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ON BAHADUR ASYMPTOTIC EFFICIENCY OF THE MAXIMUM LIKELIHOOD ESTIMATOR FOR A GENERALIZED SEMIPARAMETRIC MODEL

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Abstract. This paper studies the Bahadur asymptotic efficiency of the maximum likelihood estimator for the generalized semiparametric model $Y = f(X, \beta) + g(T) + \varepsilon$.

Key words and phrases: Bahadur asymptotic efficiency, generalized semiparametric model.

1. Introduction

Let $\mathbf{U} = (u_1, \ldots, u_n)$ be a sequence of independent, identically distributed (i.i.d.) observations. Bahadur (1960) proposed an asymptotically efficient concept for a consistent estimator based on the rate of its tail probability. It can be stated (under certain regularity conditions) that, for any consistent estimator $T_n(\mathbf{U})$,

$$\liminf_{\zeta\to 0} \liminf_{n\to\infty} \frac{1}{n\zeta^2} \log P_\beta\{|T_n(\mathbf{U}) - \beta| \geq \zeta\} \geq -\frac{I(\beta)}{2},$$

and that the maximum likelihood estimator (MLE) β_n achieves the lower bound, that is,

$$\lim_{\zeta\to 0} \lim_{n\to\infty} \frac{1}{n\zeta^2} \log P_\beta\{|\beta_n - \beta| \geq \zeta\} = -\frac{I(\beta)}{2},$$

where $I(\beta)$ is Fisher's information. In other words, for any consistent estimator T_n, $P_\beta\{|T_n(\mathbf{U}) - \beta| \geq \zeta\}$ can not tend to zero faster than the exponential rate give by $\exp\{-\frac{1}{2}n\zeta^2 I(\beta)\}$, and for MLE β_n, $P_\beta\{|\beta_n - \beta| \geq \zeta\}$ achieves this optimal exponential rate. The estimator which has reached this rate is called Bahadur asymptotically efficient (BAE). Fu (1973) showed, under regularity conditions which differ partly from Bahadur's, that a large class of consistent estimators $\{\beta_n^*\}$ is asymptotically efficient in Bahadur's sense. It also gave a simple and direct method to verify Bahadur's (1967) results. Cheng (1980) proved, under a weaker condition than Bahadur's, that the MLE in single-parameter and multi-parameter cases are BAE. Lu (1983) studied the Bahadur efficiency of the MLE for the linear model. There is considerable literature on BAE for parametric model estimation (see Fu (1982)).

This paper studies the BAE of the MLE for the generalized semiparametric model

$$Y_i = f(X_i, \beta) + g(T_i) + \varepsilon_i, \tag{1.1}$$

where $X_i = (x_{i1}, \ldots, x_{iq}), i = 1, \ldots, n$, are given row vectors, β is a $k \times 1$ vector of an unknown parameter to be estimated, $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. random variables with a common probability density function $\varphi(\cdot)$ with respect to the Lebesgue measure, $f(\cdot, \cdot)$ is an known function defined on $R^q \times R^k$, g is an unknown Hölder continuous function of known order p (see Chen (1988)) in R^1, T and ε are independent and T follows a uniform distribution on $[0, 1]$.

2. Assumptions and Statement of the Main Result

Let $\{T_i, Y_i, i = 1, \ldots, n\}$ be a sample of size n from the model (1.1). Assume T_i and ε_i are independent. Throughout this paper we denote a vector by a boldface letter, a matrix by a calligraphic letter. Define

$$\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^\tau, \quad f'(x, \beta) = \Big(\frac{\partial f(x, \beta)}{\partial \beta_i}\Big)_{k \times 1}, \quad \mathbf{g}(\mathbf{T}) = (g(T_1), \ldots, g(T_n))^\tau,$$

$$\mathbf{f}'(X, \beta) = (f'(x_1, \beta), \ldots, f'(x_n, \beta)), \qquad \mathcal{R}_\backslash^* = \mathbf{f}'(\mathcal{X}, \beta)\mathbf{f}'^\tau(\mathcal{X}, \beta),$$

$$I = I(\varphi) = \int (\psi'(x))^2 \varphi(x)\, dx < \infty, \quad \text{where } \psi(x) = \frac{\varphi'(x)}{\varphi(x)}.$$

Definition 1. For $a \in R^k$, $\|a\| = \Big(\sum_{i=1}^k a_i^2\Big)^{1/2}$, $\mathcal{B}_1 = (b_{ij})_{n_1 \times n_2}, |a| = \max_{1 \le i \le k} |a_i|$, $|\mathcal{B}_\infty|^* = \max_{\rangle,|} |\lfloor_{\rangle|}|$, $\|\mathcal{B}_\infty\|^* = \max_{\substack{\dashv \in \mathcal{R}^{\setminus \in} \\ \|\dashv\| = \infty}} \|\mathcal{B}_\infty \dashv\|$, where $\|\cdot\|$ denotes L_2-norm, and $\|\cdot\|^*$ matrix norm.

Definition 2. The estimator $\tilde{h}_n(Y_1, \ldots, Y_n)$ of β is called a *locally uniformly consistent estimator of* β if for every $\beta_0 \in R^k$, there exists $\delta > 0$ such that for each $\zeta > 0$

$$\lim_{n \to \infty} \sup_{|\beta - \beta_0| < \delta} P_\beta\{\|\tilde{h}_n - \beta\| > \zeta\} = 0.$$

Definition 3. Assume $\mathcal{R}_\backslash^{*-1}$ exists. The consistent estimator $\tilde{h}_n$ of β is said to be *Bahadur asymptotically efficient*, if for each $\beta_0 \in R^k$,

$$\limsup_{\zeta \to 0} \limsup_{n \to \infty} \frac{1}{\zeta^2} \|\mathcal{R}_\backslash^{*-1}\|^* \log P_{\beta_0}\{\|\tilde{h}_n - \beta_0\| > \zeta\} \le -I/2.$$

In the paper, we assume that $\varphi(x)$ is positive and twice differentiable in R^1, $\lim_{|x|\to\infty}\varphi(x)=\lim_{|x|\to\infty}\varphi'(x)=0$, and $\|f'(x,\beta)\|$ is bounded.

In the following we list the sufficient conditions for our main result.

1. There exist constants C_1, $C_2>0$ such that $C_1\le\mu_1\le\cdots\le\mu_k\le C_2$, where $\mu_1,\ldots,\mu_k$ are eigenvalues of $n^{-1}\mathcal{R}^*_\backslash$. Denote $R=C_2/C_1$.
2. There exist a number $\varsigma>1/2$ and a function $H_\beta(x)$ satisfying $E|H_\beta(X)|<\infty$ such that $\|f'(x,\beta^\Delta)-f'(x,\beta)\|\le H_\beta(x)\|\beta^\Delta-\beta\|^\varsigma$.
3. $\lim_{\delta\to 0}\int\sup_{|h|\le\delta}|\psi'(y+h)-\psi'(y)|\varphi(y)\,dy=0$.
4. There exists $t_0>0$ such that
$$\int\exp\{t_0|\psi(x)|\}\varphi(x)\,dx<\infty\qquad\text{and}\qquad\int\exp\{t_0|\psi'(x)|\}\varphi(x)\,dx<\infty.$$
5. There exist a measurable function $h(x)>0$ and a nondecreasing function $\gamma(t)$ which is positive for $t>0$ and $\lim_{t\to 0^+}\gamma(t)=0$ such that $\int\exp\{h(x)\}\varphi(x)\,dx<\infty$ and $|\psi(x+t)-\psi(x)|\le h(x)\gamma(t)$ whenever $|t|\le|t_0|$.

Throughout this paper, it is understood that the MLE β_{ML} of β is based on $\{Y_i=f(X_i,\beta)+\hat g_n(T_i)+\varepsilon_i,\ i=1,\ldots,n\}$, where $\hat g_n$ is an estimator for g suggested by Chen (1988) which satisfies
$$|g(T_i)-\hat g_n(T_i)|\le B_2M_n^{-p},\quad i=1,\ldots,n. \tag{2.1}$$

6. The MLE β_{ML} exists, and for each $\delta>0$, $\beta_0\in R^k$, there exist constants $K=K(\delta,\beta_0)$ and $\rho=\rho(\delta,\beta_0)>0$, such that
$$P_{\beta_0}\Big\{|\beta_{ML}-\beta_0|>\delta\Big\}\le K\exp\{-\rho\|\mathcal{R}^*_\backslash\|^*\delta^\in\}.$$

The following theorem gives our main result that β_{ML} is a BAE estimator of β.

Theorem. *Suppose conditions* 1–3 *hold,* $\tilde h_n$ *is a locally uniformly consistent estimator; then for each* $\beta_0\in R^k$
$$\liminf_{\zeta\to 0}\liminf_{n\to\infty}\frac{1}{\zeta^2}\|\mathcal{R}^{*-1}_\backslash\|^*\log P_{\beta_0}\{\|\tilde h_n-\beta_0\|>\zeta\}\ge-\frac{I}{2}. \tag{2.2}$$
If the conditions 1 *to* 6 *hold, then for each* $\beta_0\in R^k$
$$\limsup_{\zeta\to 0}\limsup_{n\to\infty}\frac{1}{\zeta^2}\|\mathcal{R}^{*-1}_\backslash\|^*\log P_{\beta_0}\Big\{\|\beta_{ML}-\beta_0\|>\zeta\Big\}\le-\frac{I}{2}. \tag{2.3}$$

Remark. The result (2.3) implies that β_{ML} is asymptotically efficient in Bahadur's sense.

3. Proof of the Main Result

We first prove the result (2.2). The proof is divided into three steps. First step: get a UMP test Φ_n^* whose power is 1/2 for the hypothesis $H_0 : \beta = \beta_0$ VS $H_1 : \beta = \beta_n$. Second step: by constructing a test $\Phi_n(\mathbf{Y})$ corresponding to $\tilde{h}_n$, show that the power of the constructed test is larger than 1/2. The last step: by using the Neyman-Pearson Fundamental Lemma, show that $E_{\beta_0}\Phi_n$, the level of Φ_n, is larger than $E_{\beta_0}\Phi_n^*$.

Proof of (2.2). For each $\zeta > 0$, set

$$\beta_n = \beta_0 + \frac{\mathcal{R}_{\backslash}^{*-1} a_n}{\|\mathcal{R}_{\backslash}^{*-1}\|^*}\zeta,$$

where $a_n \in R^k$, $a_n^\tau \mathcal{R}_{\backslash}^{*-1} a_n = \|\mathcal{R}_{\backslash}^{*-1}\|^*$ and $\|a_n\| = 1$. Let $l_i = (0, \ldots, 1, \ldots, 0)^\tau \in R^k$. It is easy to show $\|\mathcal{R}_{\backslash}^{*-1}\|^* \geq a^\tau \mathcal{R}_{\backslash}^{*-1} a \geq 1/\|\mathcal{R}_{\backslash}^*\|^*$, and

$$|\beta_n - \beta_0| \leq \|\mathcal{R}_{\backslash}^*\| \cdot \|\mathcal{R}_{\backslash}^{*-\infty}\|^* \zeta \leq \mathcal{R}\zeta.$$

Denote

$$\Gamma_n(\mathbf{Y}) = \prod_{i=1}^{n} \frac{\varphi(y_i - f(x_i, \beta_n) - g(t_i))}{\varphi(y_i - f(x_i, \beta_0) - g(t_i))},$$

$$\Delta_i = f(x_i, \beta_n) - f(x_i, \beta_0), \quad d_n = \exp\left\{\frac{I(1+\mu)\zeta^2}{2\|\mathcal{R}_{\backslash}^{*-1}\|^*}\right\} \quad (\mu > 0).$$

By the Neyman-Pearson Fundamental Lemma, there exists a test $\Phi_n^*(\mathbf{Y})$ such that

$$E_{\beta_n}\{\Phi_n^*(\mathbf{Y})\} = \frac{1}{2}.$$

Under the H_0, we have the following inequality:

$$E_{\beta_0}\{\Phi_n^*(\mathbf{Y})\} \geq \int_{\Gamma_n \leq d_n} \Phi_n^*(\mathbf{Y})\, dP_{n\beta_0} \geq \frac{1}{d_n}\left[\frac{1}{2} - \int_{\Gamma_n(\mathbf{Y}) > d_n} \Phi_n^*(\mathbf{Y})\, dP_{n\beta_n}\right].$$

If

$$\limsup_{n\to\infty} P_{\beta_n}\{\Gamma_n(\mathbf{Y}) > d_n\} \leq \frac{1}{4}, \tag{3.1}$$

then for n large enough,

$$E_{\beta_0}\{\Phi_n^*(\mathbf{Y})\} \geq \frac{1}{4d_n}. \tag{3.2}$$

Define

$$\Phi_n(Y) = \begin{cases} 1, & \text{if } |a_n^\tau(\tilde{h}_n - \beta_0)| \geq \lambda'\zeta, \\ 0, & \text{otherwise,} \end{cases} \tag{3.3}$$

where $\lambda' \in (0,1)$. Since $a_n^\tau(\beta_n - \beta_0) = \zeta$ and $\tilde{h}_n$ is a locally uniformly consistent estimator, we have

$$\liminf_{n\to\infty} E_{\beta_n}\{\Phi_n(\mathbf{Y})\} \geq \liminf_{n\to\infty} P_{\beta_n}\{\|\tilde{h}_n - \beta_n\| \leq (1-\lambda')\zeta\} = 1.$$

We have from Neyman-Pearson Fundamental Lemma that, for n large enough,

$$E_{\beta_0}\{\Phi_n(\mathbf{Y})\} \geq E_{\beta_0}\{\Phi_n^*(\mathbf{Y})\}. \tag{3.4}$$

It follows from (3.2), (3.3) and (3.4) that

$$P_{\beta_0}\{\|\tilde{h}_n - \beta_0\| \geq \lambda'\zeta\} \geq P_{\beta_0}\{|a_n^\tau(\tilde{h}_n - \beta_0)| \geq \lambda'\zeta\} = E_{\beta_0}\{\Phi_n(\mathbf{Y})\} \geq \frac{1}{4d_n}.$$

By letting $\mu \to 0$ and $\lambda' \to 1$, this would completes the proof of (2.2) if (3.1) is proved.

Now we return to prove the inequality (3.1). Let $\Delta_i = f(X_i, \beta_n) - f(X_i, \beta_0) = f'(X_i, \tilde{\beta}_n)\cdot(\beta_n - \beta)$; then by condition 2, we know that $\|\Delta_i\| \leq RC\zeta$, and

$$\sum_{i=1}^n {\Delta_i}^2 \leq \frac{2a_n^\tau \mathcal{R}_\backslash^{*-1} a_n}{{\|\mathcal{R}_\backslash^{*-1}\|^*}^2}\zeta^2 = \frac{2\zeta^2}{\|\mathcal{R}_\backslash^{*-1}\|^*}. \tag{3.5}$$

According to the Taylor expansion, for ζ sufficiently small,

$$\sum_{i=1}^n \log\frac{\varphi(Y_i)}{\varphi(Y_i+\Delta_i)} = -\sum_{i=1}^n\Big\{\psi(Y_i)\Delta_i + \frac{1}{2}(\psi'(Y_i) + R_i(Y_i))\Delta_i^2\Big\},$$

where $R_i(Y_i) = \psi'(Y_i + \theta_i\Delta_i) - \psi'(Y_i)$, $0 < \theta_i < 1$, then

$$\begin{aligned}
&P_{\beta_n}\{\Gamma_n(Y) > d_n\} \;= P_0\left\{\prod_{i=1}^n \frac{\varphi(Y_i)}{\varphi(Y_i+\Delta_i)} > d_n\right\}\\
&= P_0\left\{\sum_{i=1}^n \log\frac{\varphi(Y_i)}{\varphi(Y_i+\Delta_i)} > \frac{I(1+\mu)\zeta^2}{2\|\mathcal{R}_\backslash^{*-1}\|^*}\right\}\\
&\leq P_0\left\{\frac{1}{2}\sum_{i=1}^n I(\varphi)\Delta_i^2 > \frac{I(1+\mu/2)\zeta^2}{2\|\mathcal{R}_\backslash^{*-1}\|^*}\right\} + P_0\left\{-\sum_{i=1}^n \psi(Y_i)\Delta_i > \frac{I\mu\zeta^2}{12\|\mathcal{R}_\backslash^{*-1}\|^*}\right\}\\
&\quad + P_0\left\{-\sum_{i=1}^n \frac{1}{2}[\psi'(Y_i) + I(\varphi)]\Delta_i^2 > \frac{I\mu\zeta^2}{12\|\mathcal{R}_\backslash^{*-1}\|^*}\right\}\\
&\quad + P_0\left\{-\frac{1}{2}\sum_{i=1}^n R_i(Y_i)\Delta_i^2 > \frac{I\mu\zeta^2}{12\|\mathcal{R}_\backslash^{*-1}\|^*}\right\}\\
&= P_1 + P_2 + P_3 + P_4, \qquad \text{say.}
\end{aligned}$$

We now estimate the four probabilities P_1, P_2, P_3 and P_4, respectively.

$$P_1 = P_0 \left\{ I(\varphi) \frac{1}{\|\mathcal{R}_\backslash^{*-1}\|^*} > \frac{I(1+\mu/2)}{\|\mathcal{R}_\backslash^{*-1}\|^*} \right\} = 0. \tag{3.6}$$

It follows from the Tchebichev inequality, condition 1 and (3.5) that

$$\begin{aligned} P_2 &= P_{\mathbf{T}} \Big[P_0 \Big\{ -\sum_{i=1}^n \psi(Y_i)\Delta_i > \frac{I\mu\zeta^2}{12\|\mathcal{R}_\backslash^{*-1}\|^*} \Big| \mathbf{T} \Big\} \Big] \\ &\le P_{\mathbf{T}} \Big[\frac{144\|\mathcal{R}_\backslash^{*-1}\|^*}{I\mu^2\zeta^2} \Big] \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

$$P_3 \le P_{\mathbf{T}} \Big[\Big(\frac{6\|\mathcal{R}_\backslash^{*-1}\|^*}{I\mu\zeta^2} \Big)^2 (CR\zeta)^2 \frac{\zeta^2}{\|\mathcal{R}_\backslash^{*-1}\|^*} E_0(\psi'(Y_1) + I(\varphi))^2 \Big] \to 0, \quad \text{as } n \to \infty,$$

$$P_4 \le P_{\mathbf{T}} \Big[\frac{6\|\mathcal{R}_\backslash^{*-1}\|^*}{I\mu\zeta^2} \cdot \frac{1}{\|\mathcal{R}_\backslash^{*-1}\|^*} \zeta^2 E_0 \{ \max_{1\le i\le n} |R_i(Y_i)| \Big| \mathbf{T} \} \Big].$$

From condition 3, and for ζ be sufficiently small such that $|\Delta_i| < \delta$, then

$$E_0 \{ \max_{1\le i\le n} |R_i(Y_i)| \Big| \mathbf{T} \} \le \int \sup_{|h|<\delta} |\psi'(y+h) - \psi'(y)| \varphi(y)\, dy \le \frac{\mu}{24}. \tag{3.7}$$

Combining the results (3.6) to (3.7), we have completed the proof of (3.1). The proof of (2.2) is now complete.

Now, we outline our proof of the result (2.3). First, by using the Taylor expansion and condition 2, we get the expansion of the projection, $a^\tau(\beta_{ML} - \beta)$, of $\beta_{ML} - \beta$ on the unit sphere $\|a\| = 1$. Second we decompose $P_{\beta_0}\{|a^\tau(\beta_{ML} - \beta_0)| > \zeta\}$ into five terms. Last step, we calculate, carefully, the value of each term.

It follows from the definition of β_{ML}, the Taylor expansion and condition 2 that

$$\begin{aligned} a^\tau(\beta_{ML} - \beta) = &- ((\mathcal{F}\mathcal{I})^{-1} + \tilde{W}) \Big[\sum_{i=1}^n \psi(Y_i - f(X_i,\beta) - g(T_i)) a^\tau \mathcal{R}_\backslash^{*-1} \Big] \\ &- \sum_{i=1}^n \psi'(Y_i - f(X_i,\beta)^* - g^*(T_i)) a^\tau \mathcal{R}_\backslash^{*-1} f'(X_i,\beta)(\hat{g}_n(T_i) - g(T_i)), \end{aligned}$$

where $f(X_i,\beta)^*$ lies between $f(X_i,\beta_{ML})$ and $f(X_i,\beta)$, and $g^*(T_i)$ lies between $g(T_i)$ and $\hat{g}_n(T_i)$. Denote

$$\begin{aligned} R_i(Y_i, X_i, T_i) &= \psi'(Y_i - f(X_i,\beta)^* - g^*(T_i)) - \psi'(Y_i - f(X_i,\beta) - g(T_i)), \\ R_1^* &= \sum_{i=1}^n [I + \psi'(Y_i - f(X_i,\beta) - g(T_i))] \mathcal{R}_\backslash^{*-1} f'(X_i,\beta) f'^\tau(X_i,\beta), \\ R_2^* &= \sum_{i=1}^n R_i(Y_i, X_i, T_i) \mathcal{R}_\backslash^{*-1} f'(X_i,\beta) f'^\tau(X_i,\beta). \end{aligned}$$

Let α be sufficiently small such that $\det(-\mathcal{FI}+\mathcal{R}^*_\infty+\mathcal{R}^*_\in)\neq\prime$ when $|R_1^*+R_2^*|<\alpha$. Hence $(-\mathcal{FI}+R_1^*+R_2^*)^{-1}$ exists, and we denote it by $-((\mathcal{FI})^{-1}+\tilde{W})$. Moreover, according to continuity, there is a nondecreasing function $\eta(\alpha)>0$ such that $|\tilde{W}|<\eta(\alpha)$ and $\lim_{\alpha\to 0}\eta(\alpha)=0$ when $|R_1^*+R_2^*|<\alpha$. Hence, we have, for every $0<\lambda<1/4$, $a\in R^k$ and $\|a\|=1$, that

$$
\begin{aligned}
&P_{\beta_0}\Big\{\Big|a^\tau(\beta_{ML}-\beta_0)\Big|>\zeta\Big\}\\
&\le P_{\beta_0}\Big\{\Big|\sum_{i=1}^n\psi(Y_i-f(X_i,\beta_0)-g(T_i))a^\tau\mathcal{R}_\backslash^{*-1}f'(X_i,\beta)\Big|>(1-2\lambda)I\zeta\Big\}\\
&\quad+P_{\beta_0}\Big\{\Big|\sum_{i=1}^n a^\tau\psi(Y_i-f(X_i,\beta_0)-g(T_i))\mathcal{R}_\backslash^{*-1}f'(X_i,\beta)\Big|>\frac{\lambda\zeta}{\eta(2\alpha)}\Big\}\\
&\quad+P_{\beta_0}\{|R_1^*|>\alpha\}+P_{\beta_0}\{|R_2^*|>\alpha\}\\
&\quad+P_{\beta_0}\Big\{\sum_{i=1}^n|\hat{g}_n(T_i)-g(T_i)||\psi'(Y_i-f(X_i,\beta)^*-g^*(T_i))a^\tau\mathcal{R}_\backslash^{*-1}f'(X_i,\beta)|>\lambda\zeta\Big\}\\
&=P_1+P_2+P_3+P_4+P_5,\ \text{say.}
\end{aligned}
$$

In the following we use Lemma 1 of Lu (1983) to calculate each term above. We only calculate the probability P_4, since the probabilities P_1,P_2,P_3,P_5 are obtained similarly.

It follows from condition 4 that,

$$
\begin{aligned}
&|R_i(Y_i,X_i,T_i)|=|\psi'(Y_i-f(X_i,\beta)^*-g^*(T_i)-\psi'(Y_i-f(X_i,\beta_0)-g(T_i))|\\
&\le h(Y_i-f(X_i,\beta_0)-g(T_i))\gamma(f'(X_i,\beta)(\beta_{ML}-\beta_0)-(g^*(T_i)-g(T_i))).
\end{aligned}
$$

Denote $h_0=E_{\beta_0}h$, then

$$
\begin{aligned}
P_4\le\ &\sum_{j=1}^k\sum_{s=1}^k P_{\beta_0}\Big\{\Big|\sum_{i=1}^n h(Y_i-f(X_i,\beta_0)-g(T_i))\gamma\Big(f'(X_i,\beta)(\beta_{ML}-\beta_0)\\
&\qquad-(g^*(T_i)-g(T_i))\Big)l_j^\tau\mathcal{R}_\backslash^{*-1}f'(X_i,\beta)f'^\tau(X_i,\beta)l_s\Big|\ge\alpha\Big\}\\
\le\ &\sum_{j=1}^k\sum_{s=1}^k\Big[P_{\beta_0}\{\bigcup_{i=1}^n|\hat{g}_n(T_i)-g(T_i)|>\delta\}+P_{\beta_0}\{|\beta_{ML}-\beta_0|\ge\delta\}\\
&\qquad+P_{\beta_0}\Big\{\Big|\sum_{i=1}^n h(Y_i-f(X_i,\beta_0)-g(T_i))\gamma(C\delta+\delta)l_j^\tau\mathcal{R}_\backslash^{*-1}\\
&\qquad f'(X_i,\beta)f'^\tau(X_i,\beta)l_s\Big|\ge\alpha\Big\}\Big]\\
=\ &P_4^{(1)}+P_4^{(2)}+P_4^{(3)},\qquad\text{say.}
\end{aligned}
$$

Let $\zeta \le (2\rho/((1-2\lambda)^2 I))^{1/2}$. It follows from condition 5 and $|a^\tau \mathcal{R}_\backslash^{*-1} a| \ge 1/\|\mathcal{R}_\backslash^*\|^*$ that

$$\rho(\delta,\beta_0)\|\mathcal{R}_\backslash^*\|^* \ge \frac{(1-2\lambda)^2 I}{2a^\tau \mathcal{R}_\backslash^{*-1} a}\zeta^2 \text{ and } P_4^{(2)} \le K(\delta,\beta_0)\exp\Big\{-\frac{(1-2\lambda)^2 I}{2a^\tau \mathcal{R}_\backslash^{*-1} a}\zeta^2\Big\}. \quad (3.8)$$

Let $\sigma_h^2 = E_{\beta_0}[h(Y_i - f(X_i,\beta_0) - g(T_i)) - h_0]^2$; it follows from Lemma 1 of Lu (1983) that

$$\begin{aligned} P_4^{(3)} &\le P_{\beta_0}\Big\{\Big|\sum_{i=1}^n [h(Y_i - f(X_i,\beta_0) - g(T_i)) - h_0] \\ &\qquad\qquad l_j^\tau \mathcal{R}_\backslash^{*-1} f'(X_i,\beta) f'^\tau(X_i,\beta) l_s\Big| \ge \frac{\alpha}{2\gamma(C\delta+\delta)}\Big\} \\ &\le 2\exp\Big\{-(\frac{\alpha}{2\gamma(C\delta+\delta)})^2\Big/(2C^2 R\sigma_h^2 a^\tau \mathcal{R}_\backslash^{*-1} a)\Big(1+O_4(\frac{\alpha}{2\gamma(C\delta+\delta)})\Big)\Big\}, \end{aligned}$$

where $\Big|O_4(\alpha/(2\gamma(C\delta+\delta)))\Big| \le B_4\alpha/2\gamma(C\delta+\delta)$, B_4 depends on $h(\varepsilon_1)$ only but not n. Further, for ζ small enough,

$$P_4^{(3)} \le 2\exp\Big\{-\frac{(1-2\lambda)^2 I\zeta^2}{2a^\tau \mathcal{R}_\backslash^{*-1} a}\Big\}.$$

From (2.1) we know that

$$P_4^{(1)} = P_{\beta_0}\{\bigcup_{i=1}^n |(\hat{g}_n(T_i) - g(T_i))| > \delta\} = 0. \quad (3.9)$$

Combining the results of (3.8) to (3.9), we have

$$P_4 \le (2+K)\exp\Big\{-\frac{(1-2\lambda)^2 I\zeta^2}{2a^\tau \mathcal{R}_\backslash^{*-1} a}\Big\}.$$

This implies that we have proved

$$P_{\beta_0}\Big\{|a^\tau(\beta_{ML}-\beta_0)| > \zeta\Big\} \le (Kk^2+K+4k^2+2k)\exp\Big\{-\frac{(1-2\lambda)^2 I\zeta^2}{2a^\tau \mathcal{R}_\backslash^{*-1} a}(1+O(\zeta))\Big\}.$$

$O(\zeta) = \min\{O_1(\zeta), O_2(\zeta), O_3(\zeta), O_4(\zeta), O_5(\zeta)\}$, $|O(\zeta)| \le \mathrm{CRB}(\eta^{-1}(2\alpha)+1)$. Hence

$$\limsup_{\zeta\to 0}\limsup_{n\to\infty}\frac{a^\tau \mathcal{R}_\backslash^{*-1} a}{\zeta^2}\log P_{\beta_0}\Big\{|a^\tau(\beta_{ML}-\beta_0)| > \zeta\Big\} \le -\frac{(1-2\lambda)^2 I}{2}.$$

Since a is arbitrary, the result (2.3) follows from $\lambda \to 0$. This completes the proof of our main result (2.3).

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