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ON THE CONSISTENCY OF THE LEAST SQUARES ESTIMATOR IN MODELS SAMPLED AT RANDOM TIMES DRIVEN BY LONG MEMORY NOISE: THE RENEWAL CASE

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Abstract: In this study, we prove the strong consistency of the least squares estimator in a random sampled linear regression model with long-memory noise and an independent set of random times given by renewal process sampling. Additionally, we illustrate how to work with a random number of observations up to time $T = 1$. A simulation study is provided to illustrate the behavior of the different terms, as well as the performance of the estimator under various values of the Hurst parameter H.

Key words and phrases: Least squares estimator, long-memory noise, random times, regression model, renewal process.

1. Introduction

In many applications, data are observed at random times. This situation arises from a variety of causes, such as machinery faults or the inability to observe data in certain periods. In the financial field, the process often cannot be observed continuously (Duffie and Glynn (2004)). As a result, high-frequency financial data (a very large amount of data) tend to be sampled discretely in time, and the time separating successive observations is often random. For the random modeling of observations, the renewal case represents progressive randomness and distance from periodic sampling. Here Masry (1983) studied the problem of estimating an unknown probability density function, based on n independent observations sampled at random times. Vilar (1995) and Vilar and Vilar (2000), studied the nonparametric kernel estimator of the regression function under mixing dependence conditions, and the Ornstein–Uhlenbeck process driven by Brownian motion, respectively.

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This study offers an alternative approach to constructing trend regression models by taking into account long-memory behavior in the noise term. The interest in the long-memory noise model lies in the behavior of its covariance structure, which can cover a general class of noise. Trend analyses are important in many time series applications. Parameter estimation problems in time series, represented as a trend plus long-memory noise, are well studied; see Baillie and Chung (2002), Brockwell (2007) and Lobato and Velasco (2002), among others. In contrast, in time series models with long-memory, parameter estimation in models sampled at random times is more rare.

The concept of long-memory is very well characterized in terms of the spectral density function. However, the existence of this function is limited to stationary processes. When jointly considering a model with trend and long-memory properties, there is no stationarity property essential to defining the spectral density function in which the spectral estimate rests. In contrast, for spectral estimations, wavelet methods have been proposed for irregularly sampled real-valued data, including regression problems and long-memory estimation; see Bardet and Bertrand (2010), Efromovich (2014) and Knight, Nunes and Nason (2012).

Different applications have been considered in the financial domain, where we can detect trends by the existence of random, low, or high volatility periods. As mentioned in Duffie and Glynn (2004), certain financial data, particularly intraday, are sampled at random times, and the trading frequency (or volume) is higher during periods of faster information arrival. The authors modeled prices using a time-homogeneous continuous-time Markov process, and proposed estimating the parameters of the model by using the method of moments. Additionally, much evidence exists that financial and economic data exhibit long-memory. For example, in stock markets, further investments are often made based on the technical analysis of past prices and volume information. Aït-Sahalia and Mykland (2003) use an exponential distribution to fit the time distribution between trades for Nokia shares traded on the New York Stock Exchange.

In this study, we examine the least squares estimator (LSE) in a simple regression model, nonstationary in trend, with long-memory noise and observation measurements at random times. We also show how to deal with the number of observations needed to reach a fixed time T assuming, without loss of generality (w.l.o.g.), $T = 1$. To explain the long-memory or long-range dependence phenomenon in a model, it is common to represent it using the Hurst exponent H, which takes values in $(0, 1)$. In particular, long-range dependence can be seen when $H \in (1/2, 1)$. Mandelbrot and Van Ness (1968) studied the effect of long-range dependence. One of the most popular Gaussian stochastic processes with long-memory is fractional Brownian motion. We consider the following simple regression model:

$$Y_{\tau_{i+1}} = a\tau_i + \Delta B^H_{\tau_{i+1}}, \quad i = 0, \ldots, N(1), \tag{1.1}$$

where $a \in \mathbb{R}$ is the unknown drift parameter of the model. Long-memory is represented by the noise, defined as $\Delta B^H_{\tau_{i+1}} = B^H_{\tau_{i+1}} - B^H_{\tau_i}$. Here, $\tau := \{\tau_i, 0 \leq i\}$ is a random increasing sequence of positive random times depending on N. However, this dependence is expressed through the distribution function, and the initial value, τ_0, is also a positive random variable; see the next section for a detailed definition. Note that $N(1) = \sum_{j\geq 1} 1_{\{\tau_j \leq 1\}}$ determines the number of events in $[0, 1]$. From the definition of τ, $N(1)$ is a discrete random variable, and N represents the expected number of observations within $[0, 1]$. The process $Y := \{Y_{\tau_{i+1}}, 0 \leq i\}$, defined in equation (1.1), is nonstationary. The long-memory or long-range dependence refers to the type of noise used. However, note that the long-memory property does not necessarily hold when working with random times, as pointed out by Philippe, Robet and Vian (2020).

The LSE estimator for a, the drift parameter of the random sampled regression model with long-memory noise in (1.1), is determined by $\hat{a}_{N(1)} = \sum_{i=0}^{N(1)} \tau_i Y_{\tau_{i+1}} / \sum_{i=0}^{N(1)} \tau_i^2$. Working with random times that are not upper bounded is a challenge, because both, the observation times and the number of observations within the interval, are random. Our way of dealing with this task is to divide the problem into three stages. First, we study the almost sure convergence of $N(1)/N$ to 1. Second, we define an auxiliary least squares type estimator, $\hat{a}_N = \sum_{i=0}^{N-1} \tau_i Y_{\tau_{i+1}} / \sum_{i=0}^{N-1} \tau_i^2$, considering a fixed number $N \in \mathbb{N}$, corresponding to the sampling frequency or sampling rate and, study the convergence of $\hat{a}_N \to a$. Finally, we ensure the convergence of $|\hat{a}_N - \hat{a}_{N(1)}|$ to zero. This approach is used by Deo et al. (2006) to examine whether $N(1)$ being close to N allows us to ensure that both, $\hat{a}_{N(1)}$ and $\hat{a}_N$, are strongly consistent. In practice, our estimator is based on $N(1)$ observations, because if $N(1) < N$, then $\hat{a}_N$ cannot be computed from the data. Note that there is another type of random time, known as "jittered" or "irregular observations" in which, unlike the random time reviewed in this work, the random variables defining the jittered times are bounded. Model (1.1), with jittered random times, has been studied by Araya et al. (2023).

The remainder of the paper proceeds as follows. In Section 2, we define the random time used in the random sampled regression model with long-memory noise, we then describe our notation and present the almost sure convergence of $\tau_N \to 1$ and $N(1)/N \to 1$. Section 3 is devoted to the main results. Here, we prove the almost sure convergence of $\hat{a}_N$ and $\hat{a}_{N(1)}$ to a. In Section 4, a simulation study is presented to illustrate the performance of the estimator, considering different values of H and random times under two different sampling schemes. Finally, in Section 5, we present the proof of a technical lemma, established in Section 3.

2. Preliminaries

In this section, we introduce the basic tools and the framework used in this study. In particular, we present the random noise evaluated at random times considered throughout this work.

2.1. Random time

Let $\tau = \{\tau_i;\ i \geq 0\}$ be a strictly increasing sequence of random points over time, the distribution function of which depends on N (to avoid superscript, the dependence on N is through the distribution function), where N represents the sampling frequency or sampling rate, that is, the average number of samples obtained in $[0, 1]$.

The sequence τ, defined by the renewal process (RP), is given as follows:

$$\tau_i = \sum_{j=0}^{i} t_j \qquad i \geq 0, \tag{2.1}$$

where $\{t_j, j \geq 0\}$ is a sequence of independent and identically distributed random variables (i.i.d.), with a common distribution function $G_N(\cdot)$, that depend on N with support in $[0, \infty)$, and are absolutely continuous with density g_N, such that $G_N(0) = 0$, satisfying the following hypothesis:

H1 $\mathbb{E}\left[t_i\right] = 1/N$ for all $i \geq 0$.

H2 $\mathbb{E}\left[t_i^2\right] = \kappa_1/N^{\alpha}$, $0 < \alpha \leq 2$.

H3 $\mathbb{E}\left[t_i^4\right] = \kappa_2/N^{\beta}$, $0 < \beta \leq 2\alpha$.

Here κ_1 and κ_2 are constants not depending on N. Note that the conditions $\alpha \leq 2$ and $\beta \leq 2\alpha$ come from the Cauchy inequality.

Henceforth, $G_{N,i}$ denotes the probability distribution function associated with τ_i and its density functions $g_{N,i}$, and $N(1)$ is the number of observations needed to sample up to one. Examples of distributions satisfying **H1** to **H3** are: the beta prime distribution, with parameters $(1, N+1)$, and the exponential distribution with parameter $\lambda = N$. This distribution is a limit case for $\alpha = 2$.

Araya et al. (2019) provides a study of this model.

Remark 1. Henceforth, we write $\mathbb{E}(t_i^m) = \mathbb{E}(t_0^m)$ for $m \in \mathbb{N}$ and $i \geq 0$, because it no longer depends on i. From hypotheses **H1** and **H2**; we have

$$\mathbb{E}\left[\tau_i\right] = \frac{i+1}{N}, \quad \mathbb{E}\left[\tau_i^2\right] = (i+1)\mathbb{E}\left[t_0^2\right] + \frac{(i+1)^2-(i+1)}{N^2}, \quad i \geq 0. \tag{2.2}$$

Applying the Jensen inequality to any positive random variable X with all its finite moments, we have for $0 < b < 1$,

$$\mathbb{E}\left[X^b\right] \leq \left(\mathbb{E}\left[X\right]\right)^b. \tag{2.3}$$

Remark 2. Henceforth, C denotes a generic constant that does not depend on N, which may vary from line to line.

2.2. The noise

In this subsection we give the main properties of the process $B^H = \{B_t^H, t \geq 0\}$ with zero mean, the increments of which are considered as the noise in model (1.1).

N1 Covariance structure: $R_H(t,s) = \mathbb{E}(B_t^H B_s^H) = (1/2)\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$.

N2 We consider a finite-variance process that is self-similar with stationary increments.

For example, B^H can represent the well-known fractional Brownian motion (fBm). In the fBm framework, when $H = 1/2$, B^H is the standard Brownian motion. Other types of long-memory processes, with the same covariance structure as B^H, are the Hermite and Rosenblatt processes. For more references on these processes see Tudor (2013).

N3 The random time sequence τ, which depends on N, and the long-memory noise B^H are independent.

This latter condition is essential for the theoretical results presented in Section 3.

2.3. Almost sure convergence of $\tau_N \to 1$ and $N(1)/N \to 1$

In this section, we show how to quantify the ratio $N(1)/N$, the number of observations sampled up to time 1, and the sampling frequency or sampling rate. We show, in Section 3 and Proposition 2 that, to prove there is strong consistency, studying the behavior of $N(1)/N$ is an important task.

Remark 3. Throughout the paper, especially in Section 3, we use the following argument to ensure the convergence in probability and almost sure convergence of a given sequence of random variables.

Let $(\theta_N)_{N\geq 0}$ be a sequence of random variables. From Tshebyshev's inequality, we have

$$\mathbb{P}\left(|\theta_N| > \epsilon\right) \leq \frac{\mathbb{E}(\theta_N^m)}{\epsilon^m}, \quad m > 0. \tag{2.4}$$

If $\mathbb{E}(\theta_N^m) \leq C/N^{\gamma}$, with $\gamma > 0$, then $|\theta_N| \to 0$ in probability. Note that if $\sum_{N\geq 1}\mathbb{E}(\theta_N^m)$ converges to a finite value, using the Borel–Cantelli lemma, we get $\mathbb{P}(|\theta_N| > \epsilon\,,\text{infinitely often}) = 0$. This yields that $|\theta_N| \to 0$ a.s.

Let us consider $\tau_{N-1} = \sum_{i=0}^{N-1} t_i = 1/N\sum_{i=0}^{N-1} Nt_i$. Now, $\mathbb{E}(\tau_{N-1}) = 1$, and $Var(\tau_{N-1}) = NVar(t_0) = \kappa_1/N^{\alpha-1} - 1/N$, from hypotheses **H1** and **H2**, which tends to zero as $N \to \infty$ for $\alpha > 1$. In addition, by Remark 3 and considering the fourth central moment and hypotheses **H1** to **H3**, we have

$$\begin{aligned}
\mathbb{E}\left[(\tau_{N-1}-1)^4\right] &= \mathbb{E}\left[\left(\sum_{j=0}^{N-1}(t_j-\mathbb{E}\left[t_j\right])\right)^4\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{N-1}(t_j-\mathbb{E}\left[t_j\right])^4\right] + 6\mathbb{E}\left[\sum_{i\neq j}^{N-1}(t_i-\mathbb{E}\left[t_i\right])^2(t_j-\mathbb{E}\left[t_j\right])^2\right] \\
&= \sum_{j=0}^{N-1}\mathbb{E}\left[t_0^4\right] - 4\sum_{j=0}^{N-1}\mathbb{E}\left[t_0^3\right]\mathbb{E}\left[t_0\right] + 6\sum_{j=0}^{N-1}\mathbb{E}\left[t_0^2\right]\mathbb{E}\left[t_0\right]^2 - 4\sum_{j=0}^{N-1}\mathbb{E}\left[t_0\right]\mathbb{E}\left[t_0\right]^3 \\
&\quad + N\left(\mathbb{E}\left[t_0\right]\right)^4 + 6\sum_{0\leq i\neq j\leq N-1}\left(\mathbb{E}\left[(t_0-\mathbb{E}\left[t_0\right])^2\right]\right)^2 \\
&\leq \frac{\kappa_2}{N^{\beta-1}} + \frac{6\kappa_1}{N^{\alpha+1}} + \frac{1}{N^3} + \frac{6\kappa_1^2}{N^{2\alpha-2}} \leq \frac{C}{N^{(\beta-1)\wedge(\alpha+1)\wedge 3\wedge(2\alpha-2)}} = \frac{C}{N^{(\beta-1)\wedge(2\alpha-2)}}.
\end{aligned}$$

Then, by the Tshebyshev inequality, τ_{N-1} converges to one in probability if $\alpha > 1$ and $\beta > 1$. Additionally, from the Borel–Cantelli lemma, τ_{N-1} converges to one, almost surely if $\alpha > 3/2$ and $\beta > 2$.

Because we proved the a.s. convergence of τ_N to one, we can similarly get the a.s. convergence of $\tau_{N+N\epsilon} - \tau_N$ to ϵ, for $\alpha > 3/2$ and $\beta > 2$, as shown below. Using this fact and recalling that the random variables τ_N and $\tau_{N+N\epsilon} - \tau_N$ are independent, we have

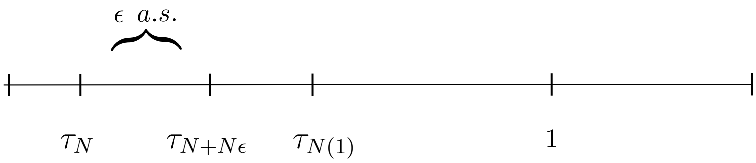


Figure 1. Representation of $\tau_N, \tau_{N+N\epsilon}$, and $\tau_{N(1)}$.

$$
\begin{aligned}
\mathbb{P}\left(\frac{N(1)}{N}-1>\epsilon\right) &= \mathbb{P}\left(N(1)>N+N\epsilon\right)=\mathbb{P}\left(\tau_{N+N\epsilon}<1\right)\\
&= \mathbb{P}\left(1-\tau_N>\tau_{N+N\epsilon}-\tau_N\right)\\
&= \mathbb{P}\left(\left(1-\tau_N>\tau_{N+N\epsilon}-\tau_N\right)\wedge\left(\tau_{N+N\epsilon}-\tau_N<\frac{\epsilon}{2}\right)\right)\\
&\quad +\mathbb{P}\left(\left(1-\tau_N>\tau_{N+N\epsilon}-\tau_N\right)\wedge\left(\tau_{N+N\epsilon}-\tau_N>\frac{\epsilon}{2}\right)\right)\\
&\leq \mathbb{P}\left(\tau_{N+N\epsilon}-\tau_N<\frac{\epsilon}{2}\right)+\mathbb{P}\left(1-\tau_N>\frac{\epsilon}{2}\right)\\
&\leq \mathbb{P}\left(\tau_{N+N\epsilon}-\tau_N-\epsilon<-\frac{\epsilon}{2}\right)+\mathbb{P}\left(1-\tau_N>\frac{\epsilon}{2}\right)\\
&\leq \frac{1}{(\epsilon/2)^4}\left[\frac{\epsilon}{N^{\beta-1}}+\frac{\epsilon^2}{N^{2\alpha-2}}\right]\leq\frac{C}{N^{(\beta-1-3\delta)\wedge(2\alpha-2-2\delta)}}, \qquad (2.5)
\end{aligned}
$$

with $\epsilon = 1/N^{\delta}$, $\delta > 0$. Analogously, using the a.s. convergence of $\tau_{N-N\epsilon} - \tau_N$ to ϵ, we have

$$
\begin{aligned}
\mathbb{P}\left(\frac{N(1)}{N}-1<-\epsilon\right) &= \mathbb{P}\left(N(1)<N-N\epsilon\right)=\mathbb{P}\left(\tau_{N-N\epsilon}>1\right)\\
&\leq \frac{C}{N^{(\beta-1-3\delta)\wedge(2\alpha-2-2\delta)}}. \qquad (2.6)
\end{aligned}
$$

Then, the convergence in probability is ensured if $\delta < ((\beta-1)/3 \wedge (\alpha-1))$; this condition is true if $\alpha > 1$ and $\beta > 1$. On the other hand, the a.s. convergence is ensured, by applying the Borel–Cantelli lemma, if $\delta < ((\beta-2)/3 \wedge (\alpha-3/2))$; that is, if $\alpha > 3/2$ and $\beta > 2$.

Several restrictions appear in the parameters α and β, which we use in our main theorem.

3. Main Results

In this section, we provide our main result. We prove that the LSE is an unbiased and strongly consistent estimator for a, the drift parameter of the random sampled regression model with long-memory noise. To estimate the parameter of interest in the model (1.1), the LSE is computed and is determined by

$$\hat{a}_{N(1)} = \frac{\sum_{i=0}^{N(1)} \tau_i Y_{\tau_{i+1}}}{\sum_{i=0}^{N(1)} \tau_i^2}. \tag{3.1}$$

Recall that, from (1.1) and (3.1), we have

$$\hat{a}_{N(1)} - a = \frac{\sum_{i=0}^{N(1)} \tau_i \Delta B^H_{\tau_{i+1}}}{\sum_{i=0}^{N(1)} \tau_i^2}. \tag{3.2}$$

Theorem 1. *Let τ be the random time defined in (2.1), and let the process $B^H = \{B_t^H, t \geq 0\}$ with a zero mean, and with increments that are considered as the noise. These satisfy hypotheses* **H1** *to* **H3** *and* **N1** *to* **N3***, respectively. Then, for $\alpha > \max\{3/2, 1/H\}$ and $\beta > 2$, the LS estimator $\hat{a}_{N(1)}$ given in (3.1) of the drift parameter a in model (1.1) is strongly consistent,*

$$\hat{a}_{N(1)} \xrightarrow[N\to\infty]{a.s.} a.$$

For $\alpha > 1$ and $\beta > 1$, convergence in probability is ensured.

Proof. Let τ be given by (2.1), and let N be the sampling frequency or sampling rate. Let $\hat{a}_N$ be the LS estimator obtained by replacing $N(1)$ with $N-1$ in (3.1); that is,

$$\hat{a}_N = \frac{\sum_{i=0}^{N-1} \tau_i Y_{\tau_{i+1}}}{\sum_{i=0}^{N-1} \tau_i^2}. \tag{3.3}$$

We consider the following decomposition from (3.1) and (3.3):

$$\hat{a}_{N(1)} - a = \hat{a}_{N(1)} - \hat{a}_N + \hat{a}_N - a.$$

Then, the proof of the main Theorem 1 is given in two steps

- First, we prove in Proposition 1 that $\hat{a}_N$ converges a.s. to a.
- Second, we control the difference $\hat{a}_{N(1)} - \hat{a}_N$ a.s. in Proposition 2.

Proposition 1. *Let τ be the random time defined in (2.1), and let the process $B^H = \{B_t^H, t \geq 0\}$ with a zero mean, and with increments that are considered as the noise. These satisfy hypotheses* **H1** *to* **H3** *and* **N1** *to* **N3***, respectively, for $\alpha > \max\{3/2, 1/H\}$ and $\beta > 2$. Then, the LS estimator $\hat{a}_N$ given in (3.3) of the drift parameter a in model (1.1) is strongly consistent,*

$$\hat{a}_N \xrightarrow[N\to\infty]{a.s.} a.$$

For $\alpha > 1$ and $\beta > 1$, the convergence in probability is ensured.

Proposition 2. *Let τ be the random time defined in (2.1), and let the process $B^H = \{B_t^H, t \geq 0\}$ with a zero mean, and with increments that are considered as the noise. These satisfy hypotheses **H1** to **H3** and **N1** to **N3**, respectively, for $\alpha > \max\{3/2, 1/H\}$ and $\beta > 2$. Consider the LS estimators $\hat{a}_N$ and $\hat{a}_{N(1)}$ of the drift parameter a given in (3.3) and (3.1), respectively, for the model (1.1). Then,*

$$|\hat{a}_{N(1)} - \hat{a}_N| \xrightarrow[N\to\infty]{a.s.} 0. \tag{3.4}$$

For $\alpha > 1$ and $\beta > 1$, the convergence in probability is ensured.

Proof of Proposition 1

Recall that from (1.1) and (3.3), we have

$$|\hat{a}_N - a| = \frac{\frac{1}{N}\sum_{i=0}^{N-1} \tau_i \Delta B_{\tau_{i+1}}^H}{\frac{1}{N}\sum_{i=0}^{N-1} \tau_i^2} := \frac{A_N}{D_N}. \tag{3.5}$$

To prove Proposition 1, we need an auxiliary lemma related to the convergence of the denominator D_N given in (3.5).

Lemma 1. *Let D_N be defined in (3.5). If $\tau = \{\tau_i;\ 0 \leq i \leq N-1\}$ are the sampling random times defined in (2.1) satisfying hypotheses **H1** to **H3**, then for $3/2 < \alpha < 2$ and $\beta > 2$,*

$$D_N \xrightarrow[N\to\infty]{a.s.} \frac{1}{3}.$$

For $\alpha > 1$ and $\beta > 1$, the convergence in probability is ensured.

The proof of this lemma is given in Appendix 5.

Hence, by Lemma 1, the asymptotic behavior of A_N as $N \to \infty$ remains to be studied.

From the definition of A_N and hypothesis **N3**, by conditioning on τ, it is straightforward to see that $\mathbb{E}[A_N] = 0$. With this in mind and working with the second and fourth moments, we have the selected choices of α and β to ensure the convergence in probability and/or the a.s. convergence of the numerator A_N. Lemma 1 gives us a.s. convergence and/or convergence in probability of the denominator D_N. Finally, an application of the Slutsky theorem ensures the a.s. convergence of $|\hat{a}_N - a| \to 0$ (in probability) for the selected choices of α and β.

Because $\mathbb{P}(|A_N| > \epsilon) \leq \mathbb{E}(A_N^2)/\epsilon^2$, following Remark 3, it is enough to control the expression $\mathbb{E}\left[A_N^2\right]$ in (3.5). Then,

$$\begin{aligned}
\mathbb{E}\left[A_N^2\right] &= \mathbb{E}\left[\frac{1}{N^2}\sum_{i=0}^{N-1}\tau_i^2\left(B^H_{\tau_{i+1}}-B^H_{\tau_i}\right)^2\right] \\
&\quad + \mathbb{E}\left[\frac{1}{N^2}\sum_{0\le i,j\le N-2;|i-j|=1}\tau_i\tau_j\left(B^H_{\tau_{i+1}}-B^H_{\tau_i}\right)\left(B^H_{\tau_{j+1}}-B^H_{\tau_j}\right)\right] \\
&\quad + \mathbb{E}\left[\frac{1}{N^2}\sum_{0\le i,j\le N-1;|i-j|\ge 2}\tau_i\tau_j\left(B^H_{\tau_{i+1}}-B^H_{\tau_i}\right)\left(B^H_{\tau_{j+1}}-B^H_{\tau_j}\right)\right] \\
&:= \mathbb{E}\left[A_N^{(1)}\right]+\mathbb{E}\left[A_N^{(2)}\right]+\mathbb{E}\left[A_N^{(3)}\right], \qquad (3.6)
\end{aligned}$$

where we split the sum into three terms associated with the distance of the indeces. First, we study the first term in (3.6). From hypotheses **N1** to **N3**:

Step 1.

$$\begin{aligned}
\mathbb{E}\left[A_N^{(1)}\right] &= \frac{1}{N^2}\mathbb{E}\left[\sum_{i=0}^{N-1}\tau_i^2\left(B^H_{\tau_{i+1}}-B^H_{\tau_i}\right)^2\right] \\
&= \frac{1}{N^2}\sum_{i=0}^{N-1}\int_0^\infty\int_0^\infty \mathbb{E}\left(z_i^2\left(B^H_{z_i+a}-B^H_{z_i}\right)^2|\tau_i=z_i,t_{i+1}=a\right)g_{N,i}(z_i)g_N(a)dz_ida \\
&= \frac{1}{N^2}\sum_{i=0}^{N-1}\int_0^\infty\int_0^\infty z_i^2a^{2H}g_{N,i}(z_i)g_N(a)dz_ida = \frac{1}{N^2}\sum_{i=0}^{N-1}\mathbb{E}\left[\tau_i^2\right]\mathbb{E}\left[t_{i+1}^{2H}\right].
\end{aligned}$$

By (2.3), hypothesis **H2**, and the independence of the r.v.'s τ_i and t_{i+1},

$$\begin{aligned}
\mathbb{E}\left[A_N^{(1)}\right] &\le \frac{C}{N^2}\sum_{i=0}^{N-1}\left[(i+1)\mathbb{E}\left[t_0^2\right]+\frac{i(i+1)}{N^2}\right]\left(\mathbb{E}\left[t_0^2\right]\right)^H \\
&\le \frac{C}{N^{\alpha(H+1)\wedge(\alpha H+1)}} = \frac{C}{N^{\alpha H+1}}. \qquad (3.7)
\end{aligned}$$

Step 2. By symmetry, w.l.o.g, we consider $j=i+1$. From hypothesis **N3**,

$$\begin{aligned}
\mathbb{E}\left[A_N^{(2)}\right] &= \frac{2}{N^2}\sum_{i=1}^{N-2}\mathbb{E}\left[\mathbb{E}\left[(\tau_{i-1}+t_i)\left(\tau_{i-1}+t_i+t_{i+1}\right)\left(B^H_{\tau_{i-1}+t_i}-B^H_{\tau_{i-1}}\right)\right.\right. \\
&\quad \cdot \left.\left.\left(B^H_{\tau_{i-1}+t_i+t_{i+1}}-B^H_{\tau_{i-1}+t_i}\right)\right|\tau_{i-1}=z_{i-1},t_i=a,t_{i+1}=b\right]\right] \\
&= \frac{2}{N^2}\sum_{i=1}^{N-2}\int_0^\infty\int_0^\infty\int_0^\infty (z_{i-1}+a)\left(z_{i-1}+a+b\right) \qquad (3.8)
\end{aligned}$$

$$\cdot \mathbb{E}\left[\left(B^H_{z_{i-1}+a+b} - B^H_{z_{i-1}+a}\right)\left(B^H_{z_{i-1}+a} - B^H_{z_{i-1}}\right)\right]$$
$$g_{N,i-1}(z_{i-1})g_N(a)g_N(b)dz_{i-1}dadb.$$

Hypothesis **N1** implies

$$\mathbb{E}\left[\left(B^H_{z_{i-1}+a+b} - B^H_{z_{i-1}+a}\right)\left(B^H_{z_{i-1}+a} - B^H_{z_{i-1}}\right)\right] = \frac{1}{2}\left[(a+b)^{2H} - a^{2H} - b^{2H}\right].$$

Then,

$$\mathbb{E}\left[A_N^{(2)}\right] = \frac{1}{N^2}\sum_{i=1}^{N-2}\int_0^\infty\int_0^\infty\int_0^\infty (z_{i-1}+a+b)(z_{i-1}+a)\left[(a+b)^{2H} - a^{2H} - b^{2H}\right]$$
$$\cdot\, g_{N,i-1}(z_{i-1})g_N(a)g_N(b)dz_{i-1}dadb. \tag{3.9}$$

We consider $\sigma(y) = y^{2H}$. Using a Taylor-type expansion, we linearise the increment of σ between the two points $a+b$ and a, as follows:

$$\begin{aligned}\sigma(a+b) - \sigma(a) &= \int_0^1 \sigma'\left[(1-\lambda)a + \lambda(a+b)\right] b d\lambda \\ &= \int_0^1 \sigma'\left[a+\lambda b\right] b d\lambda = 2H\int_0^1 \left[a+\lambda b\right]^{2H-1} b d\lambda \end{aligned} \tag{3.10}$$

From equality (3.10) and the fact that σ' is increasing and $\lambda \in (0,1)$ we have

$$\frac{1}{2}\left[(a+b)^{2H} - (a)^{2H}\right] \leq H\left[a+b\right]^{2H-1} b. \tag{3.11}$$

Then, by (3.9) and (3.11), we get

$$\begin{aligned}\mathbb{E}\left[A_N^{(2)}\right] &\leq \frac{C}{N^2}\sum_{i=1}^{N-2}\int_0^\infty\int_0^\infty\int_0^\infty (z_{i-1}+a+b)(z_{i-1}+a)\left[a+b\right]^{2H-1} b \\ &\quad \cdot g_{N,i-1}(z_{i-1})g_N(a)g_N(b)dz_{i-1}dadb. \\ &\leq \frac{C}{N^2}\sum_{i=1}^{N-2}\int_0^\infty\int_0^\infty\int_0^\infty \Big(z_{i-1}(z_{i-1}+a)\left[a+b\right]^{2H-1} b + (z_{i-1}+a) \\ &\quad \left[a+b\right]^{2H} b\Big) \cdot g_{N,i-1}(z_{i-1})g_N(a)g_N(b)dz_{i-1}dadb.\end{aligned}$$

Now, for $1 \leq 2H \leq 2$, $f(x) = x^{2H}$ is a convex function, and for $2H-1 < 1$, $g(x) = x^{2H-1}$ is a subadditive function. Thus we have

$$\mathbb{E}\left[A_N^{(2)}\right] \leq \frac{C}{N^2}\sum_{i=1}^{N-2}\int_0^\infty\int_0^\infty\int_0^\infty \big(z_{i-1}(z_{i-1}+a)\left[a^{2H-1}+b^{2H-1}\right]b$$
$$+(z_{i-1}+a)\left[a^{2H}+b^{2H}\right]b\big)\cdot g_{N,i-1}(z_{i-1})g_N(a)g_N(b)dz_{i-1}dadb.$$
$$\leq \frac{C}{N^2}\sum_{i=1}^{N-2}\int_0^\infty\int_0^\infty\int_0^\infty (z_{i-1}^2a^{2H-1}b+z_{i-1}^2b^{2H}+z_{i-1}a^{2H}b$$
$$+z_{i-1}ab^{2H}+z_{i-1}a^{2H}b+z_{i-1}b^{2H+1}+a^{2H+1}b+ab^{2H+1})$$
$$\cdot g_{N,i-1}(z_{i-1})g_N(a)g_N(b)dz_{i-1}dadb.$$

From the independence of t_i and after some algebraic manipulations,

$$\mathbb{E}\left[A_N^{(2)}\right]$$
$$\leq \frac{C}{N^2}\sum_{i=1}^{N-2}\Big\{\mathbb{E}(\tau_{i-1}^2)\mathbb{E}(t_i^{2H-1})\mathbb{E}(t_{i+1})+\mathbb{E}(\tau_{i-1}^2)\mathbb{E}(t_{i+1}^{2H})$$
$$+\mathbb{E}(\tau_{i-1})\mathbb{E}(t_i^{2H})\mathbb{E}(t_{i+1})+\mathbb{E}(\tau_{i-1})\mathbb{E}(t_i)\mathbb{E}(t_{i+1}^{2H})+\mathbb{E}(\tau_{i-1})\mathbb{E}(t_i^{2H})\mathbb{E}(t_{i+1})$$
$$+\mathbb{E}(\tau_{i-1})\mathbb{E}(t_{i+1}^{2H+1})+\mathbb{E}(t_i^{2H+1})\mathbb{E}(t_{i+1})+\mathbb{E}(t_i)\mathbb{E}(t_{i+1}^{2H+1})\Big\}.$$

By equalities (2.2) and (2.3) and hypotheses **H1** to **H3**, we can derive the following inequality for $\mathbb{E}\left[A_N^{(2)}\right]$:

$$\mathbb{E}\left[A_N^{(2)}\right] \leq \frac{C}{N^2}\sum_{i=1}^{N-2}\Bigg\{\left(\frac{i}{N^\alpha}+\frac{i^2}{N^2}\right)\frac{1}{N^{2H-1}}\frac{1}{N}+\left(\frac{i}{N^\alpha}+\frac{i^2}{N^2}\right)\frac{1}{N^{\alpha H}}$$
$$+\frac{i}{N}\frac{1}{N^{\alpha H}}\frac{1}{N}+\frac{i}{N}\frac{1}{N}\frac{1}{N^{\alpha H}}+\frac{i}{N}\frac{1}{N}\frac{1}{N^{\alpha H}}$$
$$+\frac{i}{N}\frac{1}{N^{\beta((2H+1)/4)}}+\frac{1}{N}\frac{1}{N^{\beta((2H+1)/4)}}+\frac{1}{N}\frac{1}{N^{\beta((2H+1)/4)}}\Bigg\} \tag{3.12}$$
$$\leq C\Bigg\{\frac{1}{N^{\alpha+2H}}+\frac{1}{N^{1+2H}}+\frac{1}{N^{\alpha+\alpha H}}+\frac{1}{N^{1+\alpha H}}+\frac{3}{N^{2+\alpha H}}$$
$$+\frac{1}{N^{\beta((2H+1)/4)+1}}+\frac{2}{N^{\beta((2H+1)/4)+2}}\Bigg\}.$$

Finally,

$$\mathbb{E}\left[A_N^{(2)}\right] \leq \frac{C}{N^{(1+\alpha H)\wedge(\beta((2H+1)/4)+1)}}. \tag{3.13}$$

Step 3. W.l.o.g., we assume that $i < j$. By symmetry, we have

$$\mathbb{E}\left[A_N^{(3)}\right] = \frac{2}{N^2}\mathbb{E}\left[\sum_{0\le i,j\le N-1;j-i\ge 2} \tau_i\tau_j\left(B^H_{\tau_{i+1}} - B^H_{\tau_i}\right)\left(B^H_{\tau_{j+1}} - B^H_{\tau_j}\right)\right].$$

We denote $X_{j-i-1} = X_{j-(i+1)} = \tau_j - \tau_{i+1} = \sum_{l=i+2}^{j} t_l$, which is independent of τ_{i+1} and t_{j+1}. By the independence of t_i, the r.v.'s X_{j-i-1} are distributed as $G_{N,j-i-1}$. Then, by hypothesis **N3**,

$$\begin{aligned}\mathbb{E}\left[A_N^{(3)}\right] =& \frac{2}{N^2}\sum_{0\le i,j\le N-1;j-i\ge 2} \mathbb{E}\left[\mathbb{E}\left[\tau_i\left(\tau_i + t_{i+1} + X_{j-i-1}\right)\left(B^H_{\tau_i+t_{i+1}} - B^H_{\tau_i}\right)\right.\right.\\ &\left(B^H_{\tau_i+t_{i+1}+X_{j-i-1}+t_{j+1}} - B^H_{\tau_i+t_{i+1}+X_{j-i-1}}\right)|\tau_i = z_i, t_{i+1} = a,\\ &\left.\left. X_{j-i-1} = x, t_{j+1} = b\right]\right]\\ =& \frac{2}{N^2}\sum_{0\le i,j\le N-1;j-i\ge 2}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty z_i\left(z_i + a + x\right)\\ &\mathbb{E}\left[\left(B^H_{z_i+a} - B^H_{z_i}\right)\left(B^H_{z_i+a+x+b} - B^H_{z_i+a+x}\right)\right]\\ &g_{N,i}(z_i)g_{N,j-i-1}(x)g_N(a)g_N(b)dz_i\,dx\,da\,db.\end{aligned}$$

From hypothesis **N1**,

$$\begin{aligned}&\mathbb{E}\left[\left(B^H_{z_i+a} - B^H_{z_i}\right)\left(B^H_{z_i+a+x+b} - B^H_{z_i+a+x}\right)\right]\\ &= \frac{1}{2}\left[(a + x + b)^{2H} + x^{2H} - (x + b)^{2H} - (a + x)^{2H}\right]\\ &\le \frac{1}{2}\left[(a + x + b)^{2H} - (a + x)^{2H}\right]. \qquad (3.14)\end{aligned}$$

Again, consider $\sigma(y) = y^{2H}$. Using a Taylor-type expansion, one can linearise the increment of σ between the two points $a + x + b$ and $a + x$ as

$$\begin{aligned}\sigma(a + x + b) - \sigma(a + x) &= \int_0^1 \sigma'\left[(1-\lambda)(a + x) + \lambda(a + x + b)\right] bd\lambda\\ = \int_0^1 \sigma'\left[a + x + \lambda b\right] bd\lambda &= 2H\int_0^1 \left[a + x + \lambda b\right]^{2H-1} bd\lambda.\end{aligned}$$

Now, because $f(x) = x^{2H-1}$ is an increasing function in x and $\lambda \in (0, 1)$, equality (3.15) yields

$$\sigma(a + x + b) - \sigma(a + x) \le 2H\left[a + x + b\right]^{2H-1} b.$$

Then, from the definiton of σ, we have

$$\frac{1}{2}\left[(a+x+b)^{2H}-(a+x)^{2H}\right] \leq H\left[a+x+b\right]^{2H-1} b. \tag{3.15}$$

Using the fact that $2H-1<1$, we can use the subadditive property of the function $f(x)=x^{2H-1}$:

$$\begin{aligned}
&\mathbb{E}\left[A_N^{(3)}\right] \\
&\leq \frac{C}{N^2}\sum_{0\leq i,j\leq N-1; j-i\geq 2}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty z_i\left(z_i+a+x\right)\left(a+x+b\right)^{2H-1} b \\
&g_{N,i}(z_i)\, g_{N,j-i-1}(x)\, g_N(a)\, g_N(b)\, dz_i\, dx\, da\, db \\
&\leq \frac{C}{N^2}\sum_{0\leq i,j\leq N-1; j-i\geq 2}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty z_i\left(z_i+a+x\right)\left((a+x)^{2H-1}b+b^{2H}\right) \\
&g_{N,i}(z_i)\, g_{N,j-i-1}(x)\, g_N(a)\, g_N(b)\, dz_i\, dx\, da\, db \\
&= \frac{C}{N^2}\sum_{0\leq i,j\leq N-1; j-i\geq 2}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty [z_i^2(a+x)^{2H-1}b+z_i^2 b^{2H} \\
&\quad + z_i(a+x)^{2H}b + z_i(a+x)b^{2H}] g_{N,i}(z_i)\, g_{N,j-i-1}(x)\, g_N(a)\, g_N(b)\, dz_i\, dx\, da\, db.
\end{aligned}$$

By the independence of the r.v.'s t_i and after some algebraic manipulations, we can obtain the following inequality:

$$\begin{aligned}
\mathbb{E}\left[A_N^{(3)}\right] \leq \frac{C}{N^2}\Bigg\{ &\sum_{0\leq i,j\leq N-1; j-i\geq 2} \mathbb{E}\left[\tau_i^2\right]\mathbb{E}\left[t_0\right]\mathbb{E}\left[X_{j-i-1}^{2H-1}\right] + \mathbb{E}\left[\tau_i^2\right]\mathbb{E}\left[t_0^{2H}\right] \\
&+ \mathbb{E}\left[\tau_i\right]\mathbb{E}\left[t_0\right]\mathbb{E}\left[X_{j-i-1}^{2H}\right] + \mathbb{E}\left[\tau_i\right]\mathbb{E}\left[X_{j-i-1}\right]\mathbb{E}\left[t_0^{2H}\right]\Bigg\}.
\end{aligned}$$

In addition,

$$\begin{aligned}
\mathbb{E}\left[X_{j-i-1}^{2H}\right] &= \mathbb{E}\left[\left(t_{i+2}+\cdots+t_j\right)^{2H}\right] \\
&\leq (j-i-1)^{2H-1}\mathbb{E}\left[\left(t_{i+2}^{2H}+\cdots+t_j^{2H}\right)\right] \\
&= (j-i-1)^{2H}\mathbb{E}\left[t_0^{2H}\right], \\
\text{and}\quad \mathbb{E}\left[X_{j-i-1}^{2H-1}\right] &\leq (j-i-1)^{2H-1}\mathbb{E}\left[t_0^{2H-1}\right].
\end{aligned}$$

From (2.2), (2.3), and hypotheses **H1** and **H2**, we obtain

$$\mathbb{E}\left[A_N^{(3)}\right] \leq \frac{C}{N^2}\Bigg\{ \sum_{0\leq i,j\leq N-1; j-i\geq 2} \left(\frac{i+1}{N^\alpha} + \frac{i(i+1)}{N^2}\right)\frac{1}{N}\frac{(j-i-1)^{2H-1}}{N^{2H-1}} + \left(\frac{i+1}{N^\alpha} + \frac{i(i+1)}{N^2}\right)\frac{1}{N^{\alpha H}} + \frac{(i+1)}{N}\frac{1}{N}\frac{(j-i-1)^{2H}}{N^{\alpha H}} + \frac{(i+1)}{N}\frac{1}{N^{\alpha H}}\frac{(j-i-1)}{N}\Bigg\}$$

By collecting the pieces, we have the following expression:

$$\mathbb{E}\left[A_N^{(3)}\right] \leq C\Bigg\{\frac{1}{N^{1+\alpha}} + \left(\frac{1}{N^{\alpha H}} + \frac{1}{N^{\alpha+\alpha H-1}}\right) + \frac{1}{N^{\alpha H+2-2H}} + \frac{1}{N^{1+\alpha H}} + \frac{1}{N^{\alpha+2H-1}} + \frac{1}{N^{2H}} + \frac{2}{N^{1+\alpha H}}\Bigg\} \leq \frac{C}{N^{(\alpha H)\wedge(\alpha+2H-1)}} = \frac{C}{N^{\alpha H}}. \tag{3.16}$$

Finally, for (3.6), considering the above constraints and by combining equations (3.7), (3.13), and (3.16), we get

$$\mathbb{E}\left[A_N^2\right] \leq \frac{C}{N^{\alpha H+1}} + \frac{C}{N^{(1+\alpha H)\wedge(\beta((2H+1)/4)+1)}} + \frac{C}{N^{\alpha H}} \leq \frac{C}{N^{\alpha H\wedge(\beta((2H+1)/4)+1)}}. \tag{3.17}$$

The convergence in L^2, and therefore in probability, of A_N in equation (3.5) is ensured. The a.s. convergence comes from the Tshebyshev inequality, Borel–Cantelli lemma, as in Remark 3, and setting $1/2 \leq H \leq 1$ and $\alpha > 1/H \geq 1$.

By Slutsky's Theorem, Lemma 1, and Table 4 in Appendix 5, the convergence of $|\hat{a}_N - a| \to 0$ is

- in probability for $2 > \alpha > 1$ and $\beta > 1$;
- a.s. for $2 > \alpha > \max\{3/2, 1/H\}$ and $\beta > 2$.

Remark 4. If $H = 1/2$, the i.i.d. case, by carefully examining the proofs in Section 3, starting in equation (3.6), we obtain that the statement of Theorem 1 is valid with upper bound an rate of convergence controlled by the denominator D_N.

The exponential distribution is a particular limiting case that arises when $\alpha = 2$ and $\beta = 4$. The behavior of A_N and D_N, in this particular case, is shown in Araya et al. (2019).

Proof of Proposition 2

Proof. Let us consider $\hat{a}_{N(1)}$ defined by (3.1), with $N(1)$ the number of observations up to time $T = 1$. We can ensure that $D_N - D_{N(1)}$ and $A_N - A_{N(1)}$, defined in (3.2) and (3.5), respectively, converge to zero a.s. when $N \to \infty$. In fact, note that if $N(1) < N - 1$, then $\tau_{N(1)} < 1 < \tau_{N-1}$, and if $N - 1 < N(1)$, then $\tau_{N-1} < \tau_{N(1)} < 1$. Thus,

$$|D_N - D_{N(1)}| \leq \frac{|N(1) - N + 1|}{N} \tau^2_{(N(1) \vee (N-1))} \leq \frac{|N(1) - N + 1|}{N} (\tau^2_{N-1} \vee 1).$$

From the a.s. convergence of $\tau_N \to 1$ and the convergence of $N(1)/N \to 1$, obtained in Section 2.3, equation (2.6), and using the same arguments presented in Remark 3, we have that $|D_N - D_{N(1)}|$ converges in probability to zero if $\alpha > 1$ and $\beta > 1$. The a.s. convergence is achieved when $\alpha > 3/2$ and $\beta > 2$.

We now recall the Garsia-Rodemich-Rumsey Lemma in Garsia et al. (1970), as well as Barlow and Yor (1982), which are powerful works in the study of the sample path Hölder continuity of a stochastic process adapted to random times; see Russo and Vallois (1993) and Nualart and Răscanu (2002).

Lemma 2. *Let $p \geq 1$, and $u > p - 1$. Then, there exists a constant $C_{u,p} > 0$ such that for any continuous function f on $[0, T]$ and for all $t, s \in [0, T]$, one has*

$$|f(t) - f(s)|^p \leq C_{u,p} |t - s|^{up-1} \leq \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^{up+1}} dxdy,$$

with $0/0 = 0$.

The following lemma provides the basic inequalities for the fBm.

Lemma 3. *Let $\{B_t^H : t \geq 0\}$ be a fractional Brownian motion of Hurst parameter $H \in (0, 1)$. Then for every $0 < \epsilon < H$ and $T > 0$ there exists a positive random variable $\eta_{\epsilon,T}$ such that $\mathbb{E}(|\eta_{\epsilon,T}|^p) < \infty$, for all $p \in [1, \infty)$ and for all $s, t \in [0, T]$ $|B_t^H - B_s^H| \leq \eta_{\epsilon,T} |t - s|^{H-\epsilon}$ a.s.*

See Nualart and Răscanu (2002) for details.

From (2.5), (2.6), and the Tshebyshev inequality, we obtain

$$\mathbb{P}\left(\left| \frac{N(1)}{N} - 1 \right| > N^{-\delta} \right) \leq \frac{C}{N^{(\beta-1-3\delta) \wedge (2\alpha-2-2\delta)}}. \tag{3.18}$$

If $\delta < \{((\beta - 2)/3) \wedge (\alpha - 3/2)\}$, applying the Borel–Cantelli lemma, the a.s. convergence is ensured. We define Ω_N^1 as the set of $\omega \in \Omega$ such that $|N(1)/N - 1| \leq$

$N^{-\delta}$. Then, $\mathbb{P}(\Omega_N^1) \geq 1 - C/N^{\mu}$, with $0 < \mu = \{(2\alpha - 2 - 2\delta) \wedge (\beta - 1 - 3\delta)\}$. Furthermore,

$$\mathbb{P}(|\tau_N - 1| > N^{-4\gamma}) \leq \frac{\mathbb{E}[(\tau_N - 1)^4]}{N^{-4\gamma}} \leq \frac{C}{N^{-4\gamma+\{(2\alpha-2)\wedge(\beta-1)\}}}. \tag{3.19}$$

If $\alpha > 3/2$, $\beta > 2$ and $0 < \gamma < \{(2\alpha - 3)/4 \wedge (\beta - 2)/4\}$ the Borel–Cantelli Lemma implies the a.s. convergence of $|\tau_N - 1|$ to zero. Define Ω_N^2, the set of $\omega \in \Omega$ such that $|\tau_N(\omega) - 1| \leq N^{-\gamma}$. Then, $\mathbb{P}(\Omega_N^2) \geq 1 - C/N^{\nu}$, with $0 < \nu = -4\gamma + \{(2\alpha - 2) \wedge (\beta - 1)\}$ and $\tau_N \leq 1 + N^{-\gamma}$ a.s.

In the set $\Omega_N^1 \cap \Omega_N^2$, with $\mathbb{P}(\Omega_N^1 \cap \Omega_N^2) \geq 1 - C/N^{\nu} - C/N^{\mu}$, and applying the Garsia-Rodemich-Rumsey Lemma to the random interval $[0, \tau_N(\omega)]$, we have, for any $\varepsilon > 0$,

$$\sup_{0 \leq s \leq t \leq \tau_N} |B_t^H - B_s^H| \leq C_{\nu,H}(1 + N^{-\gamma})^{H-\varepsilon} \tau_N^{H-\varepsilon} \xi_N, \tag{3.20}$$

where ξ_N is a random variable such that, for $q \geq 2/\varepsilon$, and on $\Omega_N^1 \cap \Omega_N^2$,

$$\mathbb{E}(\xi_N^q) \leq C_{q,\varepsilon}(1 + N^{-\gamma})^{\varepsilon q}. \tag{3.21}$$

Then, from the previous analysis of (3.20) and (3.21), we obtain on $\Omega_N^1 \cap \Omega_N^2$,

$$|A_N - A_{N(1)}| = \left| \frac{1}{N} \sum_{i=N(1)\wedge(N-1)}^{N(1)\vee(N-1)} \tau_i \Delta B_{\tau_i}^H \right| \leq \frac{\xi_N \, C_{\nu,H}(1+N^{-\gamma})^{1+2H-2\varepsilon}}{N^{\delta}}. \tag{3.22}$$

Let us now consider the random variable ξ_N on Ω_N^1 where, in that case, $\tau_N \leq 1 + N^{-\gamma}$ a.s. By (3.21), with $q = 2/\varepsilon$ we get, $\mathbb{E}(\xi_N^{2/\varepsilon}) \leq C_{2,\varepsilon}(1 + N^{-\gamma})^2$.

For $\rho > 0$, $\mathbb{P}(\xi_N > N^{\rho}) < \mathbb{E}(\xi_N^{2/\varepsilon})/N^{2\rho/\varepsilon} \leq C_{2,\varepsilon}(1 + N^{-\gamma})^2/N^{2\rho/\varepsilon}$, where the last quantity goes to 0 as N goes to infinity for any $\rho > 0$. We can ensure the a.s. convergence for $\rho > \varepsilon/2$.

Thus, considering Ω_N^3, the set of $\omega \in \Omega$ such that $\xi_N(\omega) \leq N^{\rho}$, we have $\mathbb{P}(\Omega_N^3) \geq 1 - C/N^{2\rho/\varepsilon}$, with $\rho > \varepsilon/2$. Define $\Omega_N = \Omega_N^1 \cap \Omega_N^2 \cap \Omega_N^3$. Then, $\mathbb{P}(\Omega_N) \geq 1 - C/N^{\nu} - C/N^{\mu} - C/N^{2\rho/\varepsilon}$ and, on Ω_N,

$$|A_N - A_{N(1)}| \leq C_{\nu,H} C_{2,\varepsilon}(1 + N^{-\gamma})^{1+2H-2\varepsilon} N^{\rho-\delta}. \tag{3.23}$$

For sufficiently small ε we can choose $(\beta-2)/3 > \varepsilon/2$ and $\alpha-3/2 > \varepsilon/2$, implying that $\rho - \delta < 0$. This implies the a.s. convergence on Ω_N of $|A_N - A_{N(1)}|$ to zero as N goes to infinity. In addition $\mathbb{P}(\Omega_N) \to 1$.

4. Simulation Study

Throughout this section, we develop a Monte Carlo simulation study to assess the finite-sample properties for the LS estimator in the linear regression model (1.1). The long-memory noise driven is by a fractional Brownian motion evaluated at deterministic times and two different random times defined by Equation (2.1

The deterministic case: We consider the model defined by equation (1.1) observed at equally spaced times, that is, $\tau_i = i/N$, for $i = 1, \ldots, N$. We consider $N = 200$.

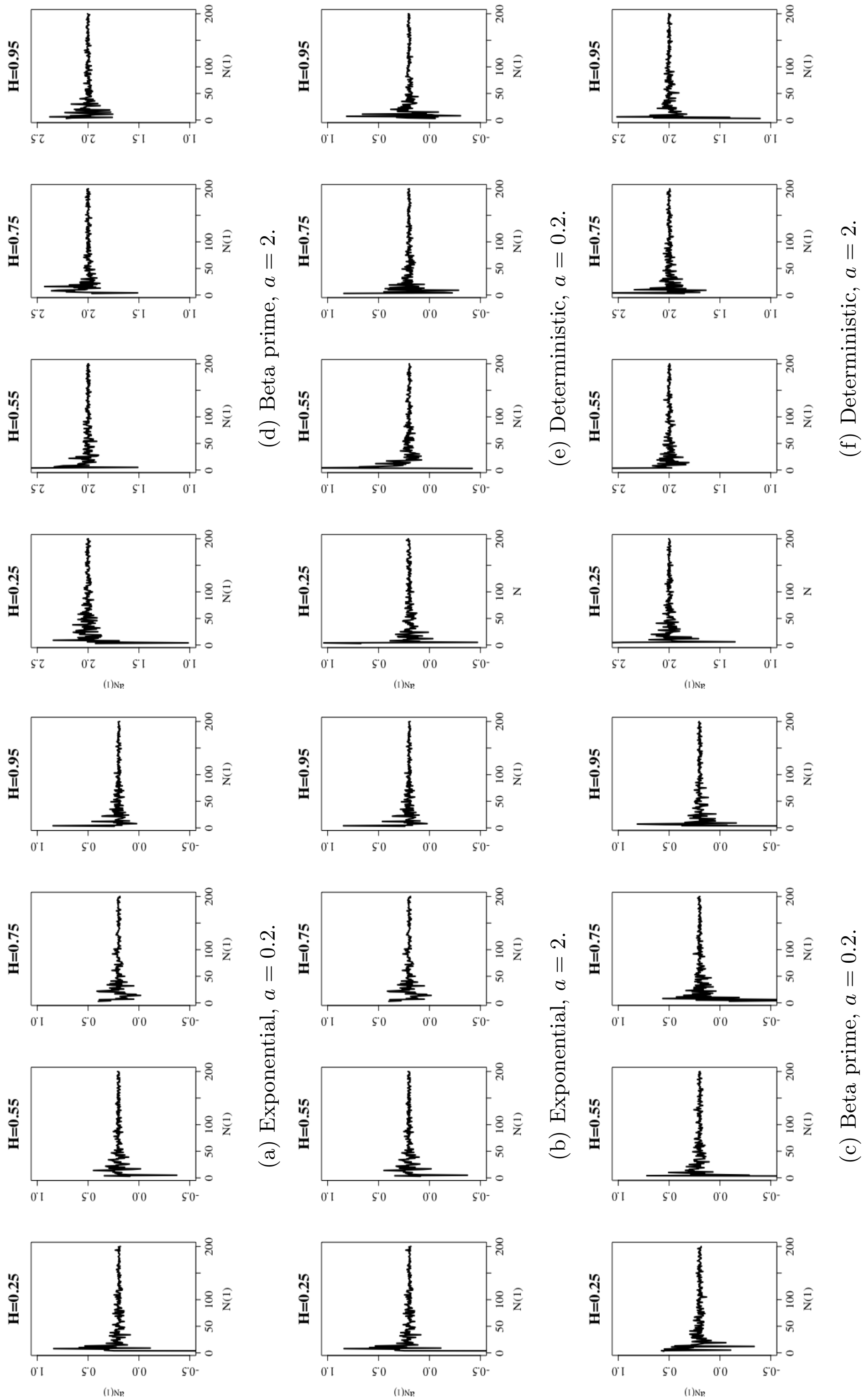
The exponential and beta prime case: The most studied renewal process is the Poisson process (e.g., Last and Penrose (2017)), which appears when t_i has an exponential distribution (λ). We consider $\lambda = 200$. For the beta prime distribution, we consider a distribution with parameters $(1, 201)$.

For all the simulations shown, we consider $M = 1{,}000$ replicates of the model with the parameters $a = 0.2$ and $a = 2$. For the exponential and beta prime cases, the number of observations is a random variable $N(1)$, representing how many observations are within the interval $[0, 1]$. We also consider different values of the Hurst parameter: $H = 0.05$, $H = 0.25$ and $H = 0.45$ (anti-persistent cases); and $H = 0.55$, $H = 0.75$, and 0.95 (long-memory cases).

Figure 2 shows the value of $\hat{a}_{N(1)}$ for different values of $N(1)$ between 3 and 200 and for different values of H. The parametric estimation stabilizes when there are approximately 100 observations per realization, either when $a = 0.2$ or $a = 2$, even if the noise is driven by an anti-persistent process ($H = 0.25$). However, for values of H less than 1/2, strong consistency is not ensured by the proposed method studied. From the results of equation (3.17) and in Table 4, we obtain an upper bound on the convergence rate of $|a_{N(1)} - a|$. When an exponential distribution of parameter $\lambda = N$ is considered ($\alpha = 2$ and $\beta = 4$), the upper bound is given by C/N for the convergence in probability. For a.s. convergence, the upper bound is C/N^{2H-1}. The latter bound coincides when the times are considered to be equally spaced. The following figure shows $\log |a_{N(1)} - a|$ as a function of $\log(N)$. The values in Table 1 correspond to the LS estimators of the parameter μ ($\hat{\mu}_{LS}$) in the equation

$$\log |a_{N(1)} - a| = \tilde{C} + \mu \log(N), \quad 3 \leq N \leq 200,$$

where $\tilde{C} = \log(C)$, and μ corresponds to the exponent in the convergence rate in probability.



(a) Exponential, $a = 0.2$.

(b) Exponential, $a = 2$.

(c) Beta prime, $a = 0.2$.

(d) Beta prime, $a = 2$.

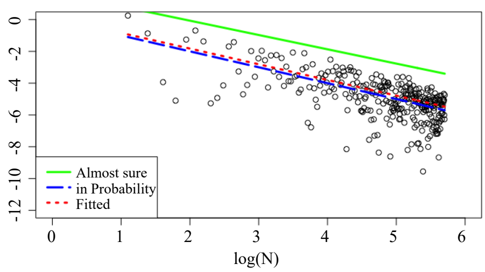
(e) Deterministic, $a = 0.2$.

(f) Deterministic, $a = 2$.

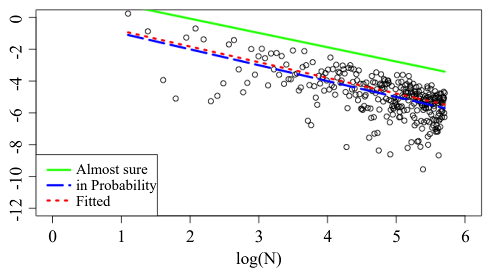
Figure 2. Rate of convergence of $\hat{a}_N$ and $\hat{a}_{N(1)}$ under deterministic, exponential, and beta prime distributions, and different values of H: (left) case $a = 0.2$, (right) case $a = 2$.

Table 1. Values of $\hat{\mu}_{LS}$.

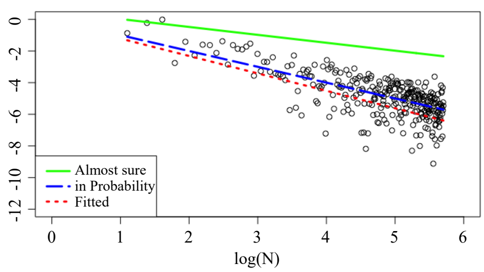
	$H = 0.75$	$H = 0.95$
Exponential random times	-1.053	-0.957
Deterministic times	-1.129	-1.031



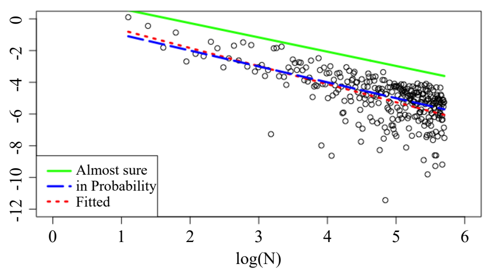
(a) Exponential, $H = 0.75$.



(b) Exponential, $H = 0.95$.



(c) Deterministic, $H = 0.75$.



(d) Deterministic, $H = 0.95$.

Figure 3. Rate of convergence bounds of $\log(|\hat{a}_{N(1)} - a|)$ under exponential and deterministic case, for $a = 0.2$.

Figure 3 compares the regression model with the estimated parameter $\hat{\mu}_{LS}$ (red dotted line) and $-\log(N)$, for $3 \leq N \leq 200$ (blue long dashed line). The green solid line represents $-(2H-1)\log(N)$, $3 \leq N \leq 200$.

The simulations in Figure 3 verify numerically our results for the upper bound on the convergence rate. Finally, the simulations show that as H increases, the slope fit improves.

Tables 2 and 3 show the mean, standard deviation (SD), and kurtosis (defined as the difference between the kurtosis of a Gaussian distribution of the simulated process) according to Equation (3.1), for different values of H, and $M = 1{,}000$ replicates of the process.

The values of the mean show that the estimator is unbiased. Note that the SD decreases as the value of H approaches one, which is expected, because the

Table 2. Exponential and beta prime: Mean, SD, and kurtosis with $a = 0.2$.

Exponential	$H = 0.05$	$H = 0.25$	$H = 0.45$	$H = 0.55$	$H = 0.75$	$H = 0.95$
Mean	0.2001	0.1995	0.1997	0.2001	0.1999	0.1998
SD	0.0074	0.0064	0.0062	0.0057	0.0055	0.0051
Kurtosis	0.2879	0.1132	0.0150	-0.04568	0.0721	-0.3278
Beta prime	$H = 0.05$	$H = 0.25$	$H = 0.45$	$H = 0.55$	$H = 0.75$	$H = 0.95$
Mean	0.2003	0.2001	0.2000	0.2001	0.2000	0.2001
SD	0.0073	0.0065	0.0061	0.0059	0.0054	0.0052
Kurtosis	0.0678	-0.1624	0.1858	0.1662	0.0249	-0.0189

Table 3. Exponential and beta prime: Mean, SD, and kurtosis with $a = 2$.

Exponential	$H = 0.05$	$H = 0.25$	$H = 0.45$	$H = 0.55$	$H = 0.75$	$H = 0.95$
Mean	1.9997	2.0002	1.9997	2.0002	2.0001	2.0002
SD	0.0072	0.0066	0.0061	0.0059	0.0057	0.0051
Kurtosis	0.1402	-0.1039	-0.0092	-0.1116	0.3593	0.0635
Beta prime	$H = 0.05$	$H = 0.25$	$H = 0.45$	$H = 0.55$	$H = 0.75$	$H = 0.95$
Mean	1.9998	2.0001	1.9999	2.0001	1.9998	2.0001
SD	0.0072	0.0067	0.0059	0.0058	0.0055	0.0050
Kurtosis	0.0670	0.1027	-0.0787	-0.0771	0.3047	0.2196

conditional variance of the noise decreases as H approaches one. In fact, for $0 \le \tau_i < \tau_j \le 1$, with $0 \le i < j < N(1)$,

$$Var\left(\left(\frac{B^H_{\tau_i} - B^H_{\tau_j}}{\tau_i}\right), \tau_j\right) = \int_0^1 \int_0^1 |t-s|^{2H} g_{N,i}(t) g_{N,j}(s) dt ds.$$

Then, for $0 \le \tau_i < \tau_j \le 1$, $Var((B^{H'}_{\tau_i} - B^{H'}_{\tau_j})/\tau_i, \tau_j) \le Var((B^H_{\tau_i} - B^H_{\tau_j})/\tau_i, \tau_j)$ when $H' > H$.

5. Proof of Lemma 1

Proof. Let us consider D_N with sampling random times as in (2.1). However, as we show below, D_N can be written in quadratic form, depending on the increments t_i, which are i.i.d. r.v.'s. We have

$$\begin{aligned} D_N &= \frac{1}{N}\sum_{k=0}^{N-1} \tau_k^2 = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{i=0}^{k} t_i\right)^2 = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{i=0}^{k}\sum_{j=0}^{k} t_i t_j \\ &= \frac{1}{N}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}(N-(i\vee j))\, t_i t_j \end{aligned}$$

$$= \frac{1}{N}\sum_{i=0}^{N-1}(N-i)\,t_i^2 + \frac{2}{N}\sum_{0\le i<j\le N-1}(N-(i\vee j))\,t_it_j.$$

Inspired by Dacunha-Castelle and Fermín (2006), we center the sequence and add terms to decompose D_N as $D_N = R_N + T_N + Q_N + U_N$, where

$$R_N = \mathbb{E}[D_N] = \frac{1}{N}\sum_{i=0}^{N-1}(N-i)\,\mathbb{E}\left[t_i^2\right] + \frac{2}{N}\sum_{0\le i<j\le N-1}(N-j)\,\mathbb{E}[t_it_j], \quad (5.1)$$

$$T_N = \frac{1}{N}\sum_{i=0}^{N-1}(N-i)\left(t_i^2 - \mathbb{E}\left[t_i^2\right]\right), \quad (5.2)$$

$$Q_N = \frac{2}{N}\sum_{0\le i<j\le N-1}\left(t_i\mathbb{E}[t_j] + \mathbb{E}[t_i]\,t_j - 2\mathbb{E}[t_i]\,\mathbb{E}[t_j]\right)(N-j), \quad (5.3)$$

$$U_N = \frac{2}{N}\sum_{0\le i<j\le N-1}\left(t_it_j - t_i\mathbb{E}[t_j] - t_j\mathbb{E}[t_i] + \mathbb{E}[t_i]\,\mathbb{E}[t_j]\right)(N-j). \quad (5.4)$$

Now, we show that R_N converges to $1/3$, and the remaining terms T_N, Q_N, and U_N converge to zero as N goes to infinity. As in the proof of Proposition 1, to prove the convergence results, we use Remark 3. From **H1** to **H3**,

$$\begin{aligned}
R_N &= \frac{1}{N}\left[\sum_{i=0}^{N-1}(N-i)\,\mathbb{E}\left[t_i^2\right] + 2\sum_{0\le i<j\le N-1}(N-j)\,\mathbb{E}[t_i]\,\mathbb{E}[t_j]\right] \\
&= \frac{1}{N}\left[\left(\sum_{i=0}^{N-1} i\right)\mathbb{E}\left[t_0^2\right] + 2\sum_{j=1}^{N-1}\sum_{i=0}^{j-1}(N-j)\,\frac{1}{N}\frac{1}{N}\right] \\
&= \frac{1}{N}\left[\left(\frac{N(N+1)}{2}\right)\mathbb{E}\left[t_0^2\right] + \frac{2}{N^2}\sum_{j=1}^{N-1}(N-j)\,j\right] \\
&= \frac{\mathbb{E}\left[t_0^2\right]}{N}\left[\frac{N(N+1)}{2}\right] + \frac{2}{N^3}\frac{N(N-1)(N+1)}{6} \\
&= \frac{\kappa_1}{2N^{1+\alpha}}\left[N(N+1)\right] + \frac{1}{3N^3}\left[(N-1)N(N+1)\right].
\end{aligned} \quad (5.5)$$

Then, R_N converges pointwise, and so in probability and a.s., to $1/3$ as N goes to infinity, for $\alpha > 1$. Second, we study the a.s. convergence of T_N to zero. Recall that from (5.2), $\mathbb{E}[T_N] = 0$. In addition, we compute

$$\begin{aligned}\mathbb{E}\left[T_N^2\right] &= \frac{1}{N^2}\left[\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}(N-i)(N-j)\mathbb{E}\left[\left(t_i^2-\mathbb{E}\left[t_i^2\right]\right)\left(t_j^2-\mathbb{E}\left[t_j^2\right]\right)\right]\right] \\ &= \frac{1}{N^2}\sum_{i=0}^{N-1}(N-i)^2 Var\left(t_0^2\right) = \frac{Var\left(t_0^2\right)}{N^2}\left(\sum_{i=1}^{N} i^2\right) \\ &= \frac{Var\left(t_0^2\right)}{6N^2}N(N+1)(2N+1) \leq \frac{C}{N^{(\beta-1)\wedge(2\alpha-1)}}. \end{aligned} \tag{5.6}$$

In (5.6), we can see that for $\alpha > 1/2$ and $\beta > 1$, T_N converges in probability to zero. Furthermore, for $\alpha > 1$ and $\beta > 2$, converges a.s. to zero. From (5.3), $Q_N = (2/N)\sum_{0\leq i<j\leq N-1}\left(t_i(1/N)+t_j(1/N)-2(1/N)(1/N)\right)(N-j)$. Then $\mathbb{E}\left[Q_N\right] = 0$. To simplify the calculation of $\mathbb{E}(Q_N^2)$, we rewrite Q_N as

$$\begin{aligned} Q_N &= \frac{2}{N}\sum_{0\leq i<j\leq N-1}\left(\frac{t_i}{N}-\frac{1}{N^2}\right)(N-j) + \frac{2}{N}\sum_{0\leq i<j\leq N-1}\left(\frac{t_j}{N}-\frac{1}{N^2}\right)(N-j) \\ &= \frac{2}{N}\sum_{i=0}^{N-2}\left(\frac{t_i}{N}-\frac{1}{N^2}\right)\left(\sum_{j=1}^{N-i-1} j\right) + \frac{2}{N}\sum_{i=1}^{N-1}\left(\frac{t_i}{N}-\frac{1}{N^2}\right)(N-i)i \\ &= \frac{2}{N}\sum_{i=0}^{N-1}\left(\frac{t_i}{N}-\frac{1}{N^2}\right)\frac{(N-i)(N-i-1)}{2} + \frac{2}{N}\sum_{i=0}^{N-1}\left(\frac{t_i}{N}-\frac{1}{N^2}\right)(N-i)i \\ &= \frac{2}{N}\sum_{i=0}^{N-1}\left(\frac{t_i}{N}-\frac{1}{N^2}\right)\left[\frac{(N-i)(N-i-1)}{2}+(N-i)i\right] \\ &= \frac{1}{N}\sum_{i=0}^{N-1}(Nt_i-1)\left(\frac{(N-i)(N+i-1)}{N^2}\right) := \frac{1}{N}\sum_{i=0}^{N-1} X_{i,N}a_{i,N}. \end{aligned} \tag{5.7}$$

Note that $X_{i,N} = Nt_i - 1$ is a triangular array of i.i.d. centered random variables, and $a_{i,N} = (N-i)(N+i-1)/N^2$ is a triangular array of constants. We study the a.s. convergence by analyzing the summability of the fourth moment of Q_N. By the independence of the random variables t_i we have

$$\begin{aligned}\mathbb{E}\left[Q_N^4\right] =& \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=0}^{N-1}X_{i,N}a_{i,N}\right)^4\right] = \frac{1}{N^4}\sum_{i=0}^{N-1}a_{i,N}^4\mathbb{E}\left[X_{i,N}^4\right] \\ &+ \frac{6}{N^4}\sum_{0\leq i<j\leq N-1}a_{i,N}^2a_{j,N}^2\mathbb{E}\left[X_{i,N}^2\right]\mathbb{E}\left[X_{j,N}^2\right] \\ &+ \frac{4}{N^4}\sum_{0\leq i<j\leq N-1}a_{i,N}a_{j,N}^3\mathbb{E}\left[X_{i,N}\right]\mathbb{E}\left[X_{j,N}^3\right]\end{aligned}$$

$$+\frac{4}{N^4}\sum_{0\le i<j<k\le N-1} a_{i,N}a_{j,N}a_{k,N}^2\mathbb{E}\left[X_{i,N}\right]\mathbb{E}\left[X_{j,N}\right]\mathbb{E}\left[X_{k,N}^2\right]$$
$$+\frac{1}{N^4}\sum_{i<j<k<l} a_{i,N}a_{j,N}a_{k,N}a_{l,N}\mathbb{E}\left[X_{i,N}\right]\mathbb{E}\left[X_{j,N}\right]\mathbb{E}\left[X_{k,N}\right]\mathbb{E}\left[X_{l,N}\right]. \tag{5.8}$$

Noting that $0<\max_i a_{i,N}\le 1$ and $\mathbb{E}\left[X_{j,N}\right]=0$, for $\forall j\in\mathbb{N}$, (5.8) yields

$$\begin{aligned}\mathbb{E}\left[Q_N^4\right] &= \frac{1}{N^4}\sum_{i=0}^{N-1}\mathbb{E}\left[X_{i,N}^4\right]+\frac{6}{N^4}\sum_{0\le i<j\le N-1}\mathbb{E}\left[X_{i,N}^2\right]\mathbb{E}\left[X_{j,N}^2\right]\\ &\le \frac{C}{N^{(\beta-1)\wedge(2\alpha-2)}}.\end{aligned}$$

Now, if $\alpha>3/2$ and $\beta>2$, the a.s. convergence of Q_N to zero is achieved. The convergence in probability is obtained for $\alpha>1$ and $\beta>1$.

Finally, studying the a.s. convergence of U_N to zero, we can see from (5.4) that $U_N=(2/N)\sum_{0\le i<j\le N-1}\left(t_i-\mathbb{E}\left[t_i\right]\right)\left(t_j-\mathbb{E}\left[t_j\right]\right)(N-j)$, and $\mathbb{E}\left[U_N\right]=0$. Studying the second moment, from the independence of the t_i

$$\begin{aligned}\mathbb{E}\left[U_N^2\right] &= \frac{4}{N^2}\sum_{0\le i<j\le N-1}\mathbb{E}\left[\left(t_i-\mathbb{E}\left[t_i\right]\right)^2\left(t_j-\mathbb{E}\left[t_j\right]\right)^2\right](N-j)^2\\ &= \frac{4}{N^2}\sum_{0\le i<j\le N-1}\mathbb{E}\left[\left(t_i-\frac{1}{N}\right)^2\left(t_j-\frac{1}{N}\right)^2\right](N-j)^2\\ &= \frac{4Var^2(t_0)}{N^2}\sum_{j=1}^{N-1}\sum_{i=0}^{j-1}(N-j)^2=\frac{4Var^2(t_0)}{N^2}\left(\sum_{j=1}^{N-1}(N-j)^2 j\right)\\ &\le \frac{4Var^2(t_0)}{N^2}\left(\sum_{j=1}^{N-1}(N-j)^2 j\right)\le CN^2Var^2(t_0)\le\frac{C}{N^{2\alpha-2}}.\end{aligned} \tag{5.9}$$

In (5.9), for $\alpha>1$, the convergence in probability is ensured, whereas for $\alpha>3/2$ the a.s. convergence is ensured.

Finally, in Table 4, we summarize the results obtained for R_N, T_N, Q_N, and U_N. These enable us to identify the restrictions on α and β to ensure the convergence in probability and a.s. convergence, recalling the methodologies in Remark 3 and the condition $\alpha<2$ from **H1** and **H2**.

Thus, D_N converges in probability to $1/3$ as N goes to infinity if $1<\alpha<2$ and $\beta>1$. Considering $3/2<\alpha<2$ and $\beta>2$, we ensure the a.s. convergence of D_N.

Table 4. Values for α and β to ensure convergence in $\mathbb{P}$ and a.s. convergence.

	convergence in $\mathbb{P}$	a.s. convergence
$\lvert R_N - 1/3\rvert \leq C/N^{\alpha-1}$	$\alpha > 1$	$\alpha > 1$
$\mathbb{E}\left[T_N^2\right] \leq C/N^{(\beta-1)\wedge(2\alpha-1)}$	$\alpha > 1/2,\ \beta > 1$	$\alpha > 1,\ \beta > 2$
$\mathbb{E}\left[Q_N^4\right] \leq C/N^{(\beta-1)\wedge(2\alpha-2)}$	$\alpha > 1,\ \beta > 1$	$\alpha > 3/2,\ \beta > 2$
$\mathbb{E}\left[U_N^2\right] \leq C/N^{2\alpha-2}$	$\alpha > 1$	$\alpha > 3/2$

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