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ORTHOGONAL ARRAYS OBTAINED BY ORTHOGONAL DECOMPOSITION OF PROJECTION MATRICES

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Abstract: This paper studies a relationship between orthogonal arrays and orthogonal decompositions of projection matrices. This relation is used for the construction of orthogonal arrays. As an application of the method, some new mixed-level orthogonal arrays of run size 36 are constructed.

Key words and phrases: Kronecker product, mixed-level orthogonal array, permutation matrix, projection matrix.

1. Introduction

An $n \times m$ matrix A, having k_i columns with p_i levels, $i = 1, \ldots, r, m = \sum_{i=1}^{r} k_i, p_i \neq p_j$, for $i \neq j$, is called an orthogonal array (OA) of strength d and size n if each $n \times d$ submatrix of A contains all possible $1 \times d$ row vectors with the same frequency. Unless stated otherwise, we consider orthogonal arrays of strength 2, using the notation $L_n(p_1^{k_1} \cdots p_r^{k_r})$ for such an array. An orthogonal array is said to have mixed-level if $r \geq 2$. Such an array is often a natural choice in practice because different factors may require different numbers of levels. Two- and three- level OA's, which form popular fractional factorials, have been discussed at great length in many standard textbooks on experimental design and analysis, for example Box and Draper (1987). The construction of mixed-level OA's has been studied by Wu (1989), Wang and Wu (1991), Wu, Zhang and Wang (1992), Hedayat, Pu and Stufken (1992) and Ryoh Fuji-Hara (1993). In this paper, an interesting relationship between orthogonal arrays and decompositions of projection matrices is presented. By exploring this relationship, we obtain a method for the construction of orthogonal arrays. Zhang (1989, 1990a, 1990b, 1991a and 1991b) has used this method to construct some mixed-level OA's of run size 36, 72, and 100. In this paper the method is further explained and some new mixed-level OA's are obtained.

Section 2 contains basic concepts and main theorems while in Section 3 we describe the method of construction. Some new mixed-level OA's of run size 36 are constructed in Section 4.

2. Basic Concepts and Main Theorems

Suppose that an experiment is carried out according to an array $A=(a_{ij})_{n\times m} = (a_1,\ldots,a_m)$, and $Y=(Y_1,\ldots,Y_n)^T$ is the experimental data vector. In the analysis of variance S_j^2, the sum of squares of the jth factor, is defined as

$$S_j^2=\sum_{i=1}^{p_j}\frac{1}{|I_{ij}|}(\sum_{s\in I_{ij}}Y_s)^2-\frac{1}{n}(\sum_{s=1}^{n}Y_s)^2, \tag{1}$$

where $I_{ij}=\{s: a_{sj}=j\}$ and $|I_{ij}|$ is the number of elements in I_{ij}. From (1), S_j^2 is a quadratic form in Y and there exists a unique symmetric matrix A_j such that $S_j^2=Y^TA_jY$. The matrix A_j is called the *matrix image* (MI) of the jth column a_j of A, denoted by $m(a_j)=A_j$. The MI of a subarray of A is defined as the sum of the MI's of all its columns. In particular, we denote the MI of A by $m(A)$. If a design is an orthogonal array, then the MI's of its columns have some interesting properties. These properties can be used to construct mixed-level OA's.

Let $(r)=(0,\ldots,r-1)^T_{1\times r}, 1_r$ be the $r\times 1$ vector of 1's and I_r the identity matrix of order r. Then

$$m(1_r)=P_r \text{ and } m((r))=\tau_r, \tag{2}$$

where $P_r=\frac{1}{r}1_r1_r^T$ and $\tau_r=I_r-P_r$.

The Kronecker product $A\otimes B$ is defined as: $A\otimes B=(a_{ij}B)_{sn\times tm}$ if $A=(a_{ij})_{n\times m}, B=(b_{ij})_{s\times t}$.

Definition 1. Suppose that p is a prime, and that a and b are OA's which have only one column, i.e., $a=L_{n_1}(p)=(a_1,\ldots,a_{n_1})^T, b=L_{n_2}(p)=(b_1,\ldots,b_{n_2})^T$. The Kronecker sum of a and b, denoted $a\oplus b$, is defined as

$$a\oplus b=L_{n_1n_2}(p)=((a_1+b_1),\ldots,(a_1+b_{n_2}),\ldots,(a_{n_1}+b_{n_2}))^T \qquad \mathrm{mod}(p)$$

For example,

$$(2)\oplus(2)=(0,1,1,0)^T, \qquad (3)\oplus(3)=(0,1,2,1,2,0,2,0,1)^T.$$

Theorem 1. *For any permutation matrix S and any array L,*

$$m(S(L\otimes 1_r))=S(m(L)\otimes P_r)S^T, \quad \textit{and} \quad m(S(1_r\otimes L))=S(P_r\otimes m(L))S^T.$$

Theorem 2. *Let A be an OA of strength 1, i.e.,*

$$A=(a_1,\ldots,a_m)=(S_1(1_{r_1}\otimes(p_1)),\ldots,S_m(1_{r_m}\otimes(p_m))),$$

where $r_ip_i=n$ and S_i is a permutation matrix, for $i=1,\ldots,m$.

The following statements are equivalent.

(1) *A is an OA of strength* 2.
(2) *The MI of A is a projection matrix.*
(3) *The MI's of any two columns of A are orthogonal, i.e.,* $m(a_i)m(a_j) = 0 (i \neq j)$.
(4) *The projection matrix* τ_n *can be decomposed as* $\tau_n = m(a_1)+\cdots+m(a_m)+\Delta$, *where* $rk(\Delta) = n-1-\sum_{j=1}^{m}(p_j-1)$ *is the rank of the matrix* Δ.

Definition 2. An OA A is said to be saturated if $\sum_{j=1}^{m}(p_j-1) = n-1$ (or, equivalently, $m(A) = \tau_n$).

Corollary 1. *Let* (L, H) *and* K *be OA's of run size* n. *Then* (K, H) *is an OA if* $m(K) \leq m(L)$, *where* $m(K) \leq m(L)$ *means that the difference* $m(K) - m(L)$ *is nonnegative definite.*

Corollary 2. *Suppose* L *and* H *are orthogonal arrays. Then* $K = (L, H)$ *is also an OA if* $m(L)$ *and* $m(H)$ *are orthogonal, i.e.,* $m(L)m(H) = 0$. *In this case* $m(K) = m(L) + m(H)$.

These theorems and corollaries can be found in Zhang (1991b, 1992 and 1993).

3. A General Method for Constructing OA's by Decompositions of the Projection Matrix τ_n

Our procedure of constructing mixed-level OA's by using decompositions of the projection matrix τ_n consists of the following three steps:

Step 1. Orthogonally decompose the projection matrix $\tau_n : \tau_n = A_1 + \cdots + A_k$, where $A_iA_j = 0 (i \neq j)$.
Step 2. Find an OA L_i such that $m(L_i) \leq A_i$.
Step 3. Lay out the new OA L by Corollaries 1 and 2: $L = (L_1, \ldots, L_{k_1})(k_1 \leq k)$.

In applying Step 1, the following orthogonal decompositions of τ_n are very useful, $\tau_{n \cdot k} = I_n \otimes \tau_k + \tau_n \otimes P_k = \tau_n \otimes P_k + P_n \otimes \tau_k + \tau_n \otimes \tau_k = \tau_n \otimes I_k + P_n \otimes \tau_k$,

$$\tau_{p \cdot r \cdot q} = \tau_p \otimes I_r \otimes P_q + P_p \otimes \tau_{rq} + \tau_p \otimes I_r \otimes \tau_q. \tag{3}$$

These equations are easy to verify from $\tau_n = I_n - P_n, P_{nk} = P_n \otimes P_k$ and $I_{nk} = I_n \otimes I_k$.

The following theorem plays a very useful role in the procedure.

Theorem 3. *Suppose* $\tau_{n_1} = \sum_j S_j A S_j^T$ *and* $\tau_{n_2} = \sum_j T_j B T_j^T$ *are orthogonal decompositions of* τ_{n_1} *and* τ_{n_2}, *respectively, where the* S_j*'s and* T_j*'s are permutation matrices and* $n = n_1 n_2$. *Then* $\tau_{n_1 n_2}$ *can be orthogonally decomposed into*

$$\tau_{n_1 n_2} = \sum_j (S_j \otimes T_j)(A \otimes P_{n_2} + I_{n_1} \otimes B)(S_j^T \otimes T_j^T). \tag{4}$$

If there exists an OA H such that $m(H) \leq I_{n_1} \otimes B + A \otimes P_{n_2}$, then

$$L = ((S_1 \otimes T_1)H, (S_2 \otimes T_2)H, \ldots)$$

is also an OA.

Proof. From (3) we have

$$\tau_{n_1 n_2} = \tau_{n_1} \otimes P_{n_2} + I_{n_1} \otimes \tau_{n_2}.$$

Since $P_{n_2} = T_j P_{n_2} T_j^T$ and $I_{n_1} = S_j I_{n_1} S_j^T$ hold for all j, we get

$$\tau_{n_1 n_2} = \sum_j (S_j A S_j^T) \otimes (T_j P_{n_2} T_j^T) + \sum_j (S_j I_{n_1} S_j^T) \otimes (T_j B T_j^T).$$

Using the matrix property $(ABC)\otimes(DEF) = (A\otimes D)(B\otimes E)(C\otimes F)$, we obtain

$$\tau_{n_1 n_2} = \sum_j (S_j \otimes T_j)(A \otimes P_{n_2} + I_{n_1} \otimes B)(S_j^T \otimes T_j^T).$$

Thus (4) holds.

Since the decompositions of both τ_{n_1} and τ_{n_2} are orthogonal, the decomposition of $\tau_{n_1 n_2}$ in (4) is orthogonal. By Theorem 1, we have

$$m((S_j\otimes T_j)H) = (S_j\otimes T_j)m(H)(S_j^T\otimes T_j^T) \leq (S_j\otimes T_j)(A\otimes P_{n_2}+I_{n_1}\otimes B)(S_j^T\otimes T_j^T),$$

So L is an OA.

4. Constructions of OA's of Run Size 36

4.1. Construction of OA $L_{36}(3 \cdot 2^{27})$

By the definition of an OA, we may assume without loss of generality that

$$L_9(3^4) = [S_1(1_3 \otimes (3)), \ldots, S_4(1_3 \otimes (3))],$$

and

$$L_4(2^3) = [Q_1((2) \otimes 1_2), \ldots, Q_3((2) \otimes 1_2)],$$

where $S_i (i = 1, \ldots, 4)$ and $Q_j (j = 1, 2, 3)$ are permutation matrices (See Table 3). Since $L_9(3^4)$ and $L_4(2^3)$ are saturated OA's, from (2), Theorem 1 and Theorem 2, we have

$$\tau_9 = \sum_{i=1}^{4} S_i(P_3 \otimes \tau_3)S_i^T,$$

and

$$\tau_4 = \sum_{i=1}^{3} Q_i(\tau_2 \otimes P_2)Q_i^T. \qquad (5)$$

From (3), we have

$$\tau_{36} = \tau_9 \otimes I_4 + P_9 \otimes \tau_4.$$

By Theorem 3, we have

$$\tau_{36} = \sum_{i=1}^{3}(S_i \otimes Q_i)(P_3 \otimes \tau_3 \otimes I_4 + P_9 \otimes \tau_2 \otimes P_2)(S_i^T \otimes Q_i^T) + [S_4(P_3 \otimes \tau_3)S_4^T] \otimes I_4.$$

Using the properties $I_4 = I_4I_4I_4, (ABC) \otimes (DEF) = (A \otimes D)(B \otimes E)(C \otimes F)$ and $I_4 = P_4 + \tau_4$, we obtain

$$\begin{aligned}\tau_{36} = &\sum_{i=1}^{3}(S_i \otimes Q_i)(P_3 \otimes (\tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2))(S_i^T \otimes Q_i^T)\\ &+(S_4 \otimes I_4)(P_3 \otimes \tau_3 \otimes P_4)(S_4^T \otimes I_4) + (S_4 \otimes I_4)(P_3 \otimes \tau_3 \otimes \tau_4)(S_4^T \otimes I_4). \quad (6)\end{aligned}$$

The above decompositions are orthogonal because of the orthogonality in each step. Now we want to find an OA whose MI is less than or equal to $\tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2$. Each of the OA's $L_{12}(2^{11}), L_{12}(3 \cdot 2^4)$ and $L_{12}(6 \cdot 2^2)$ in Table 4 contains the two columns $1_6 \otimes (2)$ and $1_3 \otimes ((2) \oplus (2))$. Deleting these two columns from the three OA's, we obtain OA's $L_{12}(2^9)$, $L_{12}(3 \cdot 2^2)$ and $L_{12}(6)$, respectively. The MI's of these arrays are less than or equal to $\tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2$, since

$$\tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2 = \tau_{12} - P_6 \otimes \tau_2 - P_3 \otimes \tau_2 \otimes \tau_2.$$

By (6) and Theorems 1, 2 and 3, we obtain OA's $L_{36}(3 \cdot 2^{27})$ as follows (See Table 1):

$$\begin{aligned}L_{36}(3 \cdot 2^{27}) = [&(S_1 \otimes Q_1)(1_3 \otimes L_{12}(2^9)), (S_2 \otimes Q_2)(1_3 \otimes L_{12}(2^9)),\\ &(S_3 \otimes Q_3)(1_3 \otimes L_{12}(2^9), (S_4 \otimes I_4)(1_3 \otimes (3) \otimes I_4)]. \quad (7)\end{aligned}$$

Furthermore, replacing the $L_{12}(2^9)$'s in (7) by $L_{12}(3 \cdot 2^2)$ and $L_{12}(6)$, we can construct OA's such as $L_{36}(3^2 \cdot 2^{20}), L_{36}(3^3 \cdot 2^{13}), L_{36}(3^4 \cdot 2^6), L_{36}(6 \cdot 3^2 \cdot 2^{11}), L_{36}(6 \cdot 3^3 \cdot 2^4), L_{36}(6^2 \cdot 3^2 \cdot 2^2)$.

4.2. Construction of $L_{36}(6^2 \cdot 3^8 \cdot 2)$

Suppose

$$L_9(3^4) = (1_3 \otimes (3), (3) \otimes 1_3, a, b).$$

By (2) and Theorem 2, we have $m(L_9(3^4)) = \tau_9$ and

$$m((a, b)) = \tau_9 - P_3 \otimes \tau_3 - \tau_3 \otimes P_3 = \tau_3 \otimes \tau_3.$$

From the definition of an OA, there exists a 9×9 permutation matrix T such that

$$(1_3 \otimes (3), (3) \otimes 1_3) = T(a, b).$$

So the MI of $(1_3 \otimes (3), (3) \otimes 1_3)$, i.e. the MI of $T(a, b)$, is $T(\tau_3 \otimes \tau_3)T^T$.

Table 1 and Table 2.

Table 1. OA $L_{36}(3 \cdot 2^{27})$	Table 2. OA $L_{36}(6^2 \cdot 3^8 \cdot 2)$
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0
1 0 0 0 1 0 1 0 1 0 1 1 1 1 0 0 0 1 0 1 1 1 1 0 0 0 1 0	0 0 0 0 0 1 2 1 2 1 1
0 1 1 1 1 0 0 0 1 1 0 0 0 1 0 1 0 1 1 1 1 0 1 0 0 1 0 0	1 2 1 2 1 0 0 0 0 0 1
1 1 1 0 1 0 0 1 0 1 1 1 0 1 0 0 1 0 1 0 0 0 1 0 1 0 1 0	1 2 1 2 1 1 2 1 2 1 0
0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 1 1 1 0 1 0 0 1 1 1 1 1 1	2 2 0 1 2 2 0 1 2 0 0
1 0 0 0 1 0 1 0 1 1 0 0 1 1 1 0 1 0 1 0 0 1 1 1 0 1 0 1	2 2 0 1 2 3 2 0 0 1 1
0 1 1 1 1 0 0 0 1 0 0 1 0 0 1 0 1 1 1 1 0 1 0 1 0 0 1 1	3 1 2 2 0 2 0 1 2 0 1
1 1 1 0 1 0 0 1 0 1 1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 1 1 1	3 1 2 2 0 3 2 0 0 1 0
0 0 0 0 0 0 0 0 0 1 0 1 1 0 0 1 1 1 1 0 1 1 0 0 1 1 1 2	4 1 1 0 2 4 0 0 2 1 0
1 0 0 0 1 0 1 0 1 1 1 1 0 0 1 1 0 0 1 1 1 0 0 1 1 0 0 2	4 1 1 0 2 5 2 1 0 0 1
0 1 1 1 1 0 0 0 1 0 1 0 1 0 0 1 1 0 0 0 1 1 1 1 1 0 0 2	5 0 2 1 1 4 0 0 2 1 1
1 1 1 0 1 0 0 1 0 0 0 1 1 1 1 1 0 0 0 1 0 1 0 0 1 1 0 2	5 0 2 1 1 5 2 1 0 0 0
0 1 0 0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 1 0 1 1 0 0 1 1 1 1	4 2 2 1 0 0 2 2 2 2 0
0 0 1 0 0 1 0 1 1 0 1 1 1 1 0 0 0 1 1 1 1 0 0 1 1 0 0 1	4 2 2 1 0 1 1 0 1 0 1
1 0 0 1 1 1 0 1 0 1 0 0 0 1 0 1 0 1 0 0 1 1 1 1 1 0 0 1	5 1 0 2 2 0 2 2 2 2 1
1 1 0 1 0 1 0 0 1 1 1 1 0 1 0 0 1 0 0 1 0 1 0 0 1 1 0 1	5 1 0 2 2 1 1 0 1 0 0
0 1 0 0 1 1 1 1 1 0 1 0 0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 2	0 1 1 1 1 2 2 0 1 2 0
0 0 1 0 0 1 0 1 1 1 0 0 1 1 1 0 1 0 0 1 1 1 1 0 0 0 1 2	0 1 1 1 1 3 1 2 2 0 1
1 0 0 1 1 1 0 1 0 0 0 1 0 0 1 0 1 1 1 1 1 0 1 0 0 1 0 2	1 0 2 0 2 2 2 0 1 2 1
1 1 0 1 0 1 0 0 1 1 1 0 1 0 1 0 0 1 1 0 0 0 1 0 1 0 1 2	1 0 2 0 2 3 1 2 2 0 0
0 1 0 0 1 1 1 1 1 1 0 1 1 0 0 1 1 1 0 1 0 0 1 1 1 1 1 0	2 0 1 2 0 4 2 2 1 0 0
0 0 1 0 0 1 0 1 1 1 1 1 0 0 1 1 0 0 1 0 0 1 1 1 0 1 0 0	2 0 1 2 0 5 1 0 2 2 1
1 0 0 1 1 1 0 1 0 0 1 0 1 0 0 1 1 0 1 1 0 1 0 1 0 0 1 0	3 2 0 0 1 4 2 2 1 0 1
1 1 0 1 0 1 0 0 1 0 0 1 1 1 1 1 0 0 0 0 1 0 0 1 0 1 1 0	3 2 0 0 1 5 1 0 2 2 0
1 0 1 1 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 1 1 1 2	2 1 2 0 1 0 1 1 1 1 0
0 1 0 1 0 0 1 1 0 0 1 1 1 1 0 0 0 1 1 0 0 1 1 1 0 1 0 2	2 1 2 0 1 1 0 2 0 2 1
1 1 1 0 0 1 1 0 0 1 0 0 0 1 0 1 0 1 1 1 0 1 0 1 0 0 1 2	3 0 1 1 2 0 1 1 1 1 1
0 0 1 1 1 1 1 0 0 1 1 1 0 1 0 0 1 0 0 0 1 0 0 1 0 1 1 2	3 0 1 1 2 1 0 2 0 2 0
1 0 1 1 0 0 1 1 1 0 1 0 0 1 1 1 1 1 1 0 1 1 0 0 1 1 1 0	4 0 0 2 1 2 1 2 0 1 0
0 1 0 1 0 0 1 1 0 1 0 0 1 1 1 0 1 0 1 1 1 0 0 1 1 0 0 0	4 0 0 2 1 3 0 1 1 2 1
1 1 1 0 0 1 1 0 0 0 0 1 0 0 1 0 1 1 0 0 1 1 1 1 1 0 0 0	5 2 1 0 0 2 1 2 0 1 1
0 0 1 1 1 1 1 0 0 1 1 0 1 0 1 0 0 1 0 1 0 1 0 0 1 1 0 0	5 2 1 0 0 3 0 1 1 2 0
1 0 1 1 0 0 1 1 1 1 0 1 1 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1	0 2 2 2 2 4 1 1 0 2 0
0 1 0 1 0 0 1 1 0 1 1 1 0 0 1 1 0 0 0 1 1 1 1 0 0 0 1 1	0 2 2 2 2 5 0 2 1 1 1
1 1 1 0 0 1 1 0 0 0 1 0 1 0 0 1 1 0 1 1 1 0 1 0 0 1 0 1	1 1 0 1 0 4 1 1 0 2 1
0 0 1 1 1 1 1 0 0 0 0 1 1 1 1 1 0 0 1 0 0 0 1 0 1 0 1 1	1 1 0 1 0 5 0 2 1 1 0

Therefore

$$\tau_9 = \sum_{i=1}^{2} T_i(\tau_3 \otimes \tau_3)T_i^T, \tag{8}$$

where $T_1 = I_9, T_2 = T$ (See Table 3).

Table 3. The permutation matrices used in the course of construction

$$S_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0 \end{bmatrix} \qquad S_4 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{bmatrix} \qquad Q_3 = \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0&0 \end{bmatrix}$$

From the decomposition of τ_{36} in (3), we have

$$\tau_{36} = I_9 \otimes \tau_4 + \tau_9 \otimes P_4.$$

It follows from Theorem 3, (5) and (8) that

$$\tau_{36} = \sum_{i=1}^{2}(T_i \otimes Q_i)(I_9 \otimes \tau_2 \otimes P_2 + \tau_3 \otimes \tau_3 \otimes P_4)(T_i^T \otimes Q_i^T) + I_9 \otimes [Q_3(\tau_2 \otimes P_2)Q_3^T].$$

Table 4. The known OA's used in this paper

The OA $L_{12}(2^{11})$	The OA $L_{12}(3 \cdot 2^4)$	The OA $L_{12}(6 \cdot 2^2)$
0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0	0 0 0
1 1 1 0 0 0 1 0 1 0 1	0 1 1 1 0	0 1 1
0 1 0 1 1 1 1 0 0 0 1	0 0 1 0 1	1 0 1
1 0 1 1 1 0 1 0 0 1 0	0 1 0 1 1	1 1 0
0 0 0 1 0 0 1 1 1 1 1	1 0 0 0 1	2 0 0
1 1 0 0 1 0 0 1 0 1 1	1 1 1 0 0	2 1 1
0 1 1 0 0 1 1 1 0 1 0	1 0 1 1 0	3 0 1
1 0 1 1 0 1 0 1 0 0 1	1 1 0 1 1	3 1 0
0 0 1 0 1 1 0 0 1 1 1	2 0 0 1 0	4 0 0
1 1 0 1 0 1 0 0 1 1 0	2 1 1 0 1	4 1 1
0 1 1 1 1 0 0 1 1 0 0	2 0 1 1 1	5 0 1
1 0 0 0 1 1 1 1 1 0 0	2 1 0 0 0	5 1 0

The OA $L_{18}(6 \cdot 3^6)$	The OA $L_9(3^4)$	The OA $L_4(2^3)$
0 0 0 0 0 0 0	0 0 0 0	0 0 0
1 0 0 2 1 2 1	0 1 1 1	0 1 1
2 0 1 2 0 1 2	0 2 2 2	1 0 1
3 0 1 1 2 2 0	1 0 1 2	1 1 0
4 0 2 1 1 0 2	1 1 2 0	
5 0 2 0 2 1 1	1 2 0 1	
4 1 0 2 2 1 0	2 0 2 1	
5 1 0 1 0 2 2	2 1 0 2	
0 1 1 1 1 1 1	2 2 1 0	
1 1 1 0 2 0 2		
2 1 2 0 1 2 0		
3 1 2 2 0 0 1		
2 2 0 1 2 0 1		
3 2 0 0 1 1 2		
4 2 1 0 0 2 1		
5 2 1 2 1 0 0		
0 2 2 2 2 2 2		
1 2 2 1 0 1 0		

Using the properties $I_9 = I_9 I_9 I_9, (ABC)\otimes(DEF) = (A\otimes D)(B\otimes E)(C\otimes F)$ and $I_9 = P_9 + \tau_9$, we obtain

$$\tau_{36} = \sum_{i=1}^{2}(T_i \otimes Q_i)(I_9 \otimes \tau_2 \otimes P_2 + \tau_3 \otimes \tau_3 \otimes P_4)(T_i^T \otimes Q_i^T) + (I_9 \otimes Q_3)(P_9 \otimes \tau_2 \otimes P_2)(I_9 \otimes Q_3^T) + (I_9 \otimes Q_3)(\tau_9 \otimes \tau_2 \otimes P_2)(I_9 \otimes Q_3^T). \quad (9)$$

Now we find an OA whose MI is $I_9\otimes\tau_2\otimes P_2+\tau_3\otimes\tau_3\otimes P_4$. The OA $L_{18}(6\cdot 3^6)$ in Table 4 contains the two columns $(3)\otimes 1_6$ and $1_3\otimes(3)\otimes 1_2$. An $L_{18}(6\cdot 3^4)$ is obtained by deleting these two columns. Then

$$m(L_{18}(6\cdot 3^4)) = \tau_{18} - \tau_3\otimes P_6 - P_3\otimes\tau_3\otimes P_2 = I_9\otimes\tau_2 + \tau_3\otimes\tau_3\otimes P_2.$$

From (9) and Theorems 1, 2 and 3, we can lay out a new OA $L_{36}(6^2\cdot 3^8\cdot 2) = [(T_1\otimes Q_1)(L_{18}(6\cdot 3^4)\otimes 1_2), (T_2\otimes Q_2)(L_{18}(6\cdot 3^4)\otimes 1_2), (I_9\otimes Q_3)(1_9\otimes(2)\otimes 1_2)]$.

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