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SPACE-TIME ESTIMATION AND PREDICTION UNDER FIXED-DOMAIN ASYMPTOTICS WITH COMPACTLY SUPPORTED COVARIANCE FUNCTIONS

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Abstract: We study the estimation and prediction of Gaussian processes with space-time covariance models belonging to the dynamical generalized Wendland ($\mathcal{DGW}$) family, under fixed-domain asymptotics. Such a class is nonseparable, has dynamical compact supports, and parameterizes differentiability at the origin similarly to the space-time Matérn class.

Our results are presented in two parts. First, we establish the strong consistency and asymptotic normality for the maximum likelihood estimator of the microergodic parameter associated with the $\mathcal{DGW}$ covariance model, under fixed-domain asymptotics. The second part focuses on optimal kriging prediction under the $\mathcal{DGW}$ model and an asymptotically correct estimation of the mean squared error using a misspecified model. Our theoretical results are, in turn, based on the equivalence of Gaussian measures under some given families of space-time covariance functions, where both space or time are compact. The technical results are provided in the online Supplementary material.

Key words and phrases: Fixed-domain asymptotics, microergodic parameter, maximum likelihood, space-time generalized wendland family

1. Introduction

1.1. Context

This study is concerned with fixed-domain asymptotics for the estimation and kriging prediction of Gaussian random fields defined over product spaces $D \times \mathcal{T}$, where D is a subset of $\mathbb{R}^d$ (d is a positive integer) and $\mathcal{T}$ is a compact interval of the real line. The most notable application refers to D as the spatial domain and to $\mathcal{T}$ as time. Although we focus on the space-time case, our results can be analogously applied to the anisotropic spatial case, where the rate of decay in the correlation in one coordinate is different from that of the remaining d coordinates.

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Here, we assume that the process is observed at n (possibly unevenly spaced) locations and repeatedly over m time points.

There might be other choices for the space-time asymptotics: for instance, for the temporal part, one might consider an increasing asymptotic framework, while keeping a fixed-domain approach for the spatial part. We are not aware of any contribution of this type, and such a setting looks challenging. Alternatively, one might consider both space and time under an increasing domain fashion. In this case, the results of Mardia and Marshall (1984) on maximum likelihood (ML) estimation apply, and the space-time asymptotics becomes a straightforward extension of the results obtained in the spatial case.

Instead, there is a lack of general results for the case of fixed-domain asymptotics. Some results have been given for specific classes of covariance functions. For instance, Zhang (2004), Wang and Loh (2011), and Kaufman and Shaby (2013) studied the asymptotic properties of the ML estimation of the microergodic parameter of the Matérn covariance model. Additionally, Stein (1999) and Kaufman and Shaby (2013) have studied the asymptotic effect of the misspecified kriging prediction on the prediction variance, under the Matérn covariance model. Recently, Bevilacqua et al. (2019) considered a fixed-domain asymptotic framework for Gaussian random fields defined over a compact set of $\mathbb{R}^d$ under the generalized Wendland ($\mathcal{GW}$) class of compactly supported correlation functions (Zastavnyi and Trigub (2002). Bevilacqua and Faouzi (2019) explored a similar problem using the generalized Cauchy class, which allows for decoupling of fractal dimensions with the Hurst effect.

The literature on space-time fixed-domain asymptotics is sparse, with the notable exception of Ip and Li (2017), who perform an asymptotic analysis based on a class of space-time covariance functions, proposed by Fuentes, Chen and Davis (2008), having both spatial and temporal margins belonging to the Matérn family (Stein (1999)). We refer to this family as the dynamical Matérn ($\mathcal{DM}$) family of space-time covariance functions. The recent paper by Porcu, Furrer and Nychka (2020) gives a thorough review of space-time covariance functions.

1.2. Our contribution

This study considers a class of nonseparable space-time covariance functions proposed by Porcu, Bevilacqua and Genton (2020). The members of this class are dynamically compactly supported, meaning that for any fixed temporal lag, the spatial margin is dynamically compactly supported; that is, there is a decreasing and continuous function, h, such that for every fixed temporal lag, t_o, the spatial margin of the space-time covariance function is compactly supported over a ball with radius $h(t_o)$ embedded in $\mathbb{R}^d$. Specifically, the spatial margin belongs to the $\mathcal{GW}$ class. For the remainder of the paper, we refer to this class as the dynamical $\mathcal{GW}$ ($\mathcal{DGW}$) class.

We study the problem of ML estimation of the $\mathcal{DGW}$ class defined over the product space $\mathcal{D} \times \mathcal{T}$, under fixed-domain asymptotics. Further, we study the problem of kriging prediction under the same asymptotic framework. The results on fixed-domain asymptotics largely rely on the equivalence of Gaussian measures (Skorokhod and Yadrenko (1973)). Thus, we derive conditions for the equivalence of such measures under either two $\mathcal{DGW}$ families with different parameters, or under a $\mathcal{DGW}$ and a $\mathcal{DM}$ family. These conditions are provided in Section B of the online Supplementary Material (`OS`). We explore the implications of these results in terms of the consistency and the asymptotic distribution of the ML estimator for the microergodic parameter. Finally, we assess the consequences of previous results in terms of the efficiency of the misspecified best linear unbiased predictors.

The remainder of the paper proceeeds as follows. Section 2 contains the necessary mathematical notation and a description of the covariance functions used in this paper. Background material on the equivalence of Gaussian measures is deferred to A in the `OS`. Section 3 provides preliminary results related to the space-time Fourier transforms of both the $\mathcal{DM}$ and the $\mathcal{DGW}$ models. We also find conditions for the equivalence of Gaussian measures under both models (see B in the `OS`). These results are the basis for Section 4, which studies the problem of consistency and asymptotic normality for the ML estimators of the parameters indexing the $\mathcal{DGW}$ family. The problem of misspecified kriging predictions under the $\mathcal{DGW}$ is then explored in Section 5. Section 6 concluldes the paper. Technical proofs are deferred to Section C in the `OS`.

2. Background Material

2.1. Preliminaries and notation

We denote by $Z = \{Z(\boldsymbol{s}, t), (\boldsymbol{s}, t) \in D \times \mathcal{T}\}$ a zero mean Gaussian random field with index set on $D \times \mathcal{T}$, with stationary covariance function $C : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ that is spatially isotropic and temporally symmetric. That is, there exists a continuous function $K : [0, \infty)^2 \to \mathbb{R}$ such that $K(0, 0) = 1$ and $C(\mathbf{h}, u) = \sigma^2 K(\|\mathbf{h}\|, |u|)$, for $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R}$, where σ^2 denotes the variance parameter. Here, $\| \cdot \|$ denotes the Euclidean norm. We denote by $\Phi_{d,T}$ the set of such functions. For the remainder of the paper, we use r for $\|\mathbf{h}\|$ and t for $|u|$. Additionally, we denote with Φ_d the family of spatially isotropic covariance functions defined on $\mathbb{R}^d$. The classes Φ_d and $\Phi_{d,T}$ are nested, with the inclusion relations

$$\Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_\infty \quad \text{and} \quad \Phi_{1,T} \supset \Phi_{2,T} \supset \cdots \supset \Phi_{\infty,T}$$

being strict, where $\Phi_\infty := \bigcap_{d\geq 1} \Phi_d$ and $\Phi_{\infty,T} := \bigcap_{d\geq 1} \Phi_{d,T}$. There is a rich mathematical theory for both classes Φ_d and $\Phi_{d,T}$. For a recent account on the class Φ_d, refer to Daley and Porcu (2014). Porcu, Gregori and Mateu (2006) provide extensive material for the class $\Phi_{d,T}$.

In particular, the results in Porcu, Gregori and Mateu (2006) (see also Gneiting and Guttorp (2010)) show that a continuous function ϕ with $\phi(0,0) = 1$ belongs to the class $\Phi_{d,T}$ if and only if there exists a probability measure F, defined on the positive quadrant of $\mathbb{R}^2$ such that

$$K(r,t) = \int_0^\infty \int_0^\infty \Omega_d(r\xi_1)\cos(t\xi_2)F(\mathrm{d}(\xi_1,\xi_2)), \quad t \geq 0, r \geq 0,$$

where $\Omega_d(t) = t^{-(d-2)/2}J_{(d-2)/2}(t)$ and J_ν is the Bessel function of the first kind of order $\nu > 0$ (Abramowitz and Stegun (1970)). Classical Fourier inversion arguments show that if K is absolutely integrable, then $K \in \Phi_{d,T}$ if and only if the function $f : [0,\infty)^2 \to \mathbb{R}$, defined by

$$f(z,\tau) = \frac{1}{(2\pi)^{(d+1)/2}} \int_0^\infty \int_0^\infty \Omega_d(z\xi_1)\cos(\tau\xi_2)\phi(\xi_1,\xi_2)\xi_1^{d-1}\mathrm{d}\xi_1\mathrm{d}\xi_2 \qquad (2.1)$$

is nonnegative and integrable. The function f is called the *isotropic spectral density* throughout.

2.2. The Matérn and generalized wendland classes of covariance functions

The Matérn class (Matérn (1986); Handcock and Stein (1993)) of continuous functions $K_{\mathcal{M}}(r;\alpha,\nu)$, $r \geq 0$, $\alpha,\nu > 0$, is defined as

$$K_{\mathcal{M}}(r;\alpha,\nu) = \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{r}{\alpha}\right)^\nu \mathcal{K}_\nu\left(\frac{r}{\alpha}\right), \qquad (2.2)$$

where $\mathcal{K}_\nu$ is a modified Bessel function of the second kind of order ν (Abramowitz and Stegun (1970)). The function $K_{\mathcal{M}}(\cdot;\alpha,\nu)$ belongs to the class Φ_∞.

We now introduce the $\mathcal{GW}$ class $K_{\mathcal{GW}}(\cdot;\beta,\mu,\kappa) : [0,\infty) \to \mathbb{R}$, defined as (Gneiting (2002); Zastavnyi (2002))

$$K_{\mathcal{GW}}(r;\beta,\mu,\kappa) := \begin{cases} \dfrac{1}{B(2\kappa,\mu+1)} \int_{r/\beta}^{1} u\left(u^2 - \left(\dfrac{r}{\beta}\right)^2\right)^{\kappa-1} (1-u)^{\mu}\,\mathrm{d}u, & 0 \le r < \beta, \\ 0, & r \ge \beta, \end{cases} \tag{2.3}$$

where $\kappa > 0$, $\beta > 0$ is the compact support parameter, and B denotes the beta function. For $\kappa = 0$, the $\mathcal{GW}$ class is defined as (Askey (1973)):

$$K_{\mathcal{GW}}(r;\beta,\mu,0) := \begin{cases} \left(1 - \dfrac{r}{\beta}\right)^{\mu}, & 0 \le r < \beta, \\ 0, & r \ge \beta. \end{cases} \tag{2.4}$$

Closed-form solutions of the integral in (2.3) can be obtained when $\kappa = k$, a nonnegative integer. In this case, (2.3) can be factorized as

$$K_{\mathcal{GW}}(r;\beta,\mu,k) = K_{\mathcal{GW}}(r;\beta,\mu+k,0)P_k(r), \qquad r \ge 0,$$

where P_k is a polynomial of order k.

The $\mathcal{GW}$ class belongs to the class Φ_d, for a fixed $d \in \mathbb{N}$, provided $\mu \ge (d+1)/2 + \kappa$.

Both $K_{\mathcal{M}}$ and $K_{\mathcal{GW}}$ are flexible models, because they allow us to parameterize the mean square and sample path differentiability of a Gaussian random field in a continuous fashion with these covariance functions. In particular, for the $\mathcal{M}$ case, given a positive integer k, the sample paths are k-times differentiable, in any direction, if and only if $\nu > k$. Similarly, for the $\mathcal{GW}$ case, the sample paths are k-times differentiable, in any direction, if and only if $\kappa > k - 0.5$. Figure 1 depicts $K_{\mathcal{GW}}(t;10,6,k)$ for $k = 0,1,2,3$ and $K_{\mathcal{M}}(t;1,\nu)$ for $\nu = 0.5, 1.5, 2.5, 3.5$.

2.3. The $\mathcal{DM}$ and $\mathcal{DGW}$ families of space-Ttme covariance functions

The $\mathcal{DM}$ family of space-time covariance functions was introduced by Fuentes, Chen and Davis (2008): the motivation for the proposal was to provide a class of space-time covariance functions that have spatial or temporal margins of the Matérn type. That is, either the spatial margin $C(\cdot,0)$ or the temporal margin $C(\mathbf{0},\cdot)$ belong to the class $K_{\mathcal{M}}(\|\cdot\|;\alpha,\nu)$, as defined in (2.2). The $\mathcal{DM}$ family is the building block for the work of Ip and Li (2017), which has largely inspired our work. To introduce the $\mathcal{DM}$ family, we follow a different path tp that of Fuentes, Chen and Davis (2008): let $\boldsymbol{\theta} = (\nu,\zeta,\upsilon,\epsilon,\sigma)^{\top}$, with $\top$ denoting the transpose of a vector. We assume ν, υ, ζ, σ are positive, and $\epsilon \in [0,1]$. We define

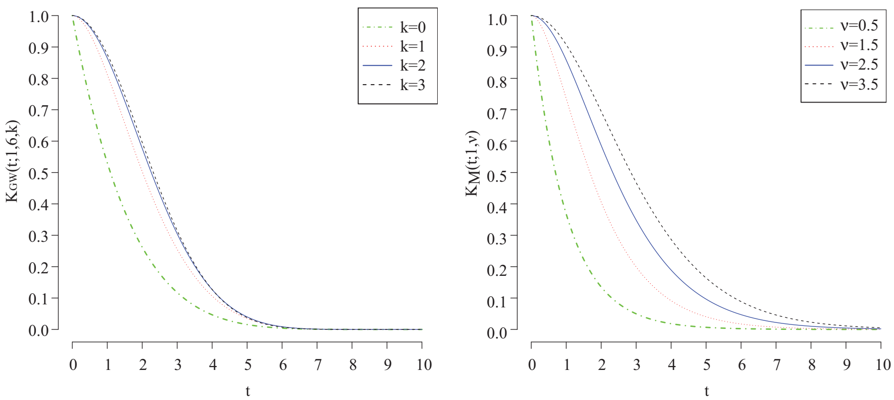

Figure 1. Left: $K_{\mathcal{GW}}(t;10,6,k)$ for $k = 0,1,2,3$, Right: $K_{\mathcal{M}}(t;1,\nu)$ for $\nu = 0.5, 1.5, 2.5, 3.5$

the parameter ℓ, which depends on $\boldsymbol{\theta}$, as

$$\ell(\boldsymbol{\theta}) = \frac{\sigma^2\zeta^{2\nu-d}\upsilon^{2\nu-1}\Gamma(\nu)}{\Gamma(\nu-(d+1)/2)}\mathbf{1}_{\{\epsilon=0\}} + \frac{\sigma^2\zeta^{2\nu-d}\upsilon^{2\nu-1}\Gamma(\nu)^2}{\Gamma(\nu-d/2)\Gamma(\nu-1/2)}\mathbf{1}_{\{\epsilon=1\}} + x\mathbf{1}_{\{\epsilon\in(0,1)\}}, \tag{2.5}$$

with $\mathbf{1}_A$ being the indicator function of any Borel set of the real line. Here, Γ denotes the gamma function (Gradshteyn and Ryzhik (2007)), and x is a positive constant that is kept fixed, and is disregarded for the rest of our exposition.

We define the $\mathcal{DM}$ class of covariance functions, $K_{\mathcal{DM}}(\cdot,\cdot;\boldsymbol{\theta}) : [0,\infty^2) \to \mathbb{R}$, through the identity

$$K_{\mathcal{DM}}(r,t;\boldsymbol{\theta}) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ut} g_{\boldsymbol{\theta}}(r,u)\mathrm{d}u, \quad (r,t) \in [0,\infty)^2, \tag{2.6}$$

where i is the imaginary unit. Here, the function $g_{\boldsymbol{\theta}}$ is defined as

$$g_{\boldsymbol{\theta}}(r,u) = \frac{\ell(\boldsymbol{\theta})\pi^{d/2}}{2^{\nu-d/2-1}\Gamma(\nu)}\left(\frac{r}{a(u)}\right)^{\nu-d/2}\left(\upsilon^2+\epsilon u^2\right)^{-\nu}\mathcal{K}_{\nu-d/2}\left(a(u)r\right),$$

with $a(u) = \sqrt{\zeta^2(\upsilon^2+u^2)/\upsilon^2+\epsilon u^2}$, $u \in \mathbb{R}$. In Equation (2.6), the parameter ζ^{-1} (spatial range) explains the rate of decay of the spatial correlation, υ^{-1} (temporal range) explains the rate of decay of the temporal correlation, and ℓ is a scale parameter proportional to the variance σ^2 (sill parameter) of the associated random field. The parameter ϵ allows us to switch from separability (when $\epsilon = 0$) to different levels of nonseparability. The arguments in Fuentes, Chen and Davis (2008) show that $K_{\mathcal{DM}}(\cdot,\cdot;\boldsymbol{\theta})$ is a member of the class $\Phi_{\infty,T}$.

In addition, Fuentes, Chen and Davis (2008) show that some special cases admit partial Fourier transforms that admit closed forms of the Matérn type.

We now follow Porcu, Bevilacqua and Genton (2020) to introduce the $\mathcal{DGW}$ class of space-time covariance functions. Let $\mu, \beta, \sigma > 0$, $\delta \in (0,2]$, $\gamma > 0$, and $\kappa \geq 0$. Let $\boldsymbol{\chi} = (\mu, \kappa, \beta, \sigma^2, \delta, \lambda, \gamma)^\top$.

The range of the parameter λ is specified below. Let us consider the function

$$h_{\delta,\gamma}(t) = \left(1 + \left(\frac{t}{\gamma}\right)^{\delta}\right)^{-1}, \quad t \geq 0. \tag{2.7}$$

We define the $\mathcal{DGW}$ class of covariance functions, denoted $K_{\mathcal{DGW}}(\cdot,\cdot;\boldsymbol{\chi})$ (Porcu, Bevilacqua and Genton (2020)), as follows:

$$K_{\mathcal{DGW}}(r,t;\boldsymbol{\chi}) = \sigma^2 \left[h_{\delta,\gamma}(t)\right]^{\lambda} K_{\mathcal{GW}}(r; \beta h_{\delta,\gamma}(t), \mu, \kappa), \qquad r,t \geq 0. \tag{2.8}$$

According to Theorem 1 in Porcu, Bevilacqua and Genton (2020) (see also Table 1 therein), $K_{\mathcal{DGW}}(\cdot,\cdot;\boldsymbol{\chi})$ belongs to the class $\Phi_{d,T}$, for some integer d, provided

$$\mu \geq \frac{d+3}{2} + \kappa + \alpha, \quad \text{and} \quad \lambda > \max\left(\frac{d+3}{2}, 2\kappa + 3\right). \tag{2.9}$$

The constant α is positive and bigger than a lower bound $\kappa_1(\delta)$ that is specified in Table 1 of Porcu, Bevilacqua and Genton (2020). Here, α is fixed and does not enter into the parameter $\boldsymbol{\chi}$. For the remainder of the paper, we suppose that α is always bigger than the lower bound $\kappa_1(\delta)$. When interpreting the parameters, we note that β is the spatial compact support when $t = 0$; that is, $K_{\mathcal{DGW}}(\cdot, 0; \boldsymbol{\chi}) = K_{\mathcal{GW}}(\cdot; \beta, \mu, \kappa)$, with $K_{\mathcal{GW}}$ as in (2.3), is compactly supported over a ball with radius β embedded in $\mathbb{R}^d$. The parameter κ determines the differentiability at the origin for the spatial margin $K_{\mathcal{DGW}}(\cdot, 0; \boldsymbol{\chi})$. The parameter $\gamma > 0$ is the temporal scale, and the parameter δ indexes the fractal dimension for the temporal sample paths. Finally, the function $h_{\delta,\gamma}$ is the temporal radius, because for every $t_o > 0$, the margin $K_{\mathcal{DGW}}(\cdot, t_o; \boldsymbol{\chi})$ is compactly supported over a ball with radius $\beta h_{\delta,\gamma}(t_o)$ embedded in $\mathbb{R}^d$. For the remainder of the paper, we use $f_{\mathcal{DM}}(\cdot,\cdot;\boldsymbol{\theta})$ and $f_{\mathcal{DGW}}(\cdot,\cdot;\boldsymbol{\chi})$ for the Fourier transforms of $K_{\mathcal{DM}}(\cdot,\cdot;\boldsymbol{\theta})$ and $K_{\mathcal{DGW}}(\cdot,\cdot;\boldsymbol{\chi})$, respectively, that are uniquely determined according to Equation (2.1).

3. Preliminary Results

3.1. Fourier transforms and tails for the $\mathcal{DGW}$ and $\mathcal{DM}$ classes

For d a positive integer and $\kappa \geq 0$, we define $\eta := (d+1)/2 + \kappa$. The next result describes the behavior of the isotropic spectral density associated with the $\mathcal{DGW}$ covariance function, $f_{\mathcal{DGW}}(\cdot,\cdot;\boldsymbol{\chi})$, defined in (2.8), and determined according to (2.1). Some further notation is needed. For given functions $g_1(x)$ and $g_2(x)$, we write $g_1(x) \asymp g_2(x)$ to mean that there exist constants c and C such that $0 < c < C < \infty$ and $c|g_2(x)| \leq |g_1(x)| \leq C|g_2(x)|$, for all x.

Note that $f \sim g$ means that the function f is asymptotically equal to the function g. We consider the function ${}_1F_2$, defined as

$$ {}_1F_2(a;b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k (c)_k k!}, \qquad z \in \mathbb{R}, $$

which is a special case of the generalized hypergeometric functions ${}_qF_p$ (Abramowitz and Stegun (1970)), with $(q)_k = \Gamma(q+k)/\Gamma(q)$, for $k \in \mathbb{N} \cup \{0\}$, being the Pochhammer symbol. Finally, for a complex number z, we use $\Im(z)$ to denote its imaginary part. We are ready to provide our first result.

Theorem 1. *Let $\mathcal{DGW}$ be the class of functions $K_{\mathcal{DGW}}(\cdot,\cdot;\boldsymbol{\chi})$ defined in Equation* (2.8)*, and let $f_{\mathcal{DGW}}(\cdot,\cdot;\boldsymbol{\chi})$ be the spectral density associated with $K_{\mathcal{DGW}}(\cdot,\cdot;\boldsymbol{\chi})$ and determined according to Equation* (2.1)*. Let $\boldsymbol{\varsigma} := (\mu,\kappa,\eta,d)^\top$. Let*

$$ \varrho_{\lambda,\eta} = \frac{2^\delta (d+\lambda-2\eta)\Gamma((\delta+1)/2)\Gamma((\delta+2)/2)\sin(\pi\delta/2)}{\gamma^\delta \pi^{3/2}}, $$

$$ \varrho_{\lambda,\eta+1} = \frac{2^\delta (d+\lambda-2\eta-2)\Gamma((\delta+1)/2)\Gamma((\delta+2)2)\sin(\pi\delta/2)}{\gamma^\delta \pi^{3/2}}, $$

$$ c_3^{\varsigma} = \frac{\Gamma(\mu+2\eta)}{\Gamma(\mu)}, \quad c_4^{\varsigma} = \frac{\Gamma(\mu+2\eta)}{\Gamma(\eta)2^{\eta-1}}, \quad c_5^{\varsigma} = \frac{\pi}{2}(\mu+\eta) $$

and

$$ L^{\varsigma} = \frac{2^{-\kappa-d+1}\Gamma(\kappa)\pi^{-d/2}\Gamma(\mu+1)\Gamma(2\kappa+d)}{B(2\kappa,\eta)\Gamma(\kappa+(d/2))\Gamma(\mu+2\eta)}. $$

Then, for $\kappa \geq 0$, $\sigma^2, \beta > 0$, $\delta \in (0,2)$, $\lambda > 2\kappa+3$, and $\mu \geq \eta+1+\alpha$, we have

1. $f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}) = -\sigma^2\beta^d\gamma^{3/2}\tau^{1/2}\sqrt{2}\pi^{-3/2}L^{\varsigma} \times \Im(\int_0^\infty \mathcal{K}_{1/2}(\gamma t\tau)_1F_2(\eta;\eta+\mu/2, \eta+\mu/2+1/2; -(z\beta(1+e^{i\pi\delta/2}t^\delta)^{-1})^2/4)/(1+e^{i\pi\delta/2}t^\delta)^{d+\lambda}t^{1/2}\mathrm{d}t)$
2. *For $\tau, z \to \infty$,*

$$ f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}) = \sigma^2\beta^{-(1+2\kappa)}L^{\varsigma}c_3^{\varsigma}z^{-2\eta} \times $$

$$\left(\left[\varrho_{\lambda,\eta}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right] + \left[\varrho_{\lambda,\eta+1}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right]\mathcal{O}(z^{-2})\right)$$
$$+ \left[\varrho_{\lambda,0}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right]\mathcal{O}(z^{-(\mu+\eta)}); \tag{3.1}$$

3. *For* $z \to \infty$, $\tau \to \infty$, $f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}) \asymp z^{-2\eta}\tau^{-1-\delta}$.

The proof of this result is deferred to Section C in the OS.

To describe the asymptotic behavior of the spectral density associated with the $\mathcal{DM}$ class, a result from Ip and Li (2017) is needed.

Theorem 2. *Let* $f_{\mathcal{DM}}(\cdot,\cdot;\boldsymbol{\theta})$ *be the spectral density function associated with the* $\mathcal{DM}$ *class in Equation* (2.6), *and being uniquely determined according to* (2.1). *Then, for* $\upsilon > 0$ *and* $\epsilon \in [0,1]$, *we have*

1. $f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta}) = \ell(\boldsymbol{\theta})(\zeta^2\upsilon^2 + \upsilon^2 z^2 + \zeta^2\tau^2 + \epsilon^2 z^2\tau^2)^{-\nu}$;

2. *As* $z,\tau \to \infty$ *and* $\epsilon \in (0,1]$,
$$\frac{1}{f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta})} \sim \ell^{-1}(\boldsymbol{\theta})(\epsilon z\tau)^{2\nu}\left(1 + \frac{\nu\zeta^2\upsilon^2}{\epsilon^2 z^2\tau^2} + \frac{\nu\upsilon^2}{\epsilon^2\tau^2} + \frac{\nu\zeta^2}{\epsilon^2 z^2} + \mathcal{O}(\tau^{-4}z^{-4})\right);$$

3. *As* $z,\tau \to \infty$ *and* $\epsilon = 0$,
$$\frac{1}{f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta})} \sim \ell^{-1}(\boldsymbol{\theta})(\upsilon^2 z^2+\zeta^2\tau^2)^{\nu}\left(1+\nu\frac{\zeta^2\upsilon^2}{\zeta^2\tau^2+\upsilon^2 z^2}+\mathcal{O}\left((\zeta^2\tau^2+\upsilon^2 z^2)^{-2}\right)\right).$$

The following section describes technical results that provide the crux for the proofs of our main results.

4. ML for $\mathcal{DGW}$ Classes

Following the arguments in Zhang (2004), an immediate consequence of Theorem 2 in the OS is that for fixed κ, δ, μ, and λ, the parameters σ^2, β, and γ cannot be estimated consistently. Instead, we show here that the microergodic parameter $\sigma^2/(\gamma^\delta\beta^{2\kappa+1})$ is consistently estimable. We then assess the asymptotic distribution of the ML estimator. Let $D \times \mathcal{T}$ be a bounded subset of $\mathbb{R}^d \times \mathbb{R}$, and let $\boldsymbol{Z}_{nm} = (Z(\boldsymbol{s}_1,t_1),\ldots,Z(\boldsymbol{s}_n,t_m))^\top$ be a finite realization of a zero mean stationary Gaussian random field $Z(\boldsymbol{s},t)$, $(\boldsymbol{s},t) \in D\times\mathcal{T}$, with a given parametric covariance function $\sigma^2 K(r,t;\boldsymbol{\tau})$, with $\sigma^2 > 0$, $\boldsymbol{\tau}$ a parameter vector, and K a member of the class $\Phi_{d,T}$, with $K(0,0;\boldsymbol{\tau}) = 1$. Here, we consider the $\mathcal{DGW}$ covariance model, that is, $K_{\mathcal{DGW}}(r,t;\boldsymbol{\chi}) = \sigma^2 K(r,t;\boldsymbol{\tau})$, where $K(r,t;\boldsymbol{\tau}) = [h_{\delta,\gamma}(t)]^\lambda K_{\mathcal{GW}}(r;\beta h_{\delta,\gamma}(t),\mu,\kappa)$, with $h_{\delta,\gamma}(t) = \left(1+(t/\gamma)^\delta\right)^{-1}$ and

$\boldsymbol{\tau} = (\mu, \kappa, \beta, \delta, \lambda, \gamma)^\top$. At the same time, in the current exposition, $\boldsymbol{\tau}$ does not contain the parameters that are fixed, but only those that are to be estimated using the ML. Specifically κ, δ, λ, and μ are assumed known and fixed; that is, we assume $\boldsymbol{\tau} = (\beta, \gamma)^\top$, the spatial and temporal scale parameters. Then, the Gaussian log-likelihood function is defined as

$$\mathcal{L}_{nm}(\sigma^2, \beta, \gamma) = -\frac{1}{2}\Bigg(nm \log(2\pi\sigma^2) + \log\left(|R_{nm}(\beta, \gamma)|\right) \tag{4.1}$$

$$+ \frac{1}{\sigma^2} \boldsymbol{Z}_{nm}^\top R_{nm}(\beta, \gamma)^{-1} \boldsymbol{Z}_{nm} \Bigg), \tag{4.2}$$

where $R_{nm}(\beta, \gamma) = [K(\|\boldsymbol{s}_i - \boldsymbol{s}_j\|, |t_l - t_k|; \beta, \gamma)]_{i,j=1;,l,k=1}^{n;m}$ is the correlation matrix. Let $\hat{\sigma}^2_{nm}$ be the ML estimator of the variance parameter obtained by maximizing $\mathcal{L}_{nm}(\sigma^2, \beta, \gamma)$ with respect to σ^2, and given by

$$\hat{\sigma}^2_{nm}(\beta, \gamma) = \frac{1}{nm} \boldsymbol{Z}_{nm}^\top R_{nm}(\beta, \gamma)^{-1} \boldsymbol{Z}_{nm}. \tag{4.3}$$

We now establish the strong consistency and the asymptotic distribution of the random variable $\hat{\sigma}^2_{nm}(\beta, \gamma)/(\gamma^\delta \beta^{2\kappa+1})$, that is, the ML estimator of the microergodic parameter.

Theorem 3. *Let $Z(\boldsymbol{s}, t)$, $(\boldsymbol{s}, t) \in D \times \mathcal{T} \subset \mathbb{R}^d \times \mathbb{R}$, for $d = 1, 2$, be a zero mean Gaussian random field with covariance model $K_{\mathcal{DGW}}(\cdot, \cdot; \boldsymbol{\chi})$, and let $\boldsymbol{\tau} = (\mu, \kappa, \beta_0, \delta, \lambda, \gamma_0)^\top$, with $\lambda > 2\kappa + 3$ and $\mu > \eta + 1 + \alpha$. For κ, δ, λ, and μ fixed and known and arbitrary β and γ, we have as $n, m \to \infty$,*

1. $\hat{\sigma}^2_{nm}(\beta, \gamma)/\gamma^\delta \beta^{2\kappa+1} \xrightarrow{a.s.} \sigma_0^2/\gamma_0^\delta \beta_0^{2\kappa+1}$, *and*
2. $\sqrt{n \times m}\left(\hat{\sigma}^2_{nm}(\beta, \gamma)/\gamma^\delta \beta^{2\kappa+1} - \sigma_0^2/\gamma_0^\delta \beta_0^{2\kappa+1}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, 2\left(\sigma_0^2/\gamma_0^\delta \beta_0^{2\kappa+1}\right)^2\right)$.

The proof is deferred to Section C in the `OS`. The second point of Theorem 3 provides the asymptotic distribution of the microergodic parameter for arbitrary dependence parameters β and γ. Nevertheless, in practical applications, both parameters must be estimated. In principle, the asymptotic distribution of the random variable $\hat{\sigma}^2_{nm}(\hat{\beta}, \hat{\gamma})/\hat{\gamma}^\delta \hat{\beta}^{2\kappa+1}$, with $\hat{\boldsymbol{\tau}} = (\hat{\beta}, \hat{\gamma})^\top$, can be obtained following the arguments in Kaufman and Shaby (2013) or Bevilacqua et al. (2019). However, to establish the strong consistency and asymptotic distribution of the sequence of random variables $\hat{\sigma}^2_{nm}(\hat{\beta}, \hat{\gamma})/\hat{\gamma}^\delta \hat{\beta}^{2\kappa+1}$, we need to prove the monotone behavior of $\hat{\sigma}^2_{nm}(\beta, \gamma)/\gamma^\delta \beta^{2\kappa+1}$ when viewed as a function of $(\beta, \gamma) \in I \times J$, with $I \times J$ a product of bounded intervals. Unfortunately, we have not been able to inspect such a monotonicity property.

In the following, to assess the quality of the approximation of Theorem 3 (Point 2), we consider a simulation study that takes into account the case when γ and β are arbitrary. We also explore the case when both are estimated using the ML.

Specifically, we consider 500 simulations, using a Cholesky decomposition, of a Gaussian random field with a $\mathcal{DGW}$ space-time covariance function observed in $[0,1]^2 \times [0,1]$. In particular, we consider x^2 location sites uniformly distributed in $[0,1]^2$ with $x = 6, 8, 10, 12, 14$ and $0, 0.1, \ldots, 0.9, 1$ temporal instants; that is, we consider $n = 36, 64, 100, 144, 196$, and $m = 11$. The increasing total number of space-time observations in the three-dimensional unit cube is $n \times m = 396, 704, 1100, 1584, 2156$, respectively.

For each simulation, we consider $\kappa = 0, 1$, $\delta = 1.75$, $\lambda = 5$, and $\mu = 5.5 + \kappa$ as known and fixed, and we set $\sigma_0^2 = 1$, $\beta_0 = 1$, and $\gamma_0 = 3$. We estimate the microergodic parameter as

$$\frac{\hat{\sigma}_i^2(x_i, y_i)}{x_i^{2\kappa+1} y_i^{\delta}} = \frac{\boldsymbol{Z}_i^{\top} R_{nm}(x_i, y_i)^{-1} \boldsymbol{Z}_i}{nm x_i^{2\kappa+1} y_i^{\delta}},$$

where $x_i = \beta_0$ and $y_i = \gamma_0$ for the case with arbitrary dependence parameters (here, we set them equal to the true dependence parameters), and $x_i = \hat{\beta}_i$ and $y_i = \hat{\gamma}_i$ for the case of parameters estimated using the ML. Here, $\boldsymbol{Z}_i$ is the data vector of simulation i.

For the first case, the ML estimation is obtained using (4.3), and for the second case, the ML estimation is obtained using the maximization, with respect to β and γ, of the log profile likelihood $\mathcal{L}_{nm}(\hat{\sigma}_{nm}^2(\beta, \gamma), \beta, \gamma)$.

Using the asymptotic distributions stated in Theorem 3, Table 1 compares the sample quantiles of order $0.05, 0.25, 0.5, 0.75$ and 0.95 and the mean and variance of

$$\sqrt{\frac{n \times m}{2}} \left(\frac{\hat{\sigma}_i^2(x_i, y_i) \beta_0^{2\kappa+1} \gamma_0^{\delta}}{\sigma_0^2 x_i^{2\kappa+1} y_i^{\delta}} - 1 \right)$$

when $x_i = \beta_0$ and $y_i = \gamma_0$ with the associated theoretical values of the standard Gaussian distribution. In the same table, we also explore the case $x_i = \widehat{\beta}_i$ and $y_i = \widehat{\gamma}_i$.

As expected, the best approximation is achieved overall when using the true dependence parameters (*i.e.*, $x_i = \beta_0$, $y_i = \gamma_0$), and in the case of $x_i = \widehat{\beta}_i$ and $y_i = \widehat{\gamma}_i$, the asymptotic distribution seems a satisfactory approximation of the sample distribution, visually improving with increasing n. Note that the variance increases when the smoothness parameter κ increases. This pattern is well known in the purely spatial case when estimating the microergodic parameter of the $\mathcal{GW}$

Table 1. For $\kappa = 0, 1$ and $\delta = 1.75$, sample quantiles, mean, and variance of $\sqrt{n \times m/2}(\hat{\sigma}_i^2(x_i, y_i)\beta_0^{2\kappa+1}\gamma_0^{\delta}/(\sigma_0^2 x_i^{2\kappa+1} y_i^{\delta}) - 1)$, $i = 1, \ldots, 500$, for $x_i = \widehat{\beta}_i, \beta_0$ and $y_i = \widehat{\gamma}_i, \gamma_0$,. when $\beta_0 = 1$ and $\gamma_0 = 3$ with $n \times m = 396, 704, 1100, 1584, 2156$, compared with the associated theoretical values of the standard Gaussian distribution.

$\kappa = 0$	(x, y)	$n \times m$	5%	25%	50%	75%	95%	Mean	Var
		396	-1.962	-0.904	0.060	0.836	2.142	0.029	1.534
	$(\widehat{\beta}, \widehat{\gamma})$	704	-1.889	-0.759	0.010	0.836	2.031	0.030	1.386
		1,100	-1.728	-0.741	0.068	0.852	1.868	0.070	1.278
		1,584	-1.642	-0.738	0.008	0.704	1.717	0.017	1.141
		2,156	-1.643	-0.720	-0.009	0.639	1.669	-0.094	1.119
		396	-1.535	-0.705	-0.014	0.720	1.788	0.022	1.061
	(β_0, γ_0)	704	-1.662	-0.733	0.032	0.704	1.758	0.001	1.060
		1,100	-1.675	-0.700	0.032	0.709	1.682	0.021	1.052
		1,584	-1.634	-0.646	0.014	0.717	1.601	0.005	1.017
		2,156	-1.645	-0.648	-0.094	0.659	1.660	-0.079	1.012
$\kappa = 1$	(x, y)	$n \times m$	5%	25%	50%	75%	95%	Mean	Var
		396	-2.179	-0.971	-0.110	0.733	2.462	-0.036	1.839
	$(\widehat{\beta}, \widehat{\gamma})$	704	-2.039	-0.806	0.041	0.877	1.938	0.015	1.510
		1,100	-1.939	-0.782	0.104	0.800	1.850	0.002	1.382
		1,584	-1.683	-0.735	-0.030	0.653	1.977	-0.002	1.270
		2,156	-1.693	-0.720	-0.009	0.679	1.723	-0.096	1.194
		396	-1.535	-0.705	-0.014	0.720	1.788	0.022	1.061
	(β_0, γ_0)	704	-1.662	-0.733	0.032	0.704	1.758	0.001	1.060
		1,100	-1.675	-0.700	0.032	0.709	1.682	0.021	1.052
		1,584	-1.634	-0.646	0.014	0.717	1.601	0.005	1.017
		2,156	-1.645	-0.648	-0.094	0.659	1.660	-0.079	1.012
	$N(0, 1)$		-1.645	-0.674	0	0.674	1.645	0	1

or Matérn covariance models. In addition, when $x_i = \beta_0$ and $y_i = \gamma_0$, the sample quantiles do not depend on κ, as expected. We repeat this numerical experiment by considering arbitrary dependence parameters sufficiently "far" from the true values , finding that the convergence can be very slow, as observed in Kaufman and Shaby (2013) and Bevilacqua et al. (2019).

5. Prediction Using the $\mathcal{DGW}$ Model

We now consider kriging prediction, under fixed domain asymptotics, of a Gaussian random field at an unknown space-time location $(\boldsymbol{s}_0, t_0) \in \mathcal{D} \times \mathcal{T}$, using the $\mathcal{DGW}$ model $K_{\mathcal{DGW}}(r, t; \boldsymbol{\chi})$. Recall that the parameter vectors $\boldsymbol{\chi}$ and $\boldsymbol{\tau}$ are defined in Sections 2.3 and 4, respectively. Specifically, we focus on two properties:

(A) asymptotic efficient prediction, and

(B)) asymptotically correct estimation of the prediction variance.

Stein (1988) shows that both asymptotic properties hold when the Gaussian measures are equivalent (see Section A in the OS). Let $P(K_{\mathcal{DGW}}(\boldsymbol{\chi}_i))$, for $i = 0, 1$, be two probability zero mean Gaussian measures with covariance belonging to the $\mathcal{DGW}$ class of space-time covariance functions, where $\boldsymbol{\chi}_i = (\sigma_i^2, \boldsymbol{\tau}_i^\top)^\top$ and $\boldsymbol{\tau}_i = (\mu, \kappa, \beta_i, \delta, \lambda, \gamma_i)^\top$, for $i = 0, 1$ is the associated set of parameters.

Under $P(K_{\mathcal{DGW}}(\boldsymbol{\chi}_0))$, and using Theorem 2 in the OS, properties **(A)** and **(B)** hold, provided

$$\frac{\sigma_0^2}{\gamma_0^\delta \beta_0^{2\kappa+1}} = \frac{\sigma_1^2}{\gamma_1^\delta \beta_1^{2\kappa+1}},$$

and $\mu > \eta + 1 + \alpha$, $\delta > (d+1)/2$, and $d = 1, 2$. Similarly, let $P(K_{\mathcal{DM}}(\boldsymbol{\theta}))$ and $P(K_{\mathcal{DGW}}(\boldsymbol{\chi}))$ be two zero mean Gaussian measures under the $\mathcal{DM}$ and $\mathcal{DGW}$ models, respectively. Under $P(K_{\mathcal{DM}}(\boldsymbol{\theta}))$, properties **(A)** and **(B)** hold when $\mu > \eta + 1 + \alpha$, Point 2 of Theorem 3 in the OS holds, and $d = 1, 2$.

Actually, Stein (1993) gives a substantially weaker condition for asymptotic efficiency prediction based on the asymptotic behavior of the ratio of the spectral densities. Let

$$\widehat{Z}_{nm}(\boldsymbol{\tau}) = \boldsymbol{c}_{nm}(\boldsymbol{\tau})^\top R_{nm}(\boldsymbol{\tau})^{-1} \boldsymbol{Z}_{nm} \tag{5.1}$$

be the best linear unbiased predictor at an unknown location $(\boldsymbol{s}_0, t_0) \in \mathcal{D} \times \mathcal{T}$, under the misspecified model $P(K_{\mathcal{DGW}}(r, t; \boldsymbol{\chi}))$, where $\boldsymbol{c}_{nm}(\boldsymbol{\tau}) = [\phi(\boldsymbol{s}_0 - \boldsymbol{s}_i, t_0 - t_j; \boldsymbol{\tau})]_{i=1,j=1}^{n,m}$ and $R_{nm}(\boldsymbol{\tau}) = [\phi(\boldsymbol{s}_i - \boldsymbol{s}_j, t_i - t_j; \boldsymbol{\tau})]_{i=1,j=1}^{n,m}$ is the correlation matrix. If the correct model is $P(K_{\mathcal{DGW}}(r, t; \boldsymbol{\chi}_0))$, then the mean squared error of the kriging predictor is given by:

$$\begin{aligned}\mathrm{Var}_{\boldsymbol{\chi}_0}\left[\widehat{Z}_{nm}(\boldsymbol{\tau}) - Z(\boldsymbol{s}_0, t_0)\right] = \sigma_0^2\Big(&1 - 2\boldsymbol{c}_{nm}(\boldsymbol{\tau})^\top R_{nm}(\boldsymbol{\tau})^{-1}\boldsymbol{c}_{nm}(\boldsymbol{\tau}_0) \\ &+\boldsymbol{c}_{nm}(\boldsymbol{\tau})^\top R_{nm}(\boldsymbol{\tau})^{-1} R_{nm}(\boldsymbol{\tau}_0) R_{nm}(\boldsymbol{\tau})^{-1}\boldsymbol{c}_{nm}(\boldsymbol{\tau}_0)\Big).\end{aligned} \tag{5.2}$$

If $\beta_0 = \beta$ and $\gamma_0 = \gamma$, that is, the true and misspecified models coincide, this expression simplifies to

$$\mathrm{Var}_{\boldsymbol{\chi}_0}\left[\widehat{Z}_{nm}(\boldsymbol{\tau}_0) - Z(\boldsymbol{s}_0, t_0)\right] = \sigma_0^2\big(1 - \boldsymbol{c}_{nm}(\boldsymbol{\tau}_0)^\top R_{nm}(\boldsymbol{\tau}_0)^{-1}\boldsymbol{c}_{nm}(\boldsymbol{\tau}_0)\big). \tag{5.3}$$

Similarly, $\mathrm{Var}_{\boldsymbol{\theta}}\left[\widehat{Z}_{nm}(\boldsymbol{\tau}) - Z(\boldsymbol{s}_0, t_0)\right]$ and $\mathrm{Var}_{\boldsymbol{\theta}}\left[\widehat{Z}_{nm}(\boldsymbol{\theta}) - Z(\boldsymbol{s}_0, t_0)\right]$ can be defined under $P(K_{\mathcal{DM}}(\boldsymbol{\theta}))$. Here, $\widehat{Z}_{nm}(\boldsymbol{\theta})$ is the best linear unbiased predictor using the $\mathcal{DM}$ model, and recall that $\boldsymbol{\theta} = (\nu, \zeta, \upsilon, \epsilon, \sigma)^\top$ is the set of correlation parameters. The following results are an application of Theorems 1 and 2 of Stein

(1993).

Theorem 4. *Let $P(K_{\mathcal{DGW}}(\boldsymbol{\chi}_i))$, for $i = 0, 1$, and $P(K_{\mathcal{DM}}(\boldsymbol{\theta}_0))$ be three Gaussian probability measures on $\mathcal{D} \times \mathcal{T} \subset \mathbb{R}^d \times \mathbb{R}$, and let $\mu > \eta + 1 + \alpha$. Then, for all $(\boldsymbol{s}_0, t_0) \in \mathcal{D} \times \mathcal{T}$:*

1. *Under $P(K_{\mathcal{DGW}}(\boldsymbol{\chi}_0))$, as $n \to \infty$,*

$$\frac{\mathrm{Var}_{\boldsymbol{\chi}_0}\big[\widehat{Z}_{nm}(\boldsymbol{\tau}_1) - Z(\boldsymbol{s}_0, t_0)\big]}{\mathrm{Var}_{\boldsymbol{\chi}_0}\big[\widehat{Z}_{nm}(\boldsymbol{\tau}_0) - Z(\boldsymbol{s}_0, t_0)\big]} \longrightarrow 1, \tag{5.4}$$

for any fixed $\beta_1 > 0$ and $\gamma_1 > 0$.

2. *Under $P(K_{\mathcal{DM}}(\boldsymbol{\theta}_0))$, if $\nu = \eta$ as $n \to \infty$,*

$$\frac{\mathrm{Var}_{\boldsymbol{\theta}_0}\big[\widehat{Z}_{nm}(\boldsymbol{\tau}_1) - Z(\boldsymbol{s}_0, t_0)\big]}{\mathrm{Var}_{\boldsymbol{\theta}_0}\big[\widehat{Z}_{nm}(\boldsymbol{\theta}) - Z(\boldsymbol{s}_0, t_0)\big]} \longrightarrow 1, \tag{5.5}$$

for any fixed $\beta_1 > 0$, $\gamma_1 > 0$, and $\boldsymbol{\theta} = (\nu, \zeta, \upsilon, \epsilon)$.

3. *Under $P(K_{\mathcal{DGW}}(\boldsymbol{\chi}_0))$, if $\sigma_0^2 \beta_0^{-(2\kappa+1)}/\gamma_0^{\delta} = \sigma_1^2 \beta_1^{-(2\kappa+1)}/\gamma_1^{\delta}$, then*

$$\frac{\mathrm{Var}_{\boldsymbol{\chi}_1}\big[\widehat{Z}_{nm}(\boldsymbol{\tau}_1) - Z(\boldsymbol{s}_0, t_0)\big]}{\mathrm{Var}_{\boldsymbol{\chi}_0}\big[\widehat{Z}_{nm}(\boldsymbol{\tau}_1) - Z(\boldsymbol{s}_0, t_0)\big]} \longrightarrow 1. \tag{5.6}$$

4. *Under $P(K_{\mathcal{DM}}(\boldsymbol{\theta}_0))$, for $\epsilon \in (0, 1]$, if $\sigma_1^2 \varrho_{\lambda,\eta} c_3^{\varsigma} \beta^{-2\eta} = \ell(\boldsymbol{\theta}_0)\epsilon^{-2\nu}$, $\nu = \eta$ and $1 + 2\kappa = \delta$, then as $n \to \infty$,*

$$\frac{\mathrm{Var}_{\boldsymbol{\chi}_1}\big[\widehat{Z}_{nm}(\boldsymbol{\tau}_1) - Z(\boldsymbol{s}_0, t_0)\big]}{\mathrm{Var}_{\boldsymbol{\theta}_0}\big[\widehat{Z}_{nm}(\boldsymbol{\tau}_1) - Z(\boldsymbol{s}_0, t_0)\big]} \longrightarrow 1. \tag{5.7}$$

As an illustration of the results in Theorem 4, we perform a small numerical experiment, focusing in particular on Points 1 and 3. Let us define the ratios (5.4) and (5.6) as $U_1(\beta_1, \gamma_1)$ and U_2, respectively. We randomly select $n_j = 36, 64, 100, 144, 196$, $(j = 1, \ldots 500)$ location sites without replacement from a fine regular grid on the unit square, and we keep these location sites fixed across the 11 temporal instants $0, 0.1, \ldots, 1$. We then compute the ratios $U_{1j}(\beta_1, \gamma_1)$ and U_{2j}, for $j = 1, \ldots, 500$, using the closed-form expressions in Equation (5.2) and (5.3), to predict the space-time location site $\boldsymbol{s}_0 = (0.53, 0.53)$ and $t_0 = 0.6$. Specifically, for $\kappa = 0, 1$, we set $\mu = 5.5 + \kappa$, $\delta = 1.75$, and $\lambda = 5$, as in the numerical experiment in Section 4. The parameters of the correct model $\boldsymbol{\chi}_0 = (\sigma_0^2, \boldsymbol{\tau}_0^{\top})^{\top}$ with

Table 2. $\bar{U}_1 = \sum_{j=1}^{500} U_{1j}(\beta_1, \gamma_1)/500$, $\bar{U}_2 = \sum_{j=1}^{500} U_{2j}/500$ when increasing the number of space-time locations for $\kappa = 0, 1$.

$m \times n$	κ=0		κ=1	
	$\bar{U}_1$	$\bar{U}_2$	$\bar{U}_1$	$\bar{U}_2$
396	1.00249	1.05611	1.00104	1.05730
704	1.00104	1.04349	1.00035	1.04338
1,100	1.00048	1.03826	1.00013	1.03781
1,584	1.00022	1.03513	1.00005	1.03500
2,156	1.00012	1.03354	1.00002	1.03337

$\boldsymbol{\tau}_0^\top = (\mu, \kappa, \beta_0, \delta, \lambda, \gamma_0)^\top$ are fixed as $\sigma_0^2 = 1$, $\beta_0 = 1$, and $\gamma_0 = 3$, and the parameters of the misspecified model $\boldsymbol{\chi}_1 = (\sigma_1^2, \boldsymbol{\tau}_1^\top)^\top$ with $\boldsymbol{\tau}_1^\top = (\mu, \kappa, \beta_1, \delta, \lambda, \gamma_1)^\top$ are fixed as $\sigma_1^2 = 1.25$, and $\gamma_1 = 3.05$; the spatial parameter is obtained using the equivalence condition, that is, $\beta_1 = \beta_0((\sigma_0^2/\sigma_1^2)(\gamma_0/\gamma_1)^\delta)^{-(1+2\kappa)}$ (see also A in `OS`). This gives $\beta_1 = 1.21436$ for $\kappa = 0$, and $\beta_1 = 1.066881$ for $\kappa = 1$.

Table 2 reports the overall mean $\bar{U}_1 = \sum_{j=1}^{500} U_{1j}(\beta_1, \gamma_1)/500$ and $\bar{U}_2 = \sum_{j=1}^{500} U_{2j}/500$ when increasing the number of spatiotemporal sites $n \times m = 396, 704, 1100, 1584, 2196$. It can be appreciated that the convergence to one of $\bar{U}_1$ is much faster than that of $\bar{U}_2$. These results are consistent with those of the purely spatial case in Bevilacqua et al. (2019). In addition, there are no significant differences between the cases $\kappa = 0, 1$.

6. Conclusion

There is a clear lack of general results on the asymptotic properties of the ML estimator under fixed-domain asymptotics, particularly in the space-time setting. This is reflected in the literature, where the results are sparse and are estabilished for *ad hoc* families of covariance functions. Similarly, we have established results that hold for the $\mathcal{DGW}$ family under fixed-domain asymptotics. Future research could examine a more realistic setting for the temporal component. A promising solution might be to embed time in the circle, so that the associated Gaussian random field becomes periodic.

Supplementary Material

The online Supplementary Material contains mathematical proofs and some graphical representations.

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