32	(5,7)	3	9	6	6	3	12	9
15	(1,8)	10	12	4	7	7	10	4	33	(5,8)	7	7	10	4	10	12	4
16	(2,3)	6	1	8	8	2	7	5	34	(6,7)	9	3	9	6	6	3	12
17	(2,4)	1	10	11	8	11	9	2	35	(6,8)	4	7	7	10	4	10	12
18	(2,5)	11	1	5	4	3	5	2	36	(7,8)	12	12	12	12	12	12	12

Theorem 1 makes it possible for the experimenter to use existing partitionable t-designs to obtain corresponding uniform designs. From the rich literature on partitionable t-designs, we have the following fact.

Fact 2. The following results are known:

1. For $v \equiv 3 \pmod 6$, there exists a partitionable 2-$(v, 3, 1)$ design (Ray-Chauduri and Wilson (1971));
2. For $v \equiv 4 \pmod{12}$, there exists a partitionable 2-$(v, 4, 1)$ design (Hanani (1974));
3. For $v \equiv 1$ or $3 \pmod 6$, there exists a partitionable 3-$(v, 3, 1)$ design (Lu (1983); Teirlinck (1991));
4. For any integer v, there exists a partitionable 3-$(4^v, 4, 1)$ design (Baker (1976); Zaicev, Zinoviev and Semakov (1973));
5. For any integer v, there exists a partitionable 3-$(2 \cdot K^v + 2, 4, 1)$ design, where K is equal to 7 or 31 (Teirlinck (1994)).

Interested readers are referred to the corresponding references for how to construct the above partitionable t-designs. In particular, some partitionable t-

designs are tabulated in the book by Colbourn and Dinita (2007). Combining Theorem 1 and Fact 2, we have the following theorem.

Theorem 2. *If the parameters satisfy one of the conditions below, then there exists a uniform design $U_n(q^m)$ under the DD:*

1. $n \equiv 3 \pmod 6$, $m = (n-1)/2$, $q = n/3$;
2. $n \equiv 4 \pmod{12}$, $m = (n-1)/3$, $q = n/4$;
3. $n = \binom{v}{2}$, $m = v-2$, $q = v(v-1)/6$, *where* $v \equiv 1$ *or* $3 \pmod 6$;
4. $n = \binom{l}{2}$, $m = (l-2)/2$, $q = (l(l-1))/12$, *where* $l \equiv 0 \pmod{4^v}$, *and* v *is a positive integer;*
5. $n = \binom{l}{2}$, $m = (l-2)/2$, $q = (l(l-1))/12$, *where* $l \equiv 2$ (mod $2\cdot 7^v$ or $2\cdot 31^v$), *and* v *is a positive integer.*

Remark 1. From the proof of Theorem 1, it can be easily seen that the coincidence number between any two rows of the design constructed using Construction 1 is zero or one. This indicates that for a uniform design $\mathrm{U}_n(q^m)$ in Theorem 2, its any sub-column design, say $\mathrm{U}_n(q^{m^*})$ with $1 < m^* < m$, is still a uniform design under the DD. For given n, m, m^*, and q, with $1 < m^* < m$, there are $\binom{m}{m^*}$ sub-column uniform designs of a $\mathrm{U}_n(q^m)$. In this case, the experimenter may adopt other design criteria, such as the generalized minimum aberration or centered L_2-discrepancy, to further discriminate them.

4. Construction of Nested Uniform Designs

Nested space-filling designs are useful for multi-fidelity computer experiments, sequential experiments, and stochastic optimization (e.g., see Qian (2009); Qian, Tang and Wu (2009); Qian, Ai and Wu (2009); Sun, Yin and Liu (2013); Sun, Liu and Qian (2014); Yang, Liu and Lin (2014); Yang et al. (2016b)). In the previous section, we derived several infinite classes of new uniform designs under the DD. In this section, we further explore nested uniform designs under the DD. Here, we find that nested uniform designs with two layers can be obtained from partitionable 3-$(v, 3, 1)$ designs, while nested uniform designs with more than two layers can be obtained from partitionable t-$(v, k, 1)$ designs with $t \geq 4$.

For simplicity of exposition, we use the notation $NU_{n_1,n_2,\ldots,n_l}(q_1^m, q_2^m, \ldots, q_l^m)$ to denote a symmetrical nested uniform design with l layers; that is, for $i = 1, \ldots, l$, each $U_{n_i}(q_i^m)$ is a symmetrical uniform design such that $U_{n_j}(q_j^m)$ is a subdesign of $U_{n_k}(q_k^m)$, for $j < k$. The following construction and theorem reveal that an $NU_{n_1,n_2}(q_1^m, q_2^m)$ with $n_1 = v-1, q_1 = (v-1)/2, n_2 = v(v-1)/2, q_2 = v(v-1)/6$, and $m = v-2$ can be obtained based on a partitionable 3-$(v,3,1)$ design. The proof can be found in the Appendix.

Construction 2.

Step 1. Based on a partitionable 3-$(v,3,1)$ design, a uniform design $U_{n_2}(q_2^m)$ with $n_2 = v(v-1)/2, q_2 = v(v-1)/6$, and $m = v-2$ can be obtained using Construction 1. For brevity, denote the resulting uniform design by $\mathbf{X}$.

Step 2. Collect the first $v-1$ rows of $\mathbf{X}$ as a subdesign of $\mathbf{X}$. For brevity, denote such a subdesign by $\mathbf{X}_1$.

Theorem 3. *The design $\mathbf{X}_1$ is a uniform design $U_{n_1}(q_1^m)$ under the DD and, therefore, the design $\mathbf{X}$ is a nested uniform design $NU_{n_1,n_2}(q_1^m, q_2^m)$, where*

$$n_1 = v-1, \quad q_1 = \frac{v-1}{2}, \quad n_2 = \frac{v(v-1)}{2}, \quad q_2 = \frac{v(v-1)}{6} \quad \textit{and} \quad m = v-2.$$

Example 3. The first eight rows of the design in Table 2 comprise a U-type design $U(8;4^7)$, denoted by $\mathbf{X}_1$, that is a uniform design under the DD. Therefore, the design listed in Table 2 is a nested uniform design $NU_{8,36}(4^7, 12^7)$, denoted by $\mathbf{X}$. By setting $a = 2$ and $b = 1$, we have that $D^2(\mathbf{X}; a, b) = 1{,}764$, which achieves the lower bound given in Fact 1. In addition, $D^2(\mathbf{X}_1; a, b) = 112$, which also achieves the lower bound of the DD.

Remark 2. (i) According to the proof of Theorem 3, it is not difficult to see that there are as many as v subdesigns of $\mathbf{X}$, all of which are uniform designs under the DD. These subdesigns respectively correspond to the row labels containing i, for $i = 0, \ldots, v-1$. For instance, the rows 9–15 together with the first row in Table 2 also comprise a uniform design $U_8(4^7)$. We choose the first $v-1$ rows of $\mathbf{X}$ as the nested subdesign, just for convenience. In practice, the experimenter may adopt other design criteria, such as the centered L_2-discrepancy, to further discriminate these v subdesigns. (ii) It is possible to conduct a level permutation to each column of a nested uniform design so that the level symbols appearing in the nested subdesign are consistent for each column, and the levels are distributed as uniformly as possible for each dimension. For instance, one may randomly choose a level from each of the sets $\{1,2,3\}, \{3,5,6\}, \{7,8,9\}$, and $\{10,11,12\}$, say $2,6,7$, and 12, and conduct the level permutation $(1,2,3,4,5,6,7,8,9,10,11,12) \rightarrow (1,5,3,4,2,8,11,6,9,10,7,12)$ to the first column of the design in Table 2, such that the four levels $2,6,7$, and 12 appear in the first column of the first eight rows. Similar practices may be applied to the remaining columns (see Table 3).

Table 3. A nested uniform design $NU_{8,36}(4^7, 12^7)$.

Run number	Row label	I	II	III	IV	V	VI	VII	Run number	Row label	I	II	III	IV	V	VI	VII
1	(0,1)	6	7	2	7	6	7	2	19	(2,6)	4	3	5	1	11	2	5
2	(0,2)	7	7	12	12	2	2	12	20	(2,7)	8	3	11	12	3	9	3
3	(0,3)	2	6	6	6	2	12	2	21	(2,8)	4	10	11	4	5	4	12
4	(0,4)	2	2	12	2	6	6	6	22	(3,4)	2	4	1	8	8	1	8
5	(0,5)	7	12	2	2	12	12	7	23	(3,5)	5	1	10	11	8	12	9
6	(0,6)	6	2	7	6	7	2	7	24	(3,6)	5	11	1	6	4	3	5
7	(0,7)	12	12	6	12	7	7	6	25	(3,7)	8	4	6	9	9	3	9
8	(0,8)	12	6	7	7	12	6	12	26	(3,8)	10	6	10	9	4	4	8
9	(1,2)	1	7	8	1	5	5	3	27	(4,5)	11	5	3	2	8	8	1
10	(1,3)	10	11	8	11	9	1	2	28	(4,6)	9	2	1	10	11	8	11
11	(1,4)	1	5	4	3	6	1	11	29	(4,7)	9	4	3	3	10	9	6
12	(1,5)	3	5	2	11	1	5	4	30	(4,8)	11	10	4	10	10	6	8
13	(1,6)	6	11	9	1	1	10	11	31	(5,6)	5	8	5	5	1	8	7
14	(1,7)	3	9	9	3	9	7	3	32	(5,7)	3	12	3	5	3	11	9
15	(1,8)	10	9	4	7	5	10	4	33	(5,8)	11	8	10	4	12	11	4
16	(2,3)	8	1	8	8	2	4	5	34	(6,7)	9	3	9	5	7	3	10
17	(2,4)	1	10	12	8	11	9	1	35	(6,8)	4	8	7	10	4	10	10
18	(2,5)	7	1	5	4	3	5	1	36	(7,8)	12	9	11	9	10	11	10

• This design is obtained by permuting the factor levels in Table 2.

Theorem 3 indicates that there is a $(v-1)$-run uniform design nested in a $v(v-1)/2$-run uniform design. The following construction and theorem further reveal that several uniform designs with larger run sizes than $v-1$ are also nested in the $v(v-1)/2$-run uniform design. The proof can be found in the Appendix.

Construction 3.

Step 1. Based on a partitionable 3-$(v, 3, 1)$ design, a uniform design $U_{n_2}(q_2^m)$ with $n_2 = v(v-1)/2, q_2 = v(v-1)/6$, and $m = v-2$ can be obtained using Construction 1. Denote the resulting uniform design by $\mathbf{X}$.

Step 2. For each $i \in \{2, \ldots, (v+1)/2 - 1\}$, denote $\mathcal{R}_i = \{1, \ldots, iv - i(i+1)/2\}$, $\mathcal{R}_{id} = \{[(i-k)-1]v - (i-k)[(i-k)-1]/2 + 1, \ldots, [(i-k)-1]v - (i-k)[(i-k)-1]/2 + k : k \in \{1, \ldots, i-1\}\}$, and $\nabla_i = \mathcal{R}_i / \mathcal{R}_{id}$; let $\mathbf{X}_i$ be the subdesign of $\mathbf{X}$ with run numbers that are those in ∇_i.

Theorem 4. *For each $i \in \{2, \ldots, (v+1)/2 - 1\}$, the design $\mathbf{X}_i$ constructed using Construction 3 is a uniform design $U_{n_i}(q_i^m)$ under the DD and, therefore, the design $\mathbf{X}$ can be viewed as a nested uniform design $NU_{n_i,n}(q_i^m, q^m)$, where*

$$n_i = iv - i^2, \quad q_i = \frac{iv - i^2}{2}, \quad n = \frac{v(v-1)}{2}, \quad q = \frac{v(v-1)}{6}, \quad \textit{and} \quad m = v - 2.$$

Example 4. In addition to the eight-run subdesign identified in Example 3, there are three further subdesigns with larger run sizes nested in Table 2: (1) the first 15 runs, with the first run removed, comprise a U-type design $U(14; 7^7)$; (2) the first 21 runs, with the first, second, and ninth runs removed, comprise a U-type design $U(18; 9^7)$; and (3) the first 26 runs, with the first, second, third, ninth, 10th, and 16th runs removed, comprise a U-type design $U(20; 10^7)$. Setting $a = 2$ and $b = 1$, the DD values of the above three subdesigns are 280, 432, and 520, respectively, all of which achieve the lower bound given in Fact 1.

The following construction and theorem reveal that a nested uniform design with more than two layers can be constructed based on a partitionable t-$(v, k, 1)$ design with $t \geq 4$. The proof can be found in the Appendix.

Construction 4.

Step 1. Based on a partitionable t-$(v, k, 1)$ design with $t \geq 4$, a uniform design $U_{n_{t-1}}(q_{t-1}^m)$ with

$$n_{t-1} = \binom{v}{t-1}, \; m = \frac{v - t + 1}{k - t + 1}, \quad \text{and} \quad q_{t-1} = \frac{v(v-1)\cdots(v-t+2)}{k(k-1)\cdots(k-t+2)}$$

can be obtained using Construction 1. Denote the resulting uniform design by $\mathbf{X}_{t-1}$.

Step 2. For $i = 1, \ldots, t-2$, let $\mathbf{X}_i$ be the first $\binom{v-(t-1)+i}{i}$ rows of $\mathbf{X}_{t-1}$.

Theorem 5. *For $i = 1, \ldots, t-1$, the design $\mathbf{X}_i$ constructed using* Construction 4 *is a uniform design $U_{n_i}(q_i^m)$ under the DD and, therefore, the design $\mathbf{X}_{t-1}$ is a nested uniform design $NU_{n_1,\ldots,n_{t-1}}(q_1^m, \ldots, q_{t-1}^m)$, where*

$$n_i = \binom{v-(t-1)+i}{i}, q_i = \binom{v-(t-1)+i}{i} \Big/ \binom{k-(t-1)+i}{i}, \; \textit{and}$$
$$m = \frac{v-t+1}{k-t+1}.$$

Remark 3. From the proof of Theorem 5, it is not difficult to see that a two-layer nested uniform design can also be obtained using Construction 4 when $t = 3$. In general, finding a partitionable t-$(v, k, 1)$ design with $t \geq 4$ is not easy, as shown by the paucity of literature on design theory. As a result, nested uniform designs with two layers may be the most commonly used in practice.

5. Construction of Sliced Uniform Designs

Sliced space-filling designs are useful for computer experiments with both qualitative and quantitative variables, ensembles of multiple computer models, and cross-validation (e.g., see Qian and Wu (2009); Xu, Haaland and Qian (2011); Qian (2012); Yang et al. (2016a)). In this section, we focus on sliced uniform designs under the DD. We find that such a design can be generated based on a special partitionable 3-$(v, 3, 1)$ design in which each of the parallel classes is a partitionable 2-$(v, 3, 1)$ design. In contrast to the nested subdesigns in the previous section, the sliced subdesigns in this section are asymmetrical uniform designs.

For simplicity of exposition, the notation $SU_n(q^m; s, n_1, q_1 \times \cdots \times q_m)$ is used to denote a sliced uniform design; that is, it is a uniform design $U_n(q^m)$ that can be partitioned into s slices, each of which is a uniform design $U_{n_1}(q_1 \times \cdots \times q_m)$. Obviously, we have the relation $n = sn_1$. The following construction and theorem reveal that an $SU_n(q^m; s, n_1, q_1 \times n_1^{m-1})$ can be constructed based on a partitionable 3-$(v, 3, 1)$ design in which each parallel class is a partitionable 2-$(v, 3, 1)$ design. The proof can be found in the Appendix.

Construction 5.

Step 1. Based on a partitionable 3-$(v, 3, 1)$ design in which each parallel class is a partitionable 2-$(v, 3, 1)$ design, a uniform design $U_n(q^m)$ with $n = v(v-1)/2, q = v(v-1)/6$, and $m = v-2$ can be generated using Construction 1. Denote the resulting uniform design by $\mathbf{X}$.

Step 2. Suppose that $\mathcal{P}_1$ (a partitionable 2-$(v, 3, 1)$ design) is the first parallel class of the partitionable 3-$(v, 3, 1)$ design, and let $\mathcal{U}_1, \ldots, \mathcal{U}_h$ be the h parallel classes in $\mathcal{P}_1$, where $h = (v-1)/2$, and there are $u = v/3$ blocks in each $\mathcal{U}_i$.

Step 3. Permute the rows of $\mathbf{X}$ such that the row labels corresponding to the $(iv - v + 1)$th$, \ldots, iv$th runs are two-subsets contained in the blocks of $\mathcal{U}_i$, for $i = 1, \ldots, h$. Denote the subdesign corresponding to the $(iv - v + 1)$th$, \ldots, iv$th runs as $\mathbf{X}_i$.

Theorem 6. *For each $i \in \{1, \ldots, h\}$, the design $\mathbf{X}_i$ constructed using Construction 5 is a uniform design $U_v(q_1 \times v^{m-1})$ under the DD and, therefore, the design $\mathbf{X}$ is a sliced uniform design $SU_n(q^m; s, v, q_1 \times v^{m-1})$, where*

$$n = \frac{v(v-1)}{2}, \quad q = \frac{v(v-1)}{6}, \quad m = v-2, \quad s = \frac{v-1}{2}, \quad \textit{and} \quad q_1 = \frac{v}{3}.$$

Example 5. It is not difficult to check that each $\mathcal{P}_i$ in Table 1 is a partitionable 2-$(9,3,1)$ design with the first, second, and 12th blocks, the third, fourth, and fifth blocks, the sixth, seventh, and eighth blocks, and the ninth, 10th, and 11th blocks as its four parallel classes. Hence, the block design in Table 1 is a partitionable 3-$(9,3,1)$ design, with each parallel class being a partitionable 2-$(9,3,1)$ design. Let

$$\begin{aligned}\mathcal{U}_1 &= \{\{0,7,8\},\{1,2,4\},\{3,5,6\}\}\,,\\ \mathcal{U}_2 &= \{\{1,5,7\},\{2,6,8\},\{0,3,4\}\}\,,\\ \mathcal{U}_3 &= \{\{2,3,7\},\{4,5,8\},\{0,1,6\}\}\,, \text{ and}\\ \mathcal{U}_4 &= \{\{4,6,7\},\{1,3,8\},\{0,2,5\}\}\end{aligned}$$

be the four parallel classes of $\mathcal{P}_1$. Let $\mathbf{X}_1$ be the subdesign with row labels $(0,7),(0,8),(7,8),(1,2),(1,4),(2,4),(5,6),(3,5)$, and $(3,6)$. Let $\mathbf{X}_2$ be the subdesign with row labels $(1,5),(1,7),(5,7),(2,6),(2,8),(6,8),(0,3),(0,4)$, and $(3,4)$. Let $\mathbf{X}_3$ be the subdesign with row labels $(2,3),(2,7),(3,7),(4,5),(4,8),(5,8),(0,1),(0,6)$, and $(1,6)$. Let $\mathbf{X}_4$ be the subdesign with row labels $(4,6),(4,7),(6,7),(1,3),(1,8),(3,8),(0,2),(0,5)$, and $(2,5)$. Then, one can obtain the design in Table 4 by permuting the rows in Table 2, where in Table 4, each $\mathbf{X}_i$ corresponds to a sliced subdesign. It is easy to verify that each $\mathbf{X}_i$ is a U-type design $U(9; 3\times 9^6)$. In addition, by setting $a=2$ and $b=1$, we have that $D^2(\mathbf{X}_1;a,b) = D^2(\mathbf{X}_2;a,b) = D^2(\mathbf{X}_3;a,b) = D^2(\mathbf{X}_4;a,b) = 90$, which achieves the lower bound of the DD.

Remark 4. (i) The first column of each slice has replicated levels. Therefore, the first column of the design may be assigned to the factor that is believed to be the most important. (ii) Unlike the nested designs in the previous section, it is almost impossible to find level permutations for our sliced design such that the factor levels are consistent for each slice and achieve maximum univariate stratification. Therefore, the sliced designs constructed using Construction 5 may only be suitable for experiments with qualitative factors. (iii) For further details on the existence and construction of partitionable 3-$(v,3,1)$ designs in which each parallel class is a partitionable 2-$(v,3,1)$ design, refer to Lu (1983, 1984); Teirlinck (1991).

6. Conclusion

Combinatorial configurations have received much attention, with numerous construction methods now available. In this study, we identify a strong relationship between uniform designs under the DD and partitionable t-designs. This

Table 4. A sliced uniform design $SU_{36}(12^7; 4, 3 \times 9^6)$.

	Row label	I	II	III	IV	V	VI	VII		Row label	I	II	III	IV	V	VI	VII
	(0,7)	12	9	3	9	6	6	3		(2,3)	6	1	8	8	2	7	5
	(0,8)	12	4	7	7	10	4	10		(2,7)	6	3	12	9	3	9	6
	(7,8)	12	12	12	12	12	12	12		(3,7)	6	6	3	12	9	3	9
	(1,2)	1	8	8	2	7	5	6		(4,5)	7	5	6	1	8	8	2
$\mathbf{X}_1$	(1,4)	1	5	4	3	5	2	11	$\mathbf{X}_3$	(4,8)	7	10	4	10	12	4	7
	(2,4)	1	10	11	8	11	9	2		(5,8)	7	7	10	4	10	12	4
	(5,6)	2	7	5	6	1	8	8		(0,1)	8	8	2	7	5	6	1
	(3,5)	2	1	10	11	8	11	9		(0,6)	8	2	7	5	6	1	8
	(3,6)	2	11	1	5	4	3	5		(1,6)	8	11	9	2	1	10	11
	(1,5)	3	5	2	11	1	5	4		(4,6)	9	2	1	10	11	8	11
	(1,7)	3	12	9	3	9	6	6		(4,7)	9	6	6	3	12	9	3
	(5,7)	3	9	6	6	3	12	9		(6,7)	9	3	9	6	6	3	12
	(2,6)	4	3	5	2	11	1	5		(1,3)	10	11	8	11	9	2	1
$\mathbf{X}_2$	(2,8)	4	10	12	4	7	7	10	$\mathbf{X}_4$	(1,8)	10	12	4	7	7	10	4
	(6,8)	4	7	7	10	4	10	12		(3,8)	10	4	10	12	4	7	7
	(0,3)	5	4	3	5	2	11	1		(0,2)	11	8	11	9	2	1	10
	(0,4)	5	2	11	1	5	4	3		(0,5)	11	9	2	1	10	11	8
	(3,4)	5	6	1	8	8	2	7		(2,5)	11	1	5	4	3	5	2

can help experimenters directly obtain the corresponding uniform designs, nested uniform designs, and sliced uniform designs by using existing combinatorial configurations with specific properties. The combinatorial construction methods presented here are easy to implement, and have been proved to be effective. Future work may consider partitionable t-(v, k, λ) designs with $\lambda \geq 2$. It is possible that additional uniform designs and complex-structured uniform designs may be obtained based on such configurations.

Acknowledgments

The authors thank the reviewers for their valuable comments and suggestions, and Professor Gennian Ge of Capital Normal University of China for suggesting the topic and the helpful discussions. Huang's research was supported in part by the National Natural Science Foundation of China (Grant Nos. 11701109, 11801331 and 11861017) and Guangxi Nature Science Foundation (Grant Nos. 2018JJB110027 and 2018AD19235). Yu's research was supported in part by the National Natural Science Foundation of China (Grant No. 11801103) and

Guangxi Nature Science Foundation (Grant No. 2017GXNSFBA198030), Liu's research was supported in part by the National Natural Science Foundation of China (Grant Nos. 11771220 and 11431006), National Ten Thousand Talents Program, Tianjin Development Program for Innovation and Entrepreneurship, and Tianjin "131" Talents Program, and Wu' research was supported in part by the National Natural Science Foundation of China (Grant No. 11671103) and Foundation of Guangxi Key Lab of Multi-Source Information Mining and Security (Grant no. 18-A-03-01). The first three authors contributed equally to this work.

Appendix: Proofs of Theorems

A1. Proof of Theorem 1

Since $k < v$, then $q > 1$ and $q > m$, which implies that the value of γ in Fact 1 is 0. According to Fact 1, it suffices to prove that the Hamming distance between any two distinct rows take the same value m, or take only two values m and $m-1$. Consider two different rows labelled by $(l_1^{(1)}, l_2^{(1)}, \ldots, l_{t-1}^{(1)})$ and $(l_1^{(2)}, l_2^{(2)}, \ldots, l_{t-1}^{(2)})$. Regarding these two labels as two subsets of the point set V, then the union of them, denoted by S, contains at least t distinct points. From the definition of a t-$(v, k, 1)$ design, S as a whole can occur at most in one block. Thus the Hamming distance can only take value m or $m-1$, which also indicates that any of the possible level-combinations between any two columns appears at most once.

A2. Proof of Theorem 3

It it sufficient to prove that $\mathbf{X}_1$ is a U-type design $U(n_1; q_1^m)$. This is because $\mathbf{X}$ is a uniform design according to Theorem 1, which implies that the Hamming distance between any two distinct rows of $\mathbf{X}$ can only take value m or $m-1$. Obviously, the Hamming distance between any two distinct rows of $\mathbf{X}_1$ can only take value m or $m-1$ because $\mathbf{X}_1$ is a subdesign of $\mathbf{X}$. Therefore, in order to prove $\mathbf{X}_1$ is a uniform design we just need to verify that $\mathbf{X}_1$ is a U-type design.

Since $\mathbf{X}_1$ is the first $v-1$ rows of $\mathbf{X}$ and the row labels of $\mathbf{X}$, i.e., 2-subsets of V are ordered lexicographically, it is easy to see that the row labels of $\mathbf{X}_1$ are those 2-tuples leading by 0. On the other hand, each parallel class of a partitionable 3-$(v, 3, 1)$ design is a 2-$(v, 3, 1)$ design. Therefore, for each parallel class there are exactly $q_1 = (v-1)/2$ blocks containing 0. This indicates that $\mathbf{X}_1$ is a q_1-level design with each level appearing $(v-1)/q_1 = 2$ times in each column. In other words, $\mathbf{X}_1$ is a U-type design $U(n_1; q_1^m)$ with

$$n_1 = v - 1, \quad q_1 = \frac{v-1}{2} \text{ and } m = v - 2.$$

A3. Proof of Theorem 4

It suffices to prove that $\mathbf{X}_i$ is a U-type design $U(n_i; q_i^m)$ for each $i \in \{2, \ldots, (v+1)/2 - 1\}$. For each parallel class, let $\mathcal{B}_j$ be the set of blocks that contain the element j, $j = 1, \ldots, v$. As demonstrated in the proof of Theorem 3, each $\mathcal{B}_j$ contains $(v-1)/2$ blocks. Furthermore, the intersection $\mathcal{B}_j \cap \mathcal{B}_l (j \neq l)$ contains only one block; otherwise, the 2-subset $\{j, k\}$ would occur in more than one block, which contradicts the fact that each parallel class of a partitionable 3-$(v, 3, 1)$ design is a 2-$(v, 3, 1)$ design.

Now consider the first $2v - 3$ rows of $\mathbf{X}$ whose row labels are leading by 0 and 1. Since the intersection $\mathcal{B}_1 \cap \mathcal{B}_2$ contains only one block, each column of the first $2v - 3$ rows has a level appearing 3 times while each of the remaining levels appears 2 times. Consequently, if we delete the first row whose row label is $(0, 1)$, then the remaining rows (rows 2 to $2v - 3$) comprise a U-type design $U(2v-4; (2v-4)/2)$. Similarly, if we consider the first $iv - i(i+1)/2$ rows of $\mathbf{X}$ for $i > 2$, then the row labels are those leading by 0 to $i-1$ and there are as many as $C_i^2 = i(i-1)/2$ levels appearing 3 times in each column of the first $iv - i(i+1)/2$ rows while each of the remaining levels appears only 2 times. Hence, if we delete the rows labeled by $\{(j, k) : 0 \leq j < k \leq i - 1\}$ from the the first $iv - i(i+1)/2$ rows, the reduced design is a U-type design with each level occurring 2 times in each column. In other words, for each $i \in \{2, \ldots, (v+1)/2 - 1\}$, denote

$$\begin{aligned}
\mathcal{R}_i &= \left\{1, \ldots, iv - \frac{i(i+1)}{2}\right\}, \\
\mathcal{R}_{id} &= \left\{[(i-k) - 1]v - \frac{(i-k)[(i-k)-1]}{2} + 1, \right. \\
&\qquad \left. \ldots, [(i-k) - 1]v - \frac{(i-k)[(i-k)-1]}{2} + k : k \in \{1, \ldots, i-1\}\right\} \text{ and} \\
\nabla_i &= \mathcal{R}_i - \mathcal{R}_{id},
\end{aligned}$$

and let $\mathbf{X}_i$ be the subdesign of $\mathbf{X}$ whose row numbers are those in ∇_i, then $\mathbf{X}_i$ is a U-type design $U(n_i; q_i^m)$ with

$$n_i = iv - i^2 \text{ and } q_i = \frac{iv - i^2}{2}.$$

A4. Proof of Theorem 5

It suffices to prove that $\mathbf{X}_i$ is a U-type design $U(n_i; q_i^m)$ for each $i \in \{1, \ldots, t-2\}$. Note that the row labels of $\mathbf{X}_i$ are those leading by $0, 1, \ldots, t-2-i$. On the other hand, for each parallel class there are $\binom{v-(t-1)+i}{i} \Big/ \binom{k-(t-1)+i}{i}$ blocks containing the set $\{0, 1, \ldots, t-2-i\}$. As a result, $\mathbf{X}_i$ is a U-type design $U(n_i; q_i^m)$ with

$$n_i = \binom{v-(t-1)+i}{i} \quad \text{and} \quad q_i = \binom{v-(t-1)+i}{i} \Big/ \binom{k-(t-1)+i}{i}.$$

A5. Proof of Theorem 6

It suffices to prove that $\mathbf{X}_i$ is a U-type design $U(v; q_1 \times v^{m-1})$ for $i = 1, \ldots, (v-1)/2$. Consider the first column of $\mathbf{X}_i$, it is easy to see that there are $v/3$ levels each of which occurs 3 times in the first column because the row labels of $\mathbf{X}_i$ are those 2-subsets contained in the blocks of $\mathcal{U}_i$. Now consider the jth column of $\mathbf{X}_i$ for $j \in \{2, \ldots, m-1\}$. Note that $\mathcal{U}_i$ is a partitionable 2-$(v, 3, 1)$ design, which implies that the union of any two row labels of $\mathbf{X}_i$ is a 3-subset that is contained in $\mathcal{U}_i$ or it is a 4-subset of V. As a consequence, if there is level occurring more than once in the jth column, then there is a 3-subset occurring at least twice in the 3-$(v, 3, 1)$ design or there is a 4-subset occurring at least once in the 3-$(v, 3, 1)$ design. This contradicts the definition of a 3-$(v, 3, 1)$ design. Therefore, there are v distinct levels in the jth column of $\mathbf{X}_i$ for $j \in \{2, \ldots, m-1\}$. In other words, the uniform design $\mathbf{X}$ is a sliced uniform design $SU_n(q^m; s, v, q_1 \times v^{m-1})$ where

$$n = \frac{v(v-1)}{2}, \quad q = \frac{v(v-1)}{6}, \quad m = v-2, \quad s = \frac{v-1}{2}, \quad \text{and} \quad q_1 = \frac{v}{3}.$$

References

Baker R. D. (1976). Partitioning the planes of $AG_{2m}(2)$ into 2-designs. *Discrete Math.* **15**, 205–211.

Colbourn, C. J. and Dinita, J. H. (2007). *CRC Handbook of Combinatorial Designs.* CRC Press, New York.

Fang, K. T. (1980). The uniform design: Application of number-theoretic methods in experimental design. *Acta Math. Appl. Sin.* **3**, 363–372.

Fang, K. T., Ge, G. N., Liu, M. Q. and Qin, H. (2004). Construction of uniform designs via super-simple resolvable t-designs. *Util. Math.* **66**, 15–32.

Fang, K. T., Li, R. and Sudjianto, A. (2006). *Design and Modeling for Computer Experiments.* Chapman and Hall/CRC, New York.

Fang, K. T., Lin, D. K. J. and Liu, M. Q. (2003). Optimal mixed-level supersaturated design.

Metrika **58**, 279–291.

Fang, K. T., Liu, M. Q., Qin, H. and Zhou, Y. D. (2018). *Theory and Application of Uniform Experimental Designs.* Springer, Singapore.

Hanani, H. (1974). On resolvable balanced incomplete block designs. *J. Combin. Theory Ser. A* **17**, 275–289.

Hickernell, F. J. and Liu, M. Q. (2002). Uniform designs limit aliasing. *Biometrika* **89**, 893-904.

Lu, J. X. (1983). On large sets of disjoint Steiner triple systems I, II, and III. *J. Combin. Theory Ser. A* **34**, 140–182.

Lu, J. X. (1984). On large sets of disjoint Steiner triple systems IV, V, and VI. *J. Combin. Theory Ser. A* **37**, 136–192.

Qian, P. Z. G. (2009). Nested Latin hypercube designs. *Biometrika* **96**, 957–970.

Qian, P. Z. G. (2012). Sliced Latin hypercube designs. *J. Amer. Statist. Assoc.* **107**, 393–399.

Qian, P. Z. G., Ai, M. and Wu, C. F. J. (2009). Construction of nested space-filling designs. *Ann. Statist.* **37**, 3616–3643.

Qian, P. Z. G., Tang, B. and Wu, C. F. J. (2009). Nested space-filling designs for computer experiments with two levels of accuracy. *Statist. Sinica* **19**, 287–300.

Qian, P. Z. G. and Wu, C. F. J. (2009). Sliced space-filling designs. *Biometrika* **96**, 945–956.

Ray-Chauduri, D. K. and Wilson, R. M. (1971). Solution of Kirkman's schoolgirl problem. *Proc. Sympos. Pure Math.* **19**, 187–203.

Santner, T. J., Williams, B. J. and Notz, W. I. (2003). *The Design and Analysis of Computer Experiments.* Springer, New York.

Sun, F. S., Chen, J. and Liu, M. Q. (2011). Connections between uniformity and aberration in general multi-level factorials. *Metrika* **73**, 305–315.

Sun, F. S., Liu, M. Q. and Qian, P. Z. G. (2014). On the construction of nested space-filling designs. *Ann. Statist.* **42**, 1394–1425.

Sun, F. S., Yin, Y. H. and Liu, M. Q. (2013). Construction of nested space-filling designs using difference matrices. *J. Statist. Plann. Inference* **143**, 160–166.

Tan, M., Fang, H. B. and Ross, D. D. (2012). *Evaluating Synergy: Statistical Design and Analysis of Drug Combination Studies*. Wiley, New York.

Tang, Y. (2005). *Combinatorial Properties and Constructions of Uniform Designs (in Chinese).* Ph.D Thesis. Soochow University, Jiangsu, China.

Teirlinck, L. (1991). A completion of Lu's determination of the spectrum of large sets of disjoint Steiner Triple systems. *J. Combin. Theory Ser. A* **57**, 302–305.

Teirlinck, L. (1994). Some new 2-resolvable Steiner quadruple systems. *Des. Codes Cryptogr.* **4**, 5–10.

Xu, X., Haaland, B. and Qian, P. Z. G. (2011). Sudoku-based space-filling designs. *Biometrika* **98**, 711–720.

Yang, J. Y., Liu, M. Q. and Lin, D. K. J. (2014). Construction of nested orthogonal Latin hypercube designs. *Statist. Sinica* **24**, 211–219.

Yang, J. Y., Chen, H., Lin, D. K. J. and Liu, M. Q. (2016a). Construction of sliced maximin-orthogonal Latin hypercube designs. *Statist. Sinica* **26**, 589–603.

Yang, X., Yang, J. F., Lin, D. K. J. and Liu, M. Q. (2016b). A new class of nested orthogonal Latin hypercube designs. *Statist. Sinica* **26**, 1249–1267.

Zaicev G. V., Zinoviev, V. A. and Semakov N. V. (1973). Interrelation of Preparata and Hamming codes and extension of Hamming codes to new double error-correcting codes. *Pro-*

ceeding of the 2nd International Symposium on Information Theory, 257–263.

Hengzhen Huang
College of Mathematics and Statistics, Guangxi Normal University, Guilin 541004, China.
E-mail: hzhuang@mailbox.gxnu.edu.cn
Huangsheng Yu
College of Mathematics and Statistics, Guangxi Normal University, Guilin 541004, China.
E-mail: yuhuangsheng1983@163.com
Min-Qian Liu
School of Statistics and Data Science, LPMC & KLMDASR, Nankai University, Tianjin 300071, China.
E-mail: mqliu@nankai.edu.cn
Dianhua Wu
College of Mathematics and Statistics, Guangxi Normal University, Guilin 541004, China.
E-mail: dhwugx@163.com

(Received May 2019; accepted December 2019)