Edgeworth expansions for a class of spectral density estimators and their applications to interval estimation

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Supplementary Material

Recall that $C \in (0, \infty)$ denote a generic constant (not depending on n). Write I_A (and also I(A)) for the indicator function of a set A. Let $\{X_t : t \in \mathbb{Z}\}$ be defined on a probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{D}_j : j \in \mathbb{Z}\}$ be a collection of sub σ-fields of $\mathcal{F}$ (cf. Götze and Hipp (1983)) as detailed in the Appendix. Set $\mathcal{D}_p^q = \sigma\langle\{\mathcal{D}_j : j \in \mathbb{Z},\ p \leqslant j \leqslant q\}\rangle$, $-\infty \leqslant p < q \leqslant \infty$. We now show that the regularity conditions hold for the linear process (2.1), with the natural choice $\mathcal{D}_j = \sigma\langle\epsilon_j\rangle$.

Proof of Proposition 1. The discussion given in the Appendix shows that conditions (C.1)-(C.5) holds with $\mathcal{D}_j = \sigma\langle\epsilon_j\rangle$, $j \in \mathbb{Z}$. Hence, we concentrate on verification of (C.6). The tapered DFT $d_j(\lambda)$ for the set of observations in the j-th block $\mathcal{X}_{j,l}$ is

$$d_j(\lambda) = \sum_{m \in \mathbb{Z}} \epsilon_m \left(\sum_{r=1}^{l} h_r \exp(\iota\lambda r) a_{r+j-m-1} \right). \tag{S6.1}$$

For $j, m \in \mathbb{Z}$, let $c_{jm} = \frac{\sum_{r=1}^{l} h_r(\cos\lambda r)a_{j-m+r-1}}{[2\pi\sum_{r=1}^{l} h_r^2]^{1/2}}$ and $s_{jm} = \frac{\sum_{r=1}^{l} h_r(\sin\lambda r)a_{j-m+r-1}}{[2\pi\sum_{r=1}^{l} h_r^2]^{1/2}}$.

Then, by (S6.1), for any $k \in \mathbb{Z}$ and $j \in \{1, \ldots, N\}$, we can write

$$Y_{jn} = \Big(2\pi\sum_{r=1}^{l} h_r^2\Big)^{-1}\Big|d_j(\lambda)\Big|^2 = \epsilon_k^2(c_{jk}^2 + s_{jk}^2) + A_{n,j,-k} + 2\epsilon_k B_{n,j,-k}, \quad \text{(S6.2)}$$

where, $A_{n,j,-k} \equiv \Big(\sum_{m\neq k}\epsilon_m c_{jm}\Big)^2 + \Big(\sum_{m\neq k}\epsilon_m s_{jm}\Big)^2$ and $B_{n,j,-k} \equiv c_{jk}\sum_{m\neq k}\epsilon_m c_{jm} + s_{jk}\sum_{m\neq k}\epsilon_m s_{jm}$ are *independent* of ϵ_k. Now setting $k = j_0 l$ in (S6.2), the sum in (S7.1) is

$$\sum_{j=j_0-m}^{j_0+m} W_{jn} = \frac{1}{l}\sum_{j=(j_0-m-1)l+1}^{(j_0+m)l} Y_{jn} = \frac{1}{l}\sum_{j=(j_0-m-1)l+1}^{(j_0+m)l}\Big[\epsilon_k^2(c_{jk}^2 + s_{jk}^2) + 2\epsilon_k B_{n,j,-k} + A_{n,j,-k}\Big]$$

$$= \epsilon_k^2\Bigg[\frac{1}{l}\sum_{j=(j_0-m-1)l+1}^{(j_0+m)l}(c_{jk}^2 + s_{jk}^2)\Bigg] + 2\epsilon_k\Bigg[\frac{1}{l}\sum_{j=(j_0-m-1)l+1}^{(j_0+m)l} B_{n,j,-k}\Bigg] + \Bigg[\frac{1}{l}\sum_{j=(j_0-m-1)l+1}^{(j_0+m)l} A_{n,j,-k}\Bigg]$$

$$\equiv e_{n,k}\epsilon_k^2 + 2B_{n,-k}\epsilon_k + A_{n,-k}. \quad \text{(say)},$$

where $e_{n,k} \equiv e_n$ is a constant (that does not depend on j_0 and hence, on k) and where $A_{n,-k}$ and $B_{n,-k}$ are random variables that are measurable with respect to (w.r.t) the σ-field $\mathcal{D}_{(-k)} \equiv \sigma\langle\epsilon_j : j \neq k\rangle$. Next we consider the asymptotic behavior of $e_{n,k}$. Note that

$$e_{n,k} \equiv e_n = \Big(2\pi l\sum_{r=1}^{l} h_r^2\Big)^{-1}\sum_{j=-(m+1)l+1}^{ml}\sum_{r=1}^{l}\sum_{s=1}^{l} h_r h_s e^{\iota\lambda(r-s)} a_{j+r-1}a_{j+s-1}$$

$$= \Big(2\pi l\sum_{r=1}^{l} h_r^2\Big)^{-1}\sum_{p=-(l-1)}^{l-1} e^{\iota\lambda p}\sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p}\Bigg[\sum_{j\in\mathbb{Z}} a_j a_{j+p} + R_n(s,p)\Bigg],$$

where, $R_n(s,p)$ is defined by : $\sum_{j=-(m+1)l+1}^{ml} a_{j+s-1}a_{j+p+s-1} = \sum_{j\in\mathbb{Z}} a_j a_{j+p} + R_n(s,p)$. Using the geometric rate of decay of the a_j's for large $|j|$ and

Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sup_{\substack{s=1,\ldots,l\\|p|\leqslant(l-1)}}|R_n(s,p)| &\leqslant \sup_{\substack{s=1,\ldots,l\\|p|\leqslant(l-1)}}\left[\sum_{j\leqslant-(m+1)l+s}|a_ja_{j+p}|+\sum_{j>ml+s}|a_ja_{j+p}|\right]\\ &\leqslant \sup_{\substack{s=1,\ldots,l\\|p|\leqslant(l-1)}}\left[\sum_{j\leqslant-ml}|a_j||a_{j+p}|+\sum_{j\geqslant ml+1}|a_j||a_{j+p}|\right]\\ &\leqslant 2\sum_{|j|>(m-1)l}a_j^2 = O\Big(c_1^{2(m-1)l}\Big)\quad \text{as } n\to\infty.\end{aligned}$$

Since the bound on $R_n(s,p)$ holds uniformly over all (s,p), by Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\Big(2\pi l\sum_{r=1}^{l}h_r^2\Big)^{-1}\Bigg|\sum_{p=-(l-1)}^{l-1}e^{\iota\lambda p}\sum_{s=1\vee(1-p)}^{l\wedge(l-p)}h_sh_{s+p}\;R_n(s,p)\Bigg|\\ &\leqslant \Big(2\pi l\sum_{r=1}^{l}h_r^2\Big)^{-1}\sum_{p=-(l-1)}^{l-1}\Big(\sum_{s=1}^{l}h_s^2\Big)\Big|R_n(s,p)\Big|\\ &\leqslant \frac{2l+1}{2\pi l}\max_{|p|\leqslant l-1}\Big|R_n(s,p)\Big| = O\Big(c_1^{2(m-1)l}\Big).\end{aligned}$$

Thus, for $m\geqslant 2$,

$$e_n = \Big(2\pi l\sum_{r=1}^{l}h_r^2\Big)^{-1}\sum_{p=-(l-1)}^{l-1}\left\{\Big(\sum_{s=1\vee(1-p)}^{l\wedge(l-p)}h_sh_{s+p}\Big)e^{\iota\lambda p}\sum_{j\in\mathbb{Z}}a_ja_{j+p}\right\}+O(c_1^l). \tag{S6.3}$$

Next let $\omega(\delta)=\sup\{|h(x)-h(y)|:|x-y|\leqslant\delta,\ x,y\in[0,1]\}$, $\delta>0$. By uniform continuity of $h(\cdot)$ on $[0,1]$, $\lim_{\delta\downarrow 0}\omega(\delta)=0$. Hence, for $h_s\equiv h\,(s/l)$, by the bounded convergence theorem,

$$\begin{aligned}
&\sup_{|p|^2\leqslant 4l}\left| l^{-1}\sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p} - \int_0^1 h^2(x)\,dx\right| \\
&\leqslant \sup_{|p|^2\leqslant 4l}\left| l^{-1}\sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p} - l^{-1}\sum_{s=1}^{l} h_s^2\right| + \left| l^{-1}\sum_{s=1}^{l} h_s^2 - \int_0^1 h^2(x)\,dx\right| \\
&\leqslant \omega\left(\frac{2\sqrt{l}}{l}\right)\cdot\frac{1}{l}\sum_{s=1}^{l}\left|h\left(\frac{s}{l}\right)\right| + \frac{4\sqrt{l}}{l}\left(\max_{x\in[0,1]} h(x)^2\right) + \left|\frac{1}{l}\sum_{s=1}^{l}\left(\frac{s}{l}\right)^2 - \int_0^1 h^2(x)\,dx\right| \\
&= o(1),\ \text{ as } n\to\infty.
\end{aligned}$$

Next, for any sequence $\{a_m\}_{m\in\mathbb{Z}}$, define $(a*a)(j) \equiv \sum_{p\in\mathbb{Z}} a_p a_{p+j},\ j\in\mathbb{Z}$ and $\hat{a}(\lambda) = \sum_{j\in\mathbb{Z}} e^{\iota\lambda j} a_j,\ \lambda\in(-\pi,\pi]$. Then the Fourier transform of $a*a$ at frequency λ is given by $\widehat{a*a}(\lambda) = |\hat{a}(\lambda)|^2$ and

$$\sum_{|p|^2\leqslant 4l} e^{\iota\lambda p}\sum_{j\in\mathbb{Z}} a_j a_{j+p} = \sum_{p\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} a_j a_{j+p} e^{\iota\lambda p} - \sum_{|p|^2>4l}\sum_{j\in\mathbb{Z}} a_j a_{j+p} e^{\iota\lambda p} = |\hat{a}(\lambda)|^2 + O\Big(c_1^{\sqrt{l}}\Big). \tag{S6.4}$$

Thus combining equations (S6.3)-(S6.4), we can write

$$\begin{aligned}
&\left|\sum_{|p|\leqslant(l-1)} e^{\iota\lambda p}\sum_{j\in\mathbb{Z}} a_j a_{j+p}\left(\frac{1}{l}\sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p}\right) - |\hat{a}(\lambda)|^2\int_0^1 h^2(x)\,dx\right| \\
&\leqslant \left|\sum_{|p|^2\leqslant 4l} e^{\iota\lambda p}\sum_{j\in\mathbb{Z}} a_j a_{j+p}\left(\frac{1}{l}\sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p}\right) - |\hat{a}(\lambda)|^2\int_0^1 h(x)^2 dx\right| \\
&\qquad + \sum_{|p|^2>4l}\sum_{j\in\mathbb{Z}} |a_j a_{j+p}|\cdot\left(\frac{1}{l}\sum_{s=1}^{l} h_s^2\right) \\
&= o(1) \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Consequently, it follows that

$$\lim_{n\to\infty} l \cdot e_n = \frac{|\hat{a}(\lambda)|^2}{2\pi} \neq 0 \quad (\text{as} \quad f(\lambda) \neq 0). \tag{S6.5}$$

Now set $d_n = l$, $m_n = (\log n)^2, n \geqslant 2$. It is easy to verify that the requirements of condition (C.6) on these sequences of constants hold, provided $a \geqslant 1/2$. Now, by (S6.5), the stationarity of $\{X_t\}$ and the Cramér's condition on $(\epsilon_1^2, \epsilon_1)$, there exists a $\kappa \in (0,1)$ such that

$$\sup_{j_0\in J_n}\sup_{t\in A_n} \mathbf{E}\left|\mathbf{E}\left(\exp\left(\iota t \sum_{j=j_0-m}^{j_0+m} W_{jn}\right) \mid \tilde{\mathcal{D}}_{(-j_0 l)}\right)\right|$$

$$\leqslant \sup_{j_0\in J_n}\sup_{t\in A_n} \mathbf{E}\left|\mathbf{E}\left(\exp\left(\iota t \sum_{j=j_0-m}^{j_0+m} W_{jn}\right) \mid \mathcal{D}_{(-j_0 l)}\right)\right|$$

$$\leqslant \sup_{j_0\in J_n}\sup_{t\geqslant l} \mathbf{E}\Big|\mathbf{E}\Big(\exp\left(\iota t[e_{n,k}\epsilon_k^2 + 2B_{n,-k}\epsilon_k + A_{n,-k}]\right) \mid \{\epsilon_j : j \neq k\}\Big)\Big|$$

$$\leqslant \sup_{t\geqslant l, u\in\mathbb{R}} \left|\mathbf{E}\exp\left(\iota[te_n\epsilon_1^2 + u\epsilon_1]\right)\right| \leqslant \sup_{t\geqslant|\hat{a}(\lambda)|^2, u\in\mathbb{R}} \left|\mathbf{E}\Big(\exp\left(\iota[t\epsilon_1^2 + u\epsilon_1]\right)\right| \leqslant (1-\kappa),$$

for n large. Hence, condition (C.6) holds for all $a \geqslant 1/2$. □

Proof of Theorem 1. For proving Theorem 1, we shall use Theorem 2.1 of Lahiri (2007), which gives conditions for valid EEs for the sum of block variables of the form $n^{-1}\sum_{j=1}^n \tilde{Y}_{jn}$ for zero mean variables $\tilde{Y}_{jn} = f_{jn}(\mathcal{X}_{j,l})$, $j = 1, \ldots, n$, where f_{jn}'s are Borel measurable functions from $\mathbb{R}^l \to \mathbb{R}$. To this end, we set $\tilde{Y}_{jn} = Y_{j,n} - E(Y_{j,n})$ for $j = 1, \ldots, N$ and $\tilde{Y}_{jn} = 0$ for $j = N+1, \ldots, n$, where recall that $N = n - l + 1$ and where

$Y_{j,n} = |d_{j,n}(\lambda)|^2$ (cf. (1.1)). Then, it is easy to see that all the conditions in Theorem 2.1 of Lahiri (2007) are satisfied, provided we show that: (i) $\lim_{n\to\infty} \frac{\mathrm{Var}(\sum_{j=1}^n \tilde{Y}_{jn})}{nl}$ exists and is nonzero, and (ii) $\mathbf{E}|\tilde{Y}_{jn} - \tilde{Y}^\dagger_{jn,m}| \leqslant \kappa^{-1} l \exp(-\kappa m)$ for all $m > \kappa^{-1}$, for some $\kappa \in (0,1)$, where $\tilde{Y}^\dagger_{jn,m}$ is a random variable that is measurable w.r.t. $\sigma\langle \mathcal{D}_i : j - m \leqslant i \leqslant j + m + l\rangle$. Since $l = o(n)$, by (C.2), $\frac{\mathrm{Var}(\sum_{j=1}^n \tilde{Y}_{jn})}{nl} = \frac{N^2 \mathrm{Var}(T_n)}{[n/l]nl} \to \sigma^2_\infty$ as $n \to \infty$. Thus, (i) holds. As for (ii), define $\tilde{Y}^\dagger_{jn,m}$ by replacing X_t's in the definition of $Y_{j,n}$ by $X^\dagger_{t,m}$'s, $j = 1, \ldots, N$ and let $\tilde{Y}^\dagger_{jn,m} = 0$ for $j = N+1, \ldots, n$. Then it follows that the $\tilde{Y}^\dagger_{jn,m}$ is measurable w.r.t. $\sigma\langle \mathcal{D}_i : j - m \leqslant i \leqslant j + m + l\rangle$ for all $j = 1, \ldots, n$. Further, by condition (C.3) and Cauchy-Schwarz and Jensen's inequalities,

$$
\begin{aligned}
&\sup_{j=1,\ldots,N} \mathbf{E}|\tilde{Y}_{jn} - \tilde{Y}^\dagger_{jn,m}| \\
&= \Big(2\pi \sum_{r=1}^l h_r^2\Big)^{-1} \mathbf{E}\left|\Big|\sum_{r=1}^l h_r X_r \exp(\iota\lambda r)\Big|^2 - \Big|\sum_{r=1}^l h_r X^\dagger_{r,m} \exp(\iota\lambda r)\Big|^2\right| \\
&\leqslant \Big(2\pi \sum_{r=1}^l h_r^2\Big)^{-1} \cdot \left\{\mathbf{E}\left(\Big|\sum_{r=1}^l h_r X_r \exp(\iota\lambda r)\Big| + \Big|\sum_{r=1}^l h_r X^\dagger_{r,m} \exp(\iota\lambda r)\Big|\right)^2\right\}^{1/2} \\
&\qquad\qquad \times \left\{\mathbf{E}\Big|\sum_{r=1}^l h_r (X_r - X^\dagger_{r,m})\Big|^2\right\}^{1/2} \\
&\leqslant Cl \cdot e^{-\kappa m},
\end{aligned}
$$

for m large. Hence, (ii) holds and the result follows from Theorem 2.1 of Lahiri (2007). □

Proof of Corollary 1. Note that for any $x_0 \in \mathbb{R}$,

$$\omega(I_{(-\infty,x_0]} : \delta) = \int \sup\{|I_{(-\infty,x_0-y]}(x) - I_{(-\infty,x_0]}(x)| : |y| \leqslant \delta\}\phi_{\sigma_\infty^2}(x)dx$$
$$\leqslant \int |I_{[x_0-\delta,x_0+\delta]}(x)\phi_{\sigma_\infty^2}(x)dx \leqslant 2\delta/\sqrt{2\pi},$$

so that, $\sup_{x_0\in\mathbb{R}} |\omega(I_{(-\infty,x_0]} : \delta)| = O(\delta)$ as $\delta \downarrow 0$. Hence the result follows from Theorem 1 with $f = I_{(-\infty,x_0]}$. □

Proof of Corollary 2. Follows from Corollary 1 above, K-th order Taylor's expansion and the identity $\mathbf{P}\big(\sqrt{b}\big(\widehat{f}_n(\lambda) - f(\lambda)\big) \leqslant x\big) = \mathbf{P}(T_n \leqslant x - B_n)$ for all $x \in \mathbb{R}$. We omit the routine details. □

Proof of Theorem 2. Let $Z_n = \sigma_n^{-1} b^{1/2}[\widehat{f}_n(\lambda) - \mathbf{E}\widehat{f}_n(\lambda)]$ and note that we can write $T_{1,n} = g_n(Z_n)$, where $g_n(x) = (x + B_{1,n})\,[1 + b^{-1/2}c_n x + B_{2,n}]^{-1}$, where $B_{j,n}$, $j = 1, 2$, are as defined in (3.3). We can further expand $g_n(x)$ uniformly over $\{|x| \leqslant \log n\}$ as follows

$$\begin{aligned} g_n(x) &= (x + B_{1,n})\left[1 - \left(\frac{xc_n}{\sqrt{b}} + B_{2,n}\right) + \left(\frac{xc_n}{\sqrt{b}} + B_{2,n}\right)^2 + R_{1,n}(x)\right] \\ &= x\left[1 - \left(\frac{xc_n}{\sqrt{b}} + B_{2,n}\right) + \left(\frac{xc_n}{\sqrt{b}} + B_{2,n}\right)^2\right] + B_{1,n}\left[1 - \left(\frac{xc_n}{\sqrt{b}} + B_{2,n}\right)\right] \\ &\qquad + R_{2,n}(x) \\ &= a_{0,n} + a_{1,n}x + a_{2,n}x^2 + a_{3,n}x^3 + R_{2,n}(x), \end{aligned}$$

where the $a_{j,n}$'s are as defined in (3.3), with $R_{2,n}(x) = (x + B_{1,n})\,R_{1,n}(x) +$

$B_{1,n}\left(b^{-1/2}c_nx + B_{2,n}\right)^2$ and $R_{1,n}(x) = o(\left|b^{-1/2}c_nx + B_{2,n}\right|^2)$. This implies

$$g_{2,n}(x) \equiv a_{1,n}^{-1}\left(g_n(x) - a_{0,n}\right) = x + \tilde{a}_{2,n}x^2 + \tilde{a}_{3,n}x^3, \tag{S6.6}$$

with $\tilde{a}_{j,n},\ j = 2,3$ defined as in (3.3). Hence, $T_{1,n} = a_{0,n} + a_{1,n}g_{2,n}\left(Z_n\right) + R_{2,n}\left(Z_n\right)$.

Next, define $r_{1,n}(t) = \frac{\sqrt{b}\kappa_{3,n}}{6}H_3(t)$ and $r_{2,n}(t) = \frac{b\kappa_{4,n}}{24}H_4(t) + \frac{b\kappa_{3,n}^2}{72}H_6(t)$, where $\kappa_{r,n}$ is the r^{th} cumulant of Z_n and $\phi(t)$ is the $N(0,1)$-density function. By Corollary 1 and (S6.6),

$$\begin{aligned}
&\mathbf{P}\left(g_{2,n}\left(Z_n\right) \leqslant u\right) = \int_{g_{2,n}(t)\leqslant u, |t|\leqslant \log n} \phi(t)\left[1 + b^{-1/2}r_{1,n}(t) + b^{-1}r_{2,n}(t)\right]dt + o\left(b^{-1}\right) \\
&= \int_{-\infty}^{u} \phi\left(g_{2,n}^{-1}(y)\right)\left[1 + b^{-1/2}r_{1,n}\left(g_{2,n}^{-1}(y)\right) + b^{-1}r_{2,n}\left(g_{2,n}^{-1}(y)\right)\right]\frac{I(|g_{2,n}^{-1}(y)| \leqslant \log n)dy}{\left|g_{2,n}'\left(g_{2,n}^{-1}(y)\right)\right|} \\
&\qquad + o\left(b^{-1}\right).
\end{aligned}$$

Check that $g_{2,n}^{-1}(y) = y - \tilde{a}_{2,n}y^2 + \tilde{a}_{3,n}y^3 + 2\tilde{a}_{2,n}^2y^3,\ |y| \leqslant C\log n$. Using this in the last equation above, one can show that

$$\mathbf{P}\Big(g_{2,n}\left(Z_n\right) \leqslant u\Big) = \int_{-\infty}^{u} \phi(t)\left[1 + s_{1,n}(t) + s_{2,n}(t)\right]dt + o\left(b^{-1}\right), \tag{S6.7}$$

where, $s_{j,n}(t)$ are polynomials that can be expressed in terms of $r_{j,n}(t)$ as $s_{1,n}(t) = \tilde{a}_{2,n}t^3 + \left(b^{-1/2}r_{1,n}(t) - 2\tilde{a}_{2,n}t\right)$ and $s_{2,n}(t) = \left(\tilde{a}_{3,n} - 2\tilde{a}_{2,n}^2\right)t^4 + \frac{\tilde{a}_{2,n}^2t^4}{2}\ H_2(t) + \left[\frac{r_{1,n}(t)}{\sqrt{b}} - 2\tilde{a}_{2,n}t\right]\tilde{a}_{2,n}t^3 - \frac{\tilde{a}_{2,n}t^2}{b}\ r_{1,n}'(t) + \left(6\tilde{a}_{2,n}^2 - 3\tilde{a}_{3,n}\right)t^2 - \frac{2\tilde{a}_{2,n}t}{\sqrt{b}}\ r_{1,n}(t) + \frac{r_{2,n}(t)}{b}$. Next define $q_{j,n}(.)$ by using the relation, $q_{j,n}(u)\phi(u) = \int_{-\infty}^{u} s_{j,n}(t)\phi(t)dt$, $j = 1,2$. Then, using the identity $\int H_j(t)\phi(t)dt = -H_{j-1}(t)\phi(t)$, for all $j \geqslant 1$, we can rewrite (S6.7) as

$$\mathbf{P}\Big(g_{2,n}\left(Z_n\right) \leqslant u\Big) = \Phi(u) + q_{1,n}(u)\phi(u) + q_{2,n}(u)\phi(u) + o\left(b^{-1}\right). \quad \text{(S6.8)}$$

Note that, $R_{2,n}\left(Z_n\right) \xrightarrow{P} 0$, and $\mathbf{P}\big(g_n\big(Z_n\big) \leqslant u\big) = \mathbf{P}\big(g_{2,n}\big(Z_n\big) \leqslant u_n\big)$, where $u_n = a_{1,n}^{-1}\left(u - a_{0,n}\right)$. Using these facts and using (S6.8), we get (3.4). This completes the proof of Theorem 2. □

For proving the results from Section 4, we need a moderate deviation bound on $\widehat{f}_n(\lambda)$. This is stated as a lemma below. Note that it is valid *without* the conditional Cramér's condition (C.6).

Lemma 1. *Suppose that conditions (C.1)-(C.5) hold. Then, for any* $\gamma \in (\sigma_\infty^2, \infty)$*, there exists a constant* $C_3 \in (0, \infty)$ *(depending only on* γ, s, κ*,* $\mathbf{E}|X_1|^{s+\kappa}$*) such that for all* $n \geqslant 2$*,*

$$\mathbf{P}\left(|T_n(\lambda)| > [(s-2)\gamma \log n]^{1/2}\right) \leqslant C_3 b^{-(s-2)/2}(\log n)^{-2}.$$

Proof of Lemma 1. Follows from Theorem 2.4 of Lahiri (2007) and the proof of Theorem 1 above. □

Proof of Theorem 4. From Theorem 2, note that $u_n - u = ua_{1,n}^{-1}(1 - a_{1,n}) -$

$a_{1,n}^{-1}a_{0,n}$. Using this and Taylor's expansion, for any *fixed* $u \in \mathbb{R}$, we have

$$\begin{aligned}\mathbf{P}(T_{1,n} \leqslant u) = \Phi(u) + (u_n - u)[\phi(u) + \{q'_{1,n}(u)\phi(u) + q_{1,n}(u)u\phi(u)\}] \\ + q'_{1,n}(u)\phi(u) + \frac{(u_n - u)^2}{2}u\phi(u) + q_{2,n}(u)\phi(u) + o(b^{-1}) \\ + O\big(|1 - a_{1,n}|^3 + |a_{0,n}|^3\big) + O\big(b^{-1/2}\{|1 - a_{1,n}|^2 + |a_{0,n}|^2\}\big),\end{aligned} \tag{S6.9}$$

This implies $\mathbf{P}\big(f(\lambda) \in I_{1,n}\big) = (1 - \alpha) + \{B_{2,n}z_{1-\alpha}\phi(z_{1-\alpha}) - B_{1,n}\phi(z_{1-\alpha}) + q_{1,n}(z_{1-\alpha})\phi(z_{1-\alpha})\}(1 + o(1))$. By similar arguments (cf. Theorem 3), it follows that for the VST based one-sided CI, the expansion for the coverage probability is given by $\mathbf{P}\big(f(\lambda) \in I_{2,n}\big) = (1 - \alpha) + \{B_{3,n}z_{1-\alpha}\phi(z_{1-\alpha}) - B_{1,n}\phi(z_{1-\alpha}) + q^{\dagger}_{1,n}(z_{1-\alpha})\phi(z_{1-\alpha})\}(1 + o(1))$, where recall that $q^{\dagger}_{1,n}(u)$ and $q_{1,n}(u)$ are of the order $O(b^{-1/2})$. □

Proof of Theorem 5. Using (S6.9) and the parity of the polynomials associated with higher order terms, we get the following expansion in the two sided case:

$$\begin{aligned}&\mathbf{P}(f(\lambda) \in J_{1,n}) = \mathbf{P}\big(T_{1,n} \leqslant z_{1-\alpha}/2) - \mathbf{P}(T_{1,n} \leqslant z_{\alpha/2}\big) \\ &= (1 - \alpha) + \Big[2B_{2,n}z_{1-\alpha}\phi(z_{1-\alpha/2}) - 2B_{1,n}\{q'_{1,n}(z_{1-\alpha/2}) + q_{1,n}(z_{1-\alpha/2})z_{1-\alpha/2}\}\phi(z_{1-\alpha/2}) \\ &\qquad + q_{2,n}(z_{1-\alpha/2})\phi(z_{1-\alpha/2})\Big](1 + o(1)).\end{aligned}$$

By similar arguments, it follows that for the VST based two-sided CI, the expansion for the coverage probability is given by $\mathbf{P}(f(\lambda) \in J_{2,n}) = (1 -$

$\alpha)+\Big[2B_{3,n}z_{1-\alpha}\phi(z_{1-\alpha/2})-2B_{1,n}\{(q^{\dagger}_{1,n})'(z_{1-\alpha/2})+q^{\dagger}_{1,n}(z_{1-\alpha/2})z_{1-\alpha/2}\}\phi(z_{1-\alpha/2})$
$+q^{\dagger}_{2,n}(z_{1-\alpha/2})\phi(z_{1-\alpha/2})\Big](1+o(1)).$

□

S7 Appendix

S7.1 Theoretical framework and general conditions for the EE

Suppose the $\{X_t : t \in \mathbb{Z}\}$ are defined on a probability space $(\Omega, \mathcal{F}, P)$. Also suppose that $\{\mathcal{D}_j : j \in \mathbb{Z}\}$ be a collection of sub σ-fields of $\mathcal{F}$. Let $\mathcal{D}_p^q = \sigma\langle\{\mathcal{D}_j : j \in \mathbb{Z},\ p \leqslant j \leqslant q\}\rangle$, $-\infty \leqslant p < q \leqslant \infty$. As mentioned before, here we shall adopt a framework similar to Lahiri (2007) for sums of block variables, which is an extension of Götze and Hipp (1983)'s framework for sums of weakly dependent random variables. Let $Y_{j,n} \equiv |d_{j,n}(\lambda)|^2$, $Z_{j,n} = Y_{j,n} - \mathbf{E}Y_{j,n}$, $1 \leqslant j \leqslant N$ and $W_{k,n} = \frac{1}{l}\sum_{j=(k-1)l+1}^{(kl\wedge N)} Z_{j,n}$, $1 \leqslant k \leqslant b_0$, where $b_0 = \lceil N/l \rceil$, the smallest integer not less than N/l and where $x \wedge y = \min(x, y)$, $x, y \in \mathbb{R}$. Let $b \equiv b_n = N/l$. Note that $T_n = \sqrt{b} \cdot \frac{1}{N}\sum_{j=1}^{N}\big(Y_{j,n} - \mathbf{E}Y_{j,n}\big) = \frac{1}{\sqrt{b}}\sum_{k=1}^{b_0} W_{k,n}$, a scaled sum of block variables. We will use the following conditions:

(C.1) We assume that there exists a constant $\kappa \in (0,1)$ such that for all

$$n > \kappa^{-1},\ \kappa\log n < l < \kappa^{-1}n^{1-\kappa} \quad \text{and} \quad \max\{|h_r| : r = 1, \ldots, l\} < \kappa^{-1}.$$

(C.2) There exist a constant $\kappa \in (0,1]$ and an integer $s \geqslant 3$ such that for all $n \geqslant \kappa^{-1}$, $\max\{\mathbf{E}|Y_{j,n}|^{(s+\kappa)} : j = 1, \ldots, N\} < \kappa^{-1}$. Further, $\lambda \in [0,\pi]$, and $\lim_{n\to\infty} \mathrm{Var}(T_n) = \sigma^2_\infty$ exists and is non-zero.

(C.3) We assume that there exists a constant $\kappa \in (0,1)$ such that for all $n, m > \kappa^{-1}$ and for all $j \geqslant 1$, there exists a $\mathcal{D}_{j-m}^{j+m}$-measurable $X^\dagger_{j,m}$ such that $\mathbf{E}|X_j - X^\dagger_{j,m}|^2 \leqslant \kappa^{-1}\exp(-\kappa m)$.

(C.4) There exists a constant $\kappa \in (0,1)$, such that for all $n, m = 1, 2, \ldots$, and $A \in \mathcal{D}^n_{-\infty}$ and $B \in \mathcal{D}^\infty_{n+m}$, $|\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)| \leqslant \kappa^{-1}\exp(-\kappa m)$.

(C.5) There exists a constant $\kappa \in (0,1)$, such that for all $i, j, k, r, m = 1, 2, \ldots$, and $A \in \mathcal{D}^j_i$ with $i < k < r < j$ and $m > \kappa^{-1}$, $\mathbf{E}\Big|\mathbf{P}(A|\mathcal{D}_j : j \notin [k,r]) - \mathbf{P}(A|\mathcal{D}_j : j \in [i-m,k) \cup (r, j+m])\Big| \leqslant \kappa^{-1}\exp(-\kappa m)$.

(C.6) There exist constants $a \in (0,\infty)$, $\kappa \in (0,1)$ and sequences $\{m_n\} \subset \mathbb{N}$ and $\{d_n\} \subset [1,\infty)$ with $m_n^{-1} + m_n b^{-1/2} = o(1)$, $d_n = O(l + b^a)$ and $d_n^2 m_n = O(b^{(1-\kappa)})$ such that for all $n \geqslant \kappa^{-1}$,

$$\max_{j_0 \in J_n} \sup_{t \in A_n} \mathbf{E}\left|\mathbf{E}\left\{\exp\left(\iota t \sum_{j=j_0-m_n}^{j_0+m_n} W_{j,n}\right) \middle| \tilde{\mathcal{D}}_{j_0}\right\}\right| \leqslant (1-\kappa), \tag{S7.1}$$

where $J_n = \{m_n + 1, \ldots, b - m_n - 1\}$, $A_n = \{t \in \mathbb{R} : \kappa d_n \leqslant |t| \leqslant [b^a + l]^{(1+\kappa)}\}$, and $\tilde{\mathcal{D}}_{j_0} = \sigma\langle\{\mathcal{D}_j : j \notin [(j_0 - \lfloor \frac{m_n}{2} \rfloor)l + 1, (j_0 + \lfloor \frac{m_n}{2} \rfloor + 1)l]\}$.

Condition (C.1) states the growth rate of the block size l and allows l to grow at a rate of $O(n^{1-\kappa})$ for arbitrarily small $\kappa > 0$. It also requires the taper-weights to be bounded, which is satisfied in most applications. The first part of (C.2) gives a sufficient condition for the existence of $(s+\kappa)$-order absolute moment of the block variables $W_{k,n}$. Part (ii) of (C.2) ensures that asymptotic variance of $T_n(\lambda)$ exists and is nonzero. By symmetry, this covers all $\lambda \in [-\pi, \pi]$. Note that the problem of existence of the asymptotic variance of $T_n(\lambda)$, when the taper weights h_r's derive from a taper function $h : [0,1] \to \mathbb{R}$ (cf. (2.2)) is well-studied. A set of sufficient conditions for this are given by (cf. Dahlhaus (1985)): (i) h is continuously differentiable on [0,1] with $\int_0^1 h^2(x)\,dx \in (0,\infty)$; (ii) $l = o(n)$ and (iii) f, f_4 are bounded and f is continuous at λ, where f_4 denotes the fourth order cumulant density of $\{X_t\}$. Note that a bounded f_4 exists if $\mathbf{E}|X_1|^{4+\kappa} < \infty$ for some $\kappa > 0$.

Next consider (C.3)-(C.6). Here (C.3) is an approximation condition that connects the variables X_j to the strong mixing property (C.4) of the auxiliary σ-fields $\mathcal{D}_j$'s. (C.5) is an approximate Markovian condition and is a variant of a similar condition used in Götze and Hipp (1983). This condition holds if the σ-fields $\mathcal{D}_j$'s have the Markov property. (C.6) is a Cramér-type condition on the block variables $W_{j,n}$ and is perhaps the most difficult one to verify. For examples of choices of $\mathcal{D}_j$ in different time series models, see Götze and Hipp (1983) and Lahiri (2003).

For the linear process $\{X_t\}$ in (2.1), we take $\mathcal{D}_j = \sigma\langle\epsilon_j\rangle$, $j \in \mathbb{Z}$. Then, the σ-fields $\mathcal{D}_j$'s are independent and consequently, the strong-mixing condition (C.4) and the approximate Markovian condition (C.5) on the $\mathcal{D}_j$'s hold trivially. To verify (C.3), we set

$$X_{j,m} = \sum_{k=-m}^{m} \epsilon_{j-k} a_k, \; j \in \mathbb{Z}, m \geqslant 1.$$

Then, it is evident from (2.1), (2.4) and the definition of $X_{j,m}$ that Condition (C.3) holds. Next consider condition (C.6). As shown in Proposition 1, by choosing the sequences $\{d_n\}$ and $\{m_n\}$ appropriately, the conditional Cramér's condition on the block-variables $W_{k,n}$ hold under the Cramér's condition (2.3) on the joint distribution of $(\epsilon_1, \epsilon_2)'$.

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