Level set and drift estimation from a reflected Brownian motion with drift

Supplementary Material

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1. Appendix A1

Lemma 2 Let $g \colon S \to \mathbb{R}$, where $S \subset \mathbb{R}^d$ is a compact set. Assume that $g \in C^2(S)$ and that λ is such that there exists $0 < \delta_1 < \lambda$ for which $\nabla g(x) \neq 0$ for all $x \in \overline{G_g(\lambda - \delta_1)} \setminus G_g(\lambda + \delta_1) := \mathcal{G}_g(\lambda, \delta_1)$. Then, for all $\varepsilon < \delta_1$,

$$d_H\big(G_g(\lambda - \varepsilon), G_g(\lambda + \varepsilon)\big) \leq \frac{3M}{m^2}\varepsilon, \tag{1}$$

where $M = \max_{\{x \in \mathcal{G}_g(\lambda,\delta_1)\}} \|\nabla g(x)\|$, and $m = \min_{\{x \in \mathcal{G}_g(\lambda,\delta_1)\}} \|\nabla g(x)\|$.

Proof. Let $x \in G_g(\lambda - \varepsilon)$, $y_t = x + t\nabla g(x)$ and $t = 3\varepsilon/m^2$. We have $\|y_t - x\| < 3\varepsilon M/m^2$. To prove (1) it is enough to verify that $y_t \in G_g(\lambda + \varepsilon)$.

From a Taylor expansion at x, we obtain that for some $\theta \in [x, y_t]$:

$$
\begin{aligned}
g(y_t) &= g(x) + \nabla g(x)^T (y_t - x) + \frac{1}{2}(y_t - x)^T H_\theta (y_t - x) \\
&> \lambda - \varepsilon + \frac{3\varepsilon}{m^2}\|\nabla g(x)\|^2 + \frac{9\varepsilon^2}{2m^4}\nabla g(x)^T H_\theta \nabla g(x),
\end{aligned}
$$

where H_θ is the Hessian matrix of g at θ. Since g is C^2, there exists a constant $C > 0$ such that $|\nabla g(x)^T H_\theta \nabla g(x)| \leq C\|\nabla g(x)\|^2$, from where it follows that for $\varepsilon < 2m^4/(9M^2C)$,

$$
g(y_t) > \lambda + 2\varepsilon - 9M^2C/2m^4\varepsilon^2 \geq \lambda + \varepsilon,
$$

and $y_t \in G_g(\lambda + \varepsilon)$, concluding the proof. $\square$

Lemma 3 Let $S \subset \mathbb{R}^d$ be a compact set and $g : S \to \mathbb{R}$ a C^2 function such that that there exists an $\varepsilon_0 > 0$ and a $c > 0$ such that $\|\nabla g(x)\| > m$ for all $x \in U$, where U is an open set containing $\overline{G_g(l_\tau - \varepsilon_0)} \setminus G_g(l_\tau + \varepsilon_0)$. Then $\{G_g(\lambda) : l_\tau - \varepsilon_0/2 \leq \lambda \leq l_\tau + \varepsilon_0/2\}$ is a P-uniformity class for all probability distributions P on S absolutely continuous w.r.t. Lebesgue measure.

Proof. It is enough to prove that there exists an $r > 0$ such that for all $l_\tau - \varepsilon_0 < \lambda < l_\tau + \varepsilon_0$, *reach*$(G_g(\lambda)) > r > 0$. By Theorem 2 and theorem 1 of Walther (1999), there exists an $r > 0$ such that for all $l_\tau - \varepsilon_0 < \lambda < l_\tau + \varepsilon_0$, $G_g(\lambda)$ satisfies the inner and outer r-rolling conditions. This together with lemma 2.3 in Pateiro-López and Rodríguez-Casal (2009) implies that *reach*$(G_g(\lambda)) > r > 0$ for all $l_\tau - \varepsilon_0/2 \leq \lambda \leq l_\tau + \varepsilon_0/2$. $\square$

The following Lemma can be derived from Lemma 2b) in Walther (1997), for the sake of completeness we keep the proof, which is a straightforward consequence of Lemma 2.

Lemma 4 Under the hypotheses of Lemma 2, for all $0 \leq \varepsilon < \varepsilon_0/2$ and all $l_\tau - \varepsilon < \lambda < l_\tau + \varepsilon$, $G_g(\lambda - \varepsilon) \setminus G_g(\lambda + \varepsilon) \subset B\big(\partial G_g(\lambda), 3\varepsilon M/m^2\big)$ where $M = \max_{\{x \in \overline{G_g(l_\tau - \varepsilon_0)} \setminus G_g(l_\tau + \varepsilon_0)\}} \|\nabla g(x)\|$ and
$m = \min_{\{x \in \overline{G_g(\lambda - \delta_1)} \setminus G_g(\lambda + \delta_1)\}} \|\nabla g(x)\|$.

Proof. By Lemma 2, for all $\varepsilon < \varepsilon_0/2$ and all $l_\tau - \varepsilon < \lambda < l_\tau + \varepsilon$,

$$d_H(G_g(\lambda + \varepsilon), G_g(\lambda - \varepsilon)) \leq 3\varepsilon M/m^2.$$

If we take $x \in G_g(\lambda - \varepsilon)$ with $g(x) \leq \lambda$ and $y \in G_g(\lambda + \varepsilon)$, then there exists a $t \in [x, y]$ (the segment joining x and y) such that $g(t) = \lambda$, and so $t \in \partial G_g(\lambda)$, which concludes the proof. $\square$

2. Appendix B

Proposition 1. *Let $D \subset \mathbb{R}^d$ be a bounded domain such that ∂D is C^2. Let $\{X_t\}_{t \geq 0}$ be the solution of*

$$X_t = X_0 + B_t + \int_0^t \mu(X_s)ds + \int_0^t \mathbf{n}(X_s)\xi(ds), \ \textit{where } X_t \in \overline{D}, \ \forall t \geq 0. \quad (2)$$

Then for all Borel set A such that $\mu_L(A \cap D) > 0$, we have that

$$\sup_{x \in D} \mathbb{E}_x(T_A) < \infty,$$

where $\mathbb{E}_x$ denotes the expectation w.r.t. $\mathbb{P}_x$, which implies Harris recurrence.

Proof. The proof is based on the ideas used to prove Proposition 1.4 (ii) in Burdzy, Chen and Marshall (2006) and the following result (whose proof can be found in Cattiaux (1992) 610–613):

$$\inf_{(x,y) \in \overline{D} \times \overline{D}} p(0, x, t, y) = c_t > 0,$$

where $p(0, x, t, y)$ is the density function introduced in Remark 1. Let A be a Borel set such that $\mu_L(A \cap D) > 0$. Then for all $t \geq 1$,

$$\mathbb{P}_x(T_A \leq t) \geq \mathbb{P}_x(T_A \leq 1) \geq \int_A p(0, x, 1, y)dy \geq c_1 \mu_L(A \cap C) = c' > 0.$$

By the Markov property, for every $x \in D$, $\mathbb{P}_x(T_A \geq k) \leq (1 - c')^k$, for all $k \geq 1$, which implies that

$$\sup_{x \in D} \mathbb{E}_x(T_A) \leq \sup_{x \in D} \sum_{k=0}^{\infty} \mathbb{P}_x(T_A \geq k) < \infty.$$

This proves $\sup_{x \in D} \mathbb{E}_x(T_A) < \infty$, $\square$

Proposition 2. *Let $D \subset \mathbb{R}^d$ be a bounded domain such that ∂D is C^2. Denote by π the invariant distribution of $\{X_t\}_{t \geq 0}$. If D is a non-trap domain for $\{X_t\}_{t \geq 0}$, then there exist positive constants α and β such that*

$$\sup_{x \in D} \left\| \mathbb{P}_x(X_t \in \cdot) - \pi(\cdot) \right\|_{TV} \leq \beta e^{-\alpha t}.$$

Proof. Let $x_0 \in D$ and $\eta > 0$ be such that $\mathcal{B}(x_0, 3\eta) \subset D$. Since $\sup_{x\in D} \mathbb{E}_x T_{\mathcal{B}(x_0,\eta)} < \infty$, by the Markov inequality there exists an n_1 such that $\inf_{x\in D} \mathbb{P}_x(T_{\mathcal{B}(x_0,\eta)} \le n_1) > 1/2$. Let $Z_t = x + B_t + \int_0^t \mu(X_s)ds$ be the d-dimensional Brownian motion with drift given by $\mu(x)$. Observe that, since $|\mu(x)| < L$, by Doob's maximal inequality, we have

$$\mathbb{P}_x\left(\sup_{s\in[0,t]} |Z_s| < \eta\right) \ge 1 - \frac{\sqrt{dt} + Lt}{\eta}.$$

Now take t_0 small enough so that $1 - (\sqrt{dt_0} + Lt_0)/\eta =: p_0 > 0$. By the strong Markov property,

$$\inf_{x\in D} \mathbb{P}_x\big(T_{\mathcal{B}(x_0,\eta)} \le n_1 \text{ and } X_t \in \mathcal{B}(x_0, 2\eta) \text{ for } t \in [T_{\mathcal{B}(x_0,\eta)}, T_{\mathcal{B}(x_0,\eta)}+t_0]\big) > \frac{1}{2}p_0.$$

Let $Y = \inf\{n \in \mathbb{N}\colon X_n \in \mathcal{B}(x_0, 2\eta)\}$, then $\inf_{x\in D} \mathbb{P}_x(Y \le n_1 + t_0) > p_0/2$. Applying the Markov property at times $k\lfloor(n_1 + t_0)\rfloor$,

$$\sup_{x\in D} \mathbb{P}_x(Y \ge k\lfloor(n_1 + t_0)\rfloor) \le (1 - p_0/2)^k,$$

from which it follows that

$$\sup_{x\in D} \mathbb{E}_x(Y) \le \sup_{x\in D} \sum_{k=0}^{\infty} k\lfloor(n_1 + t_0)\rfloor \mathbb{P}_x(Y \ge k\lfloor(n_1 + t_0)\rfloor) < \infty.$$

Applying theorem 16.0.2 of Meyn and Tweedie (1993a), we obtain, for every $n > 0$, that

$$\sup_{x\in D} \|\mathbb{P}_x(X_n \in \cdot) - \pi(\cdot)\|_{TV} \le c_3 e^{-c_4 n},$$

where c_3, c_4 are positive finite constants. Using the semigroup property of $\{X_t\}_{t\geq 0}$ and the fact that π is invariant,

$$\sup_{x\in D} \|\mathbb{P}_x(X_t \in \cdot) - \pi(\cdot)\|_{TV} = \\ \sup_{x\in D} \left| \int_D \mathbb{P}_y(X_{t-n} \in \cdot) d\mathbb{P}_x(X_n \in dy) - \int_D \mathbb{P}_y(X_{t-n} \in \cdot)\pi(y) \right| \leq \\ \sup_{x\in D} \|\mathbb{P}_x(X_n \in \cdot) - \pi(\cdot)\|_{TV},$$

for all t and n, with $t \geq n$. $\square$

3. Appendix C

Theorem 5 Assume that $T \to \infty$, $\Delta \to 0$, $h_n \to 0$, $\Delta n h_n^2 \to \infty$, and $\Delta n h_n^3 \to 0$. Then, for all $x \in int(S)$ $\hat{\mu}_{n,T}(x) \to \mu(x)$ in probability.

Proof. Let $\gamma_n \geq 2h_n$, $\gamma_n \to 0$, $\Delta \to 0$ and denote

$$I_n = \{i : X_{t_i} \in \mathcal{B}(x, h_n), \exists s_0 : t_i < s_0 \leq t_{i+1}, X_{s_0} \notin \mathcal{B}(x, \gamma_n)\}.$$

According to our model, the estimator can be written as

$$\hat{\mu}_n(x) = \frac{1}{\Delta N_x} \sum_{i=1}^{n} (B_{t_{i+1}} - B_{t_i}) \mathbb{I}_{\{X_{t_i} \in \mathcal{B}(x,h_n)\}} + \frac{1}{\Delta N_x} \sum_{i\in I_n} \int_{t_i}^{t_{i+1}} \mu(X_s) ds + \\ \frac{1}{\Delta N_x} \sum_{i\in I_n^C} \int_{t_i}^{t_{i+1}} \mu(X_s) ds + \frac{1}{\Delta N_x} \sum_{i\in I_n} \int_{t_i}^{t_{i+1}} \eta(X_s) dL_s =: A_{n,T} + B^1_{n,T} + B^2_{n,T} + C_{n,T}.$$

First will prove that $C_{n,T} \to 0$ in probability. Observe that, we can bound, using Theorem 4.2 in Saisho (1987)

$$\Big\| \int_{t_i}^{t_{i+1}} \eta(X_s) dL_s \Big\| \leq L_s[t_i, t_{i+1}] \leq C\sqrt{\Delta},$$

being C a positive constant, then $C_{n,T} \leq C\#I_n/(\sqrt{\Delta}N_x) \quad a.s.$ Let us fix $\epsilon > 0$, we will prove that

$$\mathbb{P}\Big(\frac{\#I_n}{\sqrt{\Delta}N_x} > \epsilon\Big) \to 0. \tag{3}$$

Let $A_{in} = \{\exists s_i : t_i \leq s_i \leq t_{i+1}, X_{s_i} \notin B(x, \gamma_n)\}$. Then,

$$\begin{aligned} &\mathbb{P}(A_{in} \cap \{X_{t_i} \in \mathcal{B}(x, h_n)\}) \leq \\ &\mathbb{P}\Big(\sup_{s \in [t_i, t_{i+1}]} \|X_s - X_{t_i}\| > \gamma_n - h_n | X_{t_i} \in \partial\mathcal{B}(x, h_n)\Big)\mathbb{P}(X_{t_i} \in \mathcal{B}(x, h_n)) \leq \\ &\qquad\qquad\qquad \frac{(\sqrt{2} + \nu)\sqrt{\Delta}}{h_n}\mathbb{P}(X_{t_i} \in \mathcal{B}(x, h_n)). \end{aligned} \tag{4}$$

Consider the random variable $\kappa = \lfloor \epsilon\sqrt{\Delta}N_x \rfloor$. Observe that if $\#I_n/(\sqrt{\Delta}N_x) > \epsilon$ then there exists $\{i_1, \ldots, i_\kappa\}$ where $1 \leq i_j < n-1$ for all $j = 1, \ldots, \kappa$, such that $\exists s_{i_j} : t_{i_j} < s_{i_j} \leq t_{i_j+1}$ and $X_{s_{i_j}} \notin \mathcal{B}(x, \gamma_n), X_{t_{i_j}} \in \mathcal{B}(x, h_n)$ for all $j = 1, \ldots, \kappa$. Let us denote $m_n = 2(n\epsilon\pi h_n^2 g(x)\sqrt{\Delta})$, observe that $m_n \to \infty$, and from (4) we get

$$\begin{aligned} \mathbb{P}\Big(\frac{\#I_n}{\sqrt{\Delta}N_x} > \epsilon\Big) &\leq \mathbb{P}\Big(\frac{\#I_n}{\sqrt{\Delta}N_x} > \epsilon, \mathbb{I}_{\{\kappa \leq m_n\}}\Big) + \mathbb{P}\Big(\kappa > m_n\Big) \\ &\leq \sum_{j=1}^{m_n} \frac{(\sqrt{2} + \nu)\sqrt{\Delta}}{h_n}\mathbb{P}\Big(X_{t_{i_j}} \in \mathcal{B}(x, h_n)\Big) + \mathbb{P}\Big(\kappa > m_n\Big). \end{aligned} \tag{5}$$

By the Ergodic theorem $\kappa/(\epsilon n\pi h_n^2 g(x)\sqrt{\Delta}) \to 1$ a.s., then with probability one, for n large enough, $\kappa \leq m_n$ from where it follows that $\mathbb{P}(\kappa > m_n) \to 0$. Lastly, again by ergodicity, we have that

$$\frac{1}{m_n \pi h_n^2} \sum_{j=1}^{m_n} \mathbb{P}\left(X_{t_{i_j}} \in \mathcal{B}(x, h_n)\right) \to g(x), \tag{6}$$

from $h_n^3 n\Delta \to 0$ we get (3) from (5) and (6).

The proof will be complete if under our asymptotic scheme, we have

$$A_{n,T} \to 0, \quad \text{ in probability,} \tag{7}$$

$$B_{n,T}^1 \to 0 \quad \text{ in probability,} \tag{8}$$

$$B_{n,T}^2 \to \mu(x) \quad \text{in probability.} \tag{9}$$

Since μ is Lipschitz and $\gamma_n \to 0$, (9) follows.

Regarding $B_{n,T}^1$ observe that $\int_{t_i}^{t_{i+1}} \mu(X_s)ds \leq \max_{x\in S} \|\mu(x)\|\Delta$ and then from (3) we get $B_{n,T}^1 \to 0$ in probability.

Let us consider now (7). Each random variable $\mathbb{I}_{\{X_{t_i} \in \mathcal{B}(x, h_{n,T})\}}$ is $\mathcal{F}_{t_i}$ measurable, due to the independence of $B_{t_{i+1}} - B_{t_i}$ w.r.t. $\mathcal{F}_{t_i}$. Then $\mathbb{E}(B_{t_{i+1}} - B_{t_i} | \mathcal{F}_{t_i}) = \mathbb{E}(B_{t_{i+1}} - B_{t_i}) = 0$, giving $\mathbb{E}(A_{n,T}) = 0$. (In fact this proves that the numerator in $A_{n,T}$ is a martingale.) We now turn to the computation of the variance. First, by the ergodic theorem, we obtain that

$$\frac{N_x}{n\pi h_n^2} \to g(x), \ a.s. \tag{10}$$

Defining

$$\hat{A}_{n,T} = \frac{1}{a_n(x)} \sum_{i=1}^{n-1} (B_{t_{i+1}} - B_{t_i}) \mathbb{I}_{\{X_{t_i} \in \mathcal{B}(x,h_n)\}},$$

with $a_n(x) = \Delta n \pi h_n^2 g(x)$, by (10) we know that $A_{n,T}$ and $\hat{A}_{n,T}$ have the same limit in probability. Furthermore

$$\begin{aligned}
\mathbb{E}((\hat{A}_{n,T})^2) &= \frac{1}{a_n(x)^2} \mathbb{E}\left(\sum_{i=1}^{n-1} \mathbb{I}_{\{X_{t_i} \in \mathcal{B}(x,h_n)\}} (B_{t_{i+1}} - B_{t_i})\right)^2 \\
&= \frac{1}{a_n(x)^2} \sum_{i=1}^{n-1} \mathbb{E}\left(\mathbb{I}_{\{X_{t_i} \in \mathcal{B}(x,h_n)\}} (B_{t_{i+1}} - B_{t_i})^2\right)
\end{aligned}$$

since the cross–terms are zero.

We then conclude that

$$\begin{aligned}
\mathbb{E}((\hat{A}_{n,T})^2) &= \frac{1}{(\Delta n \pi h_n^2 g(x))^2} \sum_{i=1}^{n-1} P\left(\mathbb{I}_{\{X_{t_i} \in \mathcal{B}(x,h_n)\}}\right) \Delta \\
&\leq \frac{1}{\Delta n \pi h_n^2 g(x)^2} \frac{1}{n \pi h_n^2} \sum_{i=1}^{n-1} \mathbb{P}(X_{t_i} \in \mathcal{B}(x, h_n)).
\end{aligned}$$

By ergodicity, we have

$$\frac{1}{n \pi h_n^2} \sum_{i=1}^{n-1} \mathbb{P}\left(X_{t_i} \in \mathcal{B}(x, h_n)\right) \to g(x),$$

then, taking into account (10), we obtain

$$\mathbb{E}((A_{n,T})^2) \lessapprox \frac{1}{\Delta n \pi h_n^2 g(x)} \to 0.$$

□

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