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Robust Estimation of Dispersion Parameter in Discretely Observed Diffusion Processes

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Supplementary Material

In this supplementary note, some technical lemmas and the proofs of Theorems 2.1, 2.2, and 2.3 are provided. In particular, we deal with the case of $\alpha > 0$ and refer to Kessler (1997) for the case of $\alpha = 0$. In what follows, we denote $a_{i-1} = a(X_{t_{i-1}^n})$, $b_{i-1}(\sigma) = b(X_{t_{i-1}^n}, \sigma)$ and $c_{i-1}(\sigma) = b(X_{t_{i-1}^n}, \sigma)^2$ for notational simplicity. Also, we drop k_0 and k_1 in $V_{n,i}^{\alpha}(\sigma; k_0, k_1)$ without confusion. Further, let

$$\begin{aligned} Z_i &= \frac{1}{\sqrt{h_n}}\big(W_{t_i^n} - W_{t_{i-1}^n}\big), \\ \Delta_i &= \int_{t_{i-1}^n}^{t_i^n} \big\{a(X_s) - a(X_{t_{i-1}^n})\big\} ds + \int_{t_{i-1}^n}^{t_i^n} \big\{b(X_s, \sigma_0) - b(X_{t_{i-1}^n}, \sigma_0)\big\} dW_s, \\ \Delta_{1,i} &= \Delta_i - \frac{1}{2} b_{i-1}(\sigma_0)\partial_x b_{i-1}(\sigma_0)(Z_i^2 - 1)h_n, \end{aligned} \tag{1}$$

provided $\partial_x b$ exists. We shall use the relation $A_n \lesssim B_n$, where A_n and B_n are nonnegative, to mean that $A_n \le C B_n$ for some constant $C > 0$. For example, $A \lesssim 1$ means that A is bounded by some constant C.

S1 Proof of Theorem 2.1

Lemma 1. *Suppose that* **A1** *and* **A3** *holds. Then, for $k \ge 1$,*

$$\max_{i \le n} E|\Delta_i|^{2k} \lesssim h_n^{2k}. \tag{S1.1}$$

If, in addition, $\partial_x^2 a$ and $\partial_x^3 b$ exist and belong to $\mathcal{P}$, then

$$\max_{i \le n} E|\Delta_{1,i}|^{2k} \lesssim h_n^{3k}. \tag{S1.2}$$

Proof. By Lemma 6 of Kessler (1997), we have that for $t_{i-1}^n \le t \le t_i^n$,

$$E\big|X_t - X_{t_{i-1}^n}\big|^k \lesssim h_n^{k/2}. \tag{S1.3}$$

Using this and Theorem 6.3 in Friedman (1975), one can prove (S1.1).

In order to show (S1.2), note that

$$\int_{t_{i-1}^n}^{t_i^n} \int_{t_{i-1}^n}^{s} dW_z dW_s = \frac{h_n}{2}\big(Z_i^2 - 1\big).$$

Then, by (1.17) in Chapter 5 of Kloeden and Platen (1999), we can write that

$$
\begin{aligned}
\Delta_{1,i} &= \int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{s} L_\sigma a(X_z)dzds + \int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{s} L^\sigma a(X_z)dW_zds \\
&\quad + \int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{s} L_\sigma b(X_z,\sigma)dzdW_s + \int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{s}\int_{t_{i-1}^n}^{z} L_\sigma L^\sigma b(X_u,\sigma)dudW_zdW_s \\
&\quad + \int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{s}\int_{t_{i-1}^n}^{z} L^\sigma L^\sigma b(X_u,\sigma)dW_udW_zdW_s \\
&:= D_{1,i}+D_{2,i}+D_{3,i}+D_{4,i}+D_{5,i},
\end{aligned}
$$

where L_σ is the one given in (2.4) and $L^\sigma = b(x,\sigma)\partial_x$. We first deal with $E|D_{4,i}|^{2k}$. It follows from Theorem 6.3 in Friedman (1975) and Jensen's inequality that

$$
\begin{aligned}
E\left|D_{4,i}\right|^{2k} &\lesssim h_n^{k-1}\int_{t_{i-1}^n}^{t_i^n} E\left\{\int_{t_{i-1}^n}^{s}\int_{t_{i-1}^n}^{z} L_\sigma L^\sigma b(X_u,\sigma)dudW_z\right\}^{2k} ds \\
&\lesssim h_n^{k-1}\int_{t_{i-1}^n}^{t_i^n} h_n^{k-1}\int_{t_{i-1}^n}^{s} E\left\{\int_{t_{i-1}^n}^{z} L_\sigma L^\sigma b(X_u,\sigma)du\right\}^{2k} dzds \\
&\lesssim h_n^{k-1}\int_{t_{i-1}^n}^{t_i^n} h_n^{k-1}\int_{t_{i-1}^n}^{s} h_n^{2k-1}\int_{t_{i-1}^n}^{z} E\big|L_\sigma L^\sigma b(X_u,\sigma)\big|^{2k} dudzds \lesssim h_n^{4k}. \quad \text{(S1.4)}
\end{aligned}
$$

Analogous to the above, it can be shown that

$$
E|D_{1,i}|^{2k} \lesssim h_n^{4k},\ E|D_{2,i}|^{2k} \lesssim h_n^{3k},\ E|D_{3,i}|^{2k} \lesssim h_n^{3k} \text{ and } E|D_{5,i}|^{2k} \lesssim h_n^{3k},
$$

which together with (S1.4) yield (S1.2). □

Lemma 2. *Suppose that* **A1** *and* **A3** *hold. Then, if $f:\mathbb{R}\times\Theta\to\mathbb{R}$ belongs to $\mathcal{P}$ and $nh_n^p\to 0$ for some $p>1$,*

$$
\sup_{\sigma\in\Theta}\max_{i\le n}\big|f^j(X_{t_{i-1}^n},\sigma)Z_i^k\Delta_i^l h_n^m\big| = o(1) \quad a.s.,
$$

where $j,k,l\in\{0,1,2,\cdots\}$ and $m>-l$.

Proof. Due to **A3** and (S1.1), we have that for any $\epsilon,\kappa>0$,

$$
\begin{aligned}
\sum_{n=1}^{\infty} P\Big(\max_{i\le n}\big|(1+|X_{t_{i-1}^n}|)^C Z_i^k\Delta_i^l h_n^m\big| > \epsilon\Big) &\le \sum_{n=1}^{\infty}\frac{n}{\epsilon^\kappa}\max_{i\le n} E\big|(1+|X_{t_{i-1}^n}|)^C Z_i^k\Delta_i^l h_n^m\big|^\kappa \\
&\lesssim \sum_{i=1}^{\infty} nh_n^{(l+m)\kappa} = \sum_{n=1}^{\infty} o\big(n^{1-(l+m)\kappa/p}\big).
\end{aligned}
$$

By choosing κ such that $1-(l+m)\kappa/p<-1$, the lemma is asserted. □

Lemma 3. *Suppose that* **A1**–**A3** *hold; let $f(x,\sigma)\in\mathcal{P}:\mathbb{R}\times\Theta\to\mathbb{R}$ be differentiable with respect to x and σ. Then, if $\partial_x f$, $\partial_\sigma f$ and $g:\mathbb{R}\to\mathbb{R}$ belong to $\mathcal{P}$ and $nh_n^p\to 0$ for some $p>1$,*

$$
\sup_{\sigma\in\Theta}\Big|\frac{1}{n}\sum_{i=1}^{n} g(Z_i)f(X_{t_{i-1}^n},\sigma) - E(g(Z))\int f(x,\sigma)d\mu_0(x)\Big| = o(1) \quad a.s..
$$

Proof. First, we show that for each $\sigma \in \Theta$,

$$\frac{1}{n}\sum_{i=1}^{n} f(X_{t_{i-1}^n}, \sigma) \xrightarrow{a.s.} \int f(x,\sigma)d\mu_0(x) \quad \text{as } n \to \infty. \tag{S1.5}$$

In view of ergodic property, we have

$$\frac{1}{nh_n}\int_0^{nh_n} f(X_s, \sigma)ds \xrightarrow{a.s.} \int f(x,\sigma)d\mu_0(x) \quad \text{as } n \to \infty. \tag{S1.6}$$

By using Jensen's inequality, Cauchy's inequality and (S1.3), we have that for any $r > 0$,

$$\begin{aligned}
&E\Big|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_n}\int_{t_{i-1}^n}^{t_i^n}\big\{f(X_{t_{i-1}^n},\sigma) - f(X_s,\sigma)\big\}ds\Big|^{2r} \\
&\le \frac{1}{nh_n}\sum_{i=1}^{n}\int_{t_{i-1}^n}^{t_i^n}\Big\{E|X_{t_{i-1}^n} - X_s|^{4r}\Big\}^{\frac{1}{2}}\Big\{\int_0^1 E\big|\partial_x f(X_s + u(X_{t_{i-1}^n} - X_s),\sigma)\big|^{4r}du\Big\}^{\frac{1}{2}}ds \\
&= O(h_n^r) = o(n^{-r/p}).
\end{aligned}$$

Hence, it follows that for any $\epsilon > 0$ and $r > p$,

$$\sum_{n=1}^{\infty} P\Big(\Big|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_n}\int_{t_{i-1}^n}^{t_i^n}\{f(X_{t_{i-1}^n},\sigma) - f(X_s,\sigma)\}ds\Big| > \epsilon\Big) \le \sum_{n=1}^{\infty} o(n^{-r/p}) < \infty,$$

which together with (S1.6) establishes (S1.5).

Next, let $S_i(\sigma) := \{g(Z_i) - E(g(Z))\}f(X_{t_{i-1}^n},\sigma)$ and $\mathcal{G}_i^n := \sigma\{W_s : s \le t_i^n\}$. Then, $\{(S_i(\sigma), \mathcal{G}_i^n)\}_{i=1}^n$ becomes a martingale difference, and thus, using Burkholder's inequality and Jensen's inequality, we have that for any $r > 1$,

$$E\Big|\frac{1}{n}\sum_{i=1}^{n} S_i(\sigma)\Big|^{2r} \le \frac{1}{n^{r+1}}\sum_{i=1}^{n} E|S_i(\sigma)|^{2r} = O\big(n^{-r}\big).$$

Therefore, we have

$$\Big|\frac{1}{n}\sum_{i=1}^{n} g(Z_i)f(X_{t_{i-1}^n},\sigma) - E(g(Z))\frac{1}{n}\sum_{i=1}^{n} f(X_{t_{i-1}^n},\sigma)\Big| = o(1) \quad a.s., \tag{S1.7}$$

and so the point wise convergence in the lemma is asserted from (S1.5). To establish the uniform convergence, it suffices to show that $\{n^{-1}\sum_{i=1}^n g(Z_i)f(X_{t_{i-1}^n},\sigma)\}$ is equicontinuous with probability one. Observe that

$$\begin{aligned}
&\sup_{\sigma_1,\sigma_2\in\Theta}\frac{1}{|\sigma_1 - \sigma_2|}\frac{1}{n}\sum_{i=1}^{n}\Big|g(Z_i)f(X_{t_{i-1}^n},\sigma_1) - g(Z_i)f(X_{t_{i-1}^n},\sigma_2)\Big| \\
&\le \sup_{\sigma\in\Theta}\frac{1}{n}\sum_{i=1}^{n}\Big|g(Z_i)\partial_\sigma f(X_{t_{i-1}^n},\sigma)\Big| \lesssim \frac{1}{n}\sum_{i=1}^{n} g(Z_i)(1 + |X_{t_{i-1}^n}|^{2C}) \\
&= O(1) \quad a.s.,
\end{aligned}$$

where the convergence of the last line is due to (S1.5) and (S1.7). Hence, we can ensure the lemma by the compactness of Θ. $\square$

Lemma 4. *Suppose that* **A1**–**A4** *and* **A6(i)** *hold; let $f(x,\sigma)\in\mathcal{P}:\mathbb{R}\times\Theta\to\mathbb{R}$ be differentiable with respect to x and σ. If $\partial_x f$ and $\partial_\sigma f$ belong to $\mathcal{P}$ and $nh_n^p\to 0$ for some $p>1$, then for any $k\in\{0,1,2,\cdots\}$,*

$$\sup_{\sigma\in\Theta}\Big|\frac{1}{n}\sum_{i=1}^n f(X_{t_{i-1}^n},\sigma)Z_i^{2k}e^{-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2}-\frac{(2k)!}{2^k k!}\int\left(1+\alpha\frac{c(x,\sigma_0)}{c(x,\sigma)}\right)^{-\frac{2k+1}{2}}f(x,\sigma)d\mu_0(x)\Big|=o(1)\quad a.s..$$

Proof. Let

$$\begin{aligned}T_i(\sigma)\;&:=\;f(X_{t_{i-1}^n},\sigma)Z_i^{2k}e^{-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2}-E\Big[f(X_{t_{i-1}^n},\sigma)Z_i^{2k}e^{-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2}\Big|\mathcal{G}_{i-1}^n\Big]\\&=\;f(X_{t_{i-1}^n},\sigma)Z_i^{2k}e^{-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2}-\frac{(2k)!}{2^k k!}\Big(1+\alpha\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}\Big)^{-\frac{2k+1}{2}}f(X_{t_{i-1}^n},\sigma).\end{aligned}$$

For any $r\in\{1,2,\cdots\}$, since $\max_{i\le n}\sup_{\sigma\in\Theta}E|T_i(\sigma)|^{2r}\lesssim 1$ by **A3**, it follows from Burkholder's inequality and Jensen's inequality that

$$E\Big|\frac{1}{n}\sum_{i=1}^n T_i(\sigma)\Big|^{2r}\le\frac{1}{n^{r+1}}\sum_{i=1}^n E|T_i(\sigma)|^{2r}=O\big(n^{-r}\big).\tag{S1.8}$$

The point wise convergence is therefore proved by (S1.5) and (S1.8). Moreover, since

$$\begin{aligned}&\sup_{\sigma_1,\sigma_2\in\Theta}\frac{1}{|\sigma_1-\sigma_2|}\frac{1}{n}\sum_{i=1}^n\Big|f(X_{t_{i-1}^n},\sigma_1)Z_i^{2k}e^{-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_1)}Z_i^2}-f(X_{t_{i-1}^n},\sigma_2)Z_i^{2k}e^{-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_2)}Z_i^2}\Big|\\&\le\sup_{\sigma\in\Theta}\frac{1}{n}\sum_{i=1}^n\Big|\partial_\sigma f(X_{t_{i-1}^n},\sigma)+\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^2}Z_i^2 f(X_{t_{i-1}^n},\sigma)\Big|Z_i^{2k}e^{-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_2)}Z_i^2}\\&\lesssim\frac{1}{n}\sum_{i=1}^n(1+|X_{t_{i-1}^n}|^{2C})(1+Z_i^2)Z_i^{2k}=O(1)\quad a.s.,\end{aligned}$$

$\big\{n^{-1}\sum_{i=1}^n f(X_{t_{i-1}^n},\sigma)Z_i^{2k}e^{-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2}\big\}$ is equicontinuous with probability one, and thus the uniform convergence is established. □

Lemma 5. *Suppose that* **A1**–**A4** *and* **A6(i)** *with some $2k$ hold. For any $k_0\in\{1,\cdots,k\}$ and $k_1\in\{0\}\cup\mathbb{N}$, if $nh_n^p\to 0$ for some $p>1$, then*

$$\sup_{\sigma\in\Theta}\Big|\frac{1}{n}\sum_{i=1}^n V_{n,i}^\alpha(\sigma)-V(\sigma)\Big|=o(1)\quad a.s.,$$

where

$$V(\sigma):=\int\frac{1}{c(x,\sigma)^{\frac{\alpha}{2}}}\left\{\frac{1}{\sqrt{1+\alpha}}-\left(1+\frac{1}{\alpha}\right)\left(1+\alpha\frac{c(x,\sigma_0)}{c(x,\sigma)}\right)^{-\frac{1}{2}}\right\}d\mu_0(x).$$

Proof. Since

$$X_{t_i^n}=X_{t_{i-1}^n}+a_{i-1}h_n+b_{i-1}(\sigma_0)Z_i\sqrt{h_n}+\Delta_i,\tag{S1.9}$$

we can express that

$$\begin{aligned} X_{t_i^n} - r_{k_0}(h_n, X_{t_{i-1}^n}, \sigma) &= b_{i-1}(\sigma_0) Z_i \sqrt{h_n} + \Delta_i - \sum_{j=2}^{k_0} \frac{h_n^j}{j!} L_\sigma^i X_{t_{i-1}^n} \\ &:= b_{i-1}(\sigma_0) Z_i \sqrt{h_n} + J_i(\sigma), \end{aligned} \tag{S1.10}$$

and therefore

$$\begin{aligned} &\frac{(X_{t_i^n} - r_{k_0}(h_n, X_{t_{i-1}^n}, \sigma))^2}{c_{i-1}(\sigma) h_n} \Big\{ 1 + \sum_{j=1}^{k_1} h_n^j d_j(X_{t_{i-1}^n}, \sigma) \Big\} \\ &= \Big\{ \frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)} Z_i^2 + \frac{2 b_{i-1}(\sigma_0) Z_i J_i(\sigma) \sqrt{h_n} + J_i(\sigma)^2}{c_{i-1}(\sigma) h_n} \Big\} \Big\{ 1 + \sum_{j=1}^{k_1} h_n^j d_j(X_{t_{i-1}^n}, \sigma) \Big\} \\ &:= \frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)} Z_i^2 + K_i(\sigma). \end{aligned} \tag{S1.11}$$

Note that by Lemma 2,

$$\max_{i \le n} \sup_{\sigma \in \Theta} |K_i(\sigma)| = o(1) \quad a.s.. \tag{S1.12}$$

Let

$$U_i^\alpha(\sigma) := \frac{1}{c_{i-1}(\sigma)^{\frac{\alpha}{2}}} \Big[\frac{1}{\sqrt{1+\alpha}} - \Big(1 + \frac{1}{\alpha}\Big) \exp\Big(-\frac{\alpha}{2} \frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)} Z_i^2 \Big) \Big]. \tag{S1.13}$$

Then, we have that

$$\begin{aligned} \big| V_{n,i}^\alpha(\sigma) - U_i^\alpha(\sigma) \big| &= \frac{1}{c_{i-1}(\sigma)^{\frac{\alpha}{2}}} \Big| \frac{1}{\sqrt{1+\alpha}} \sum_{j=1}^{k_1} h_n^j d_j^\alpha(X_{t_{i-1}^n}, \sigma) \\ &\qquad - \Big(1 + \frac{1}{\alpha}\Big) \Big[\Big\{ 1 + \sum_{j=1}^{k_1} h_n^j d_j^\alpha(X_{t_{i-1}^n}, \sigma) \Big\} e^{-\frac{\alpha}{2} K_i(\sigma)} - 1 \Big] e^{-\frac{\alpha}{2} \frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)} Z_i^2} \Big| \\ &\lesssim h_n (1 + |X_{t_{i-1}^n}|)^C \big(1 + e^{-\frac{\alpha}{2} K_i(\sigma)}\big) + \big| e^{-\frac{\alpha}{2} K_i(\sigma)} - 1 \big|, \end{aligned}$$

and thus it follows from Lemma 2 and (S1.12) that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \sup_{\sigma \in \Theta} \big| V_{n,i}^\alpha(\sigma) - U_i^\alpha(\sigma) \big| &\lesssim (1 + e^{\max_{i \le n} \sup_{\sigma \in \Theta} \frac{\alpha}{2} |K_i(\sigma)|}) \max_{i \le n} (1 + |X_{t_{i-1}^n}|)^C h_n \\ &\quad + \max_{i \le n} \sup_{\sigma \in \Theta} |K_i(\sigma)| e^{\max_{i \le n} \sup_{\sigma \in \Theta} \frac{\alpha}{2} |K_i(\sigma)|} \\ &= o(1) \quad a.s.. \end{aligned} \tag{S1.14}$$

Moreover, by Lemmas 3 and 4, we have

$$\sup_{\sigma \in \Theta} \Big| \frac{1}{n} \sum_{i=1}^n U_i^\alpha(\sigma) - V(\sigma) \Big| = o(1) \quad a.s..$$

Combining the above and (S1.14), we obtain the lemma. □

Proof of Theorem 2.1. Note that

$$\left(\frac{1}{x}\right)^{\alpha/2}\left\{\frac{1}{\sqrt{1+\alpha}}-\left(1+\frac{1}{\alpha}\right)\left(1+\alpha\frac{x_0}{x}\right)^{-1/2}\right\},\quad x>0,x_0>0$$

is minimized only at $x=x_0$. Then, $V(\sigma)$ has a unique minimum at $\sigma=\sigma_0$ by **A5**. Since Θ is compact and the contrast function converges uniformly to $V(\sigma)$ with probability one, we can establish the theorem by standard arguments. □

S2 Proof of Theorem 2.2

Lemma 6. *Suppose that* **A1**, **A3**–**A4** *and* **A6(ii)** *with some* $2k$ *hold. For any* $k_0\in\{1,\cdots,k\}$ *and* $k_1\in\{0\}\cup\mathbb{N}$, *if* $nh_n^2\to 0$, *then*

$$\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^n\partial_\sigma V_{n,i}^\alpha(\sigma_0)-\frac{1}{\sqrt{n}}\sum_{i=1}^n\partial_\sigma U_i^\alpha(\sigma_0)\Big|=o_P(1).$$

Proof. Let $R_{1,i-1}(\sigma)$ be such that

$$\frac{\partial}{\partial\sigma}\Big[\frac{1}{c_{i-1}(\sigma)^{\frac{\alpha}{2}}}\Big\{1+\sum_{j=1}^{k_1}h_n^jd_j^\alpha(X_{t_{i-1}^n},\sigma)\Big\}\Big]=-\frac{\alpha}{2}\frac{\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+1}}+h_nR_{1,i-1}(\sigma),$$

then, from (S1.11), we have

$$\begin{aligned}&\partial_\sigma V_{n,i}^\alpha(\sigma)\\
=&\Big\{-\frac{\alpha}{2}\frac{\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+1}}+h_nR_{1,i-1}(\sigma)\Big\}\Big\{\frac{1}{\sqrt{1+\alpha}}-\Big(1+\frac{1}{\alpha}\Big)\exp\Big(-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2\underbrace{-\frac{\alpha}{2}K_i(\sigma)}_{(*)}\Big)\Big\}\\
&+\frac{1+\alpha}{2}\frac{1}{c_{i-1}(\sigma)^{\frac{\alpha}{2}}}\Big\{1+\sum_{j=1}^{k_1}h_n^jd_j^\alpha(X_{i-1},\sigma)\Big\}\Big\{-\frac{c_{i-1}(\sigma_0)\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^2}Z_i^2+\partial_\sigma K_i(\sigma)\Big\}\\
&\times\exp\Big(-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2\underbrace{-\frac{\alpha}{2}K_i(\sigma)}_{(*)}\Big).\end{aligned}$$

Further, let $\widetilde{U}_i^\alpha(\sigma)$ be the term without $(*)$ in the RHS of the above equation. We first show that $\big|\sum_{i=1}^n\{\partial_\sigma V_{n,i}^\alpha(\sigma_0)-\widetilde{U}_i^\alpha(\sigma)\}\big|=o_P(\sqrt{n})$. For this, we decompose $\partial_\sigma V_{n,i}^\alpha(\sigma_0)-\widetilde{U}_i^\alpha(\sigma_0)$ into $I_i+II_i+III_i$, where

$$\begin{aligned}I_i&=\frac{1+\alpha}{2}\frac{\partial_\sigma c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}}(Z_i^2-1)e^{-\frac{\alpha}{2}Z_i^2}\big(1-e^{-\frac{\alpha}{2}K_i(\sigma_0)}\big),\\
II_i&=\Big[\frac{1+\alpha}{2}\frac{1}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}}}\Big\{\frac{\partial_\sigma c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)}Z_i^2-\partial_\sigma K_i(\sigma_0)\Big\}\sum_{j=1}^{k_1}h_n^jd_j^\alpha(X_{i-1},\sigma_0)\\
&\quad+\Big(1+\frac{1}{\alpha}\Big)h_nR_{1,i-1}(\sigma_0)\Big]e^{-\frac{\alpha}{2}Z_i^2}\big(1-e^{-\frac{\alpha}{2}K_i(\sigma_0)}\big),\\
III_i&=-\frac{1+\alpha}{2}\frac{1}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}}}\partial_\sigma K_i(\sigma_0)e^{-\frac{\alpha}{2}Z_i^2}\big(1-e^{-\frac{\alpha}{2}K_i(\sigma_0)}\big).\end{aligned}$$

To deal with $n^{-1/2}\sum_{i=1}^n I_i$, we note that

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{\partial_\sigma c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}}(Z_i^2-1)e^{-\frac{\alpha}{2}Z_i^2}K_i(\sigma_0) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{2\partial_\sigma c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}b_{i-1}(\sigma_0)}(Z_i^3-Z_i)e^{-\frac{\alpha}{2}Z_i^2}\frac{\Delta_i}{\sqrt{h_n}} \\ &+\frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{\partial_\sigma c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}}(Z_i^2-1)e^{-\frac{\alpha}{2}Z_i^2}R_{2,i}(\sigma_0) \\ &:= \sum_{i=1}^n\left(I_i^a+I_i^b\right), \qquad \text{(S2.1)}\end{aligned}$$

where

$$R_{2,i}(\sigma)=K_i(\sigma)-\frac{2b_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}\frac{Z_i\Delta_i}{\sqrt{h_n}}.$$

By simple calculations and (S1.1), we have

$$\sup_{\sigma\in\Theta}\max_{i\le n}\left\{E|R_{2,i}(\sigma)|^{2k}\vee E|\partial_\sigma R_{2,i}(\sigma)|^{2k}\right\}\lesssim h_n^{2k}. \qquad \text{(S2.2)}$$

Observe that from (1),

$$\begin{aligned}\sum_{i=1}^n I_i^a &= \frac{\sqrt{h_n}}{\sqrt{n}}\sum_{i=1}^n \frac{\partial_\sigma c_{i-1}(\sigma_0)\partial_\sigma b_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}}(Z_i^3-Z_i)(Z_i^2-1)e^{-\frac{\alpha}{2}Z_i^2} \qquad \text{(S2.3)} \\ &+\frac{1}{\sqrt{nh_n}}\sum_{i=1}^n \frac{2\partial_\sigma c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}b_{i-1}(\sigma_0)}(Z_i^3-Z_i)e^{-\frac{\alpha}{2}Z_i^2}\Delta_{1,i}.\end{aligned}$$

Since

$$\frac{\sqrt{h_n}}{\sqrt{n}}\sum_{i=1}^n \frac{\partial_\sigma c_{i-1}(\sigma_0)\partial_\sigma b_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}}E\Big[(Z_i^3-Z_i)(Z_i^2-1)e^{-\frac{\alpha}{2}Z_i^2}\Big|\mathcal{G}_{i-1}^n\Big]=0 \qquad \text{(S2.4)}$$

and, by Lemma 2,

$$\begin{aligned}&\frac{h_n}{n}\sum_{i=1}^n\left\{\frac{\partial_\sigma c_{i-1}(\sigma_0)\partial_\sigma b_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}}\right\}^2E\Big[(Z_i^3-Z_i)^2(Z_i^2-1)^2e^{-\alpha Z_i^2}\Big|\mathcal{G}_{i-1}^n\Big] \qquad \text{(S2.5)} \\ &\lesssim \max_{i\le n}(1+|X_{i-1}|)^C h_n = o(1) \quad a.s.,\end{aligned}$$

according to Lemma 9 of Genon-Catalot and Jacod (1993), the first term of the righthand side of (S2.3) converges to zero in probability. Also, by using Cauchy's inequality, Lemma 1 and (S2.2), it can be shown that

$$\frac{1}{\sqrt{nh_n}}\sum_{i=1}^n E\left|\frac{2\partial_\sigma c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}b_{i-1}(\sigma_0)}(Z_i^3-Z_i)e^{-\frac{\alpha}{2}Z_i^2}\Delta_{1,i}\right|\lesssim\sqrt{n}h_n, \qquad \text{(S2.6)}$$

$$E\sum_{i=1}^n|I_i^b|\lesssim\sqrt{n}h_n \qquad \text{(S2.7)}$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}E\Big|\frac{\partial_\sigma c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}}(Z_i^2-1)e^{-\frac{\alpha}{2}Z_i^2}K_i(\sigma_0)^2\Big| \tag{S2.8}$$
$$\lesssim \frac{1}{\sqrt{n}h_n}\sum_{i=1}^{n}E\Big|(1+|X_{i-1}|)^C(Z_i^2-1)Z_i^2\Delta_i^2\Big| + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}E\Big|(1+|X_{i-1}|)^C(Z_i^2-1)R_{2,i}(\sigma_0)^2\Big|$$
$$\lesssim \sqrt{n}h_n.$$

Since

$$1-e^{-\frac{\alpha}{2}K_i(\sigma_0)} = \frac{\alpha}{2}K_i(\sigma_0) - \frac{\alpha^2}{8}K_i(\sigma_0)^2e^{\zeta_i},$$

where $|\zeta_i| \le \alpha|K_i(\sigma_0)|/2$, by combining (S2.4)-(S2.8) and the fact that $e^{\max_i|K_i(\sigma_0)|} = O(1)$ *a.s.*, it follows that

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}I_i &= \frac{\alpha(1+\alpha)}{4}\sum_{i=1}^{n}\left(I_i^a+I_i^b\right) - \frac{\alpha^2(1+\alpha)}{16}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial_\sigma c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}}(Z_i^2-1)e^{-\frac{\alpha}{2}Z_i^2}K_i(\sigma_0)^2e^{\zeta_i}\\ &= o_P(1).\end{aligned} \tag{S2.9}$$

Now note that

$$\partial_\sigma K_i(\sigma) = -2\frac{b_{i-1}(\sigma_0)\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^2}\frac{Z_i\Delta_i}{\sqrt{h_n}} + \partial_\sigma R_{2,i}(\sigma).$$

Using this equation, (S2.2) and a similar method as for (S2.9), we can show that $\sum_{i=1}^{n}II_i = o_P(\sqrt{n})$ and $\sum_{i=1}^{n}III_i = o_P(\sqrt{n})$. Hence, we have

$$\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\partial_\sigma V_{n,i}^{\alpha}(\sigma_0) - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widetilde{U}_i^{\alpha}(\sigma_0)\Big| = o_P(1). \tag{S2.10}$$

Next, letting

$$R_{3,i}(\sigma) := \widetilde{U}_i^{\alpha}(\sigma) - \partial_\sigma U_i^{\alpha}(\sigma) - \frac{1+\alpha}{2}\frac{1}{c_{i-1}(\sigma)^{\frac{\alpha}{2}}}\partial_\sigma K_i(\sigma)e^{-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2},$$

then it can be seen that $|R_{3,i}(\sigma)| \lesssim h_n(1+|X_{i-1}|)^C\left(1+Z_i^2+\partial_\sigma K_i(\sigma)\right)$. Using again (1) and Lemma 9 of Genon-Catalot and Jacod (1993), we can establish that

$$\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widetilde{U}_i^{\alpha}(\sigma_0) - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\partial_\sigma U_i^{\alpha}(\sigma_0)\Big| = o_P(1),$$

which together with (S2.10) completes the proof. □

Lemma 7. *Suppose that* **A1**–**A4** *and* **A6(ii)** *with some* $2k$ *hold. If* $nh_n^p \to 0$ *for some* $p>1$, *then*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\partial_\sigma U_i^{\alpha}(\sigma_0) \xrightarrow{d} N(0,\Sigma_\alpha^*),$$

where

$$\Sigma_\alpha^* = \frac{1}{4}\left\{2\frac{(1+\alpha)^2(1+2\alpha^2)}{(1+2\alpha)^2\sqrt{1+2\alpha}} - \frac{\alpha^2}{1+\alpha}\right\}\int\frac{\left(\partial_\sigma c(x,\sigma_0)\right)^2}{c(x,\sigma_0)^{\alpha+2}}d\mu_0(x). \tag{S2.11}$$

Proof. According to Theorems 3.2 and 3.4 in Hall and Heyde (1980), it suffices to verify that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} E\Big[\partial_\sigma U_i^\alpha(\sigma_0)\Big|\mathcal{G}_{i-1}^n\Big] \xrightarrow{P} 0, \tag{S2.12}$$

$$\frac{1}{n}\sum_{i=1}^{n} E\Big[\big(\partial_\sigma U_i^\alpha(\sigma_0)\big)^2\Big|\mathcal{G}_{i-1}^n\Big] \xrightarrow{P} \Sigma_\alpha^*, \tag{S2.13}$$

$$\frac{1}{n^2}\sum_{i=1}^{n} E\Big[\big(\partial_\sigma U_i^\alpha(\sigma_0)\big)^4\Big|\mathcal{G}_{i-1}^n\Big] \xrightarrow{P} 0. \tag{S2.14}$$

Note that

$$\partial_\sigma U_i^\alpha(\sigma_0) = \frac{1}{2}\frac{\partial_\sigma c_{i-1}(\sigma_0)}{c_{i-1}(\sigma_0)^{\frac{\alpha}{2}+1}}\Big\{(1+\alpha)(1-Z_i^2)e^{-\frac{\alpha}{2}Z_i^2} - \frac{\alpha}{\sqrt{1+\alpha}}\Big\}. \tag{S2.15}$$

Then, it follows from the fact that $E\big[e^{-\frac{\alpha}{2}Z_i^2}\big|\mathcal{G}_{i-1}^n\big] = (1+\alpha)^{-\frac{1}{2}}$ and $E\big[Z_i^2 e^{-\frac{\alpha}{2}Z_i^2}\big|\mathcal{G}_{i-1}^n\big] = (1+\alpha)^{-\frac{3}{2}}$ that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} E\Big[\partial_\sigma U_i^\alpha(\sigma_0)\Big|\mathcal{G}_{i-1}^n\Big] = 0,$$

which yields (S2.12). Next, by (S2.15), we have that

$$\frac{1}{n}\sum_{i=1}^{n} E\Big[\big(\partial_\sigma U_i^\alpha(\sigma_0)\big)^2\Big|\mathcal{G}_{i-1}^n\Big] = \frac{1}{4}\left\{2\frac{(1+\alpha)^2(1+2\alpha^2)}{(1+2\alpha)^2\sqrt{1+2\alpha}} - \frac{\alpha^2}{1+\alpha}\right\}\frac{1}{n}\sum_{i=1}^{n}\frac{(\partial_\sigma c_{i-1}(\sigma_0))^2}{c_{i-1}(\sigma_0)^{\alpha+2}},$$

where we have used the fact that $E\big[\big(1-Z_i^2\big)^2 e^{-\alpha Z_i^2}\big|\mathcal{G}_{i-1}^n\big] = 2\big(1+2\alpha^2\big)\big(1+2\alpha\big)^{-\frac{5}{2}}$ and $E\big[\big(1-Z_i^2\big)e^{-\frac{\alpha}{2}Z_i^2}\big|\mathcal{G}_{i-1}^n\big] = \alpha\big(1+\alpha\big)^{-\frac{3}{2}}$. Hence, (S2.13) is asserted by Lemma 3. Finally, using again Lemma 3 and the fact that

$$\frac{1}{n^2}\sum_{i=1}^{n} E\Big[\big(\partial_\sigma U_i^\alpha(\sigma_0)\big)^4\Big|\mathcal{G}_{i-1}^n\Big] \lesssim \frac{1}{n^2}\sum_{i=1}^{n}\big(1+|X_{i-1}|^{2C}\big),$$

(S2.14) is verified. □

Lemma 8. *Suppose that* **A1**–**A4** *and* **A6(ii)** *with some* $2k$ *hold. For any* $k_0 \in \{1,\cdots,k\}$ *and* $k_1 \in \{0\}\cup\mathbb{N}$*, if* $nh_n^p \to 0$ *for some* $p>1$*, then*

$$\frac{1}{n}\sum_{i=1}^{n}\partial_\sigma^2 V_{n,i}^\alpha(\sigma_0) \xrightarrow{P} \frac{\alpha^2+2}{4(1+\alpha)^{\frac{3}{2}}}\int\frac{(\partial_\sigma c(x,\sigma_0))^2}{c(x,\sigma_0)^{\frac{\alpha}{2}+2}}d\mu_0(x).$$

Proof. Let

$$R_{4,i}(\sigma) := \Big\{1+\sum_{j=1}^{k_1}h_n^j d_j^\alpha(X_{i-1},\sigma)\Big\}\Big\{-\frac{c_{i-1}(\sigma_0)\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^2}Z_i^2 + \partial_\sigma K_i(\sigma)\Big\} + \frac{c_{i-1}(\sigma_0)\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^2}Z_i^2$$

and

$$R_{5,i}(\sigma) := \frac{h_n}{\sqrt{1+\alpha}}R_{1,i-1}(\sigma) + (1+\alpha)\Big\{\frac{1}{2}\frac{R_{4,i}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}}} - \frac{h_n}{\alpha}R_{1,i-1}(\sigma)\Big\}\exp\Big(-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2 - \frac{\alpha}{2}K_i(\sigma)\Big).$$

Then, we can rewrite

$$\begin{aligned}\partial_\sigma V_{n,i}^\alpha(\sigma) \quad = \quad & -\frac{1}{2}\frac{\alpha}{\sqrt{1+\alpha}}\frac{\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+1}} \\ & +\frac{1+\alpha}{2}\Big\{\frac{\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+1}}-\frac{c_{i-1}(\sigma_0)\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+2}}Z_i^2\Big\}\exp\Big(-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2-\frac{\alpha}{2}K_i(\sigma)\Big) \\ & +R_{5,i}(\sigma),\end{aligned}$$

and thus we have

$$\begin{aligned}&\partial_\sigma^2 V_{n,i}^\alpha(\sigma) \\ =& -\frac{1}{2}\frac{\alpha}{\sqrt{1+\alpha}}\Big\{\frac{\partial_\sigma^2 c_{i-1}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+1}}-(1+\frac{\alpha}{2})\frac{(\partial_\sigma c_{i-1}(\sigma))^2}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+2}}\Big\} \\ &+\frac{1+\alpha}{2}\Big[\frac{\partial_\sigma^2 c_{i-1}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+1}}-(1+\frac{\alpha}{2})\frac{(\partial_\sigma c_{i-1}(\sigma))^2}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+2}}-\Big\{\frac{\partial_\sigma^2 c_{i-1}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+2}}-(2+\frac{\alpha}{2})\frac{(\partial_\sigma c_{i-1}(\sigma))^2}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+3}}\Big\}c_{i-1}(\sigma_0)Z_i^2\Big] \\ &\qquad\times\exp\Big(-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2\Big)\underbrace{\Big\{1-\frac{\alpha}{2}K_i(\sigma)e^{\zeta_i(\sigma)}\Big\}}_{(*)} \\ &+\frac{\alpha(1+\alpha)}{4}\Big\{\frac{\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+1}}-\frac{c_{i-1}(\sigma_0)\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^{\frac{\alpha}{2}+2}}Z_i^2\Big\}\Big\{\frac{c_{i-1}(\sigma_0)\partial_\sigma c_{i-1}(\sigma)}{c_{i-1}(\sigma)^2}Z_i^2\underbrace{-\partial_\sigma K_i(\sigma)}_{(*)}\Big\} \\ &\qquad\times\exp\Big(-\frac{\alpha}{2}\frac{c_{i-1}(\sigma_0)}{c_{i-1}(\sigma)}Z_i^2\Big)\underbrace{\Big\{1-\frac{\alpha}{2}K_i(\sigma)e^{\zeta_i(\sigma)}\Big\}+\partial_\sigma R_{5,i}(\sigma)}_{(*)} \\ :=& \widehat{U}_i^\alpha(\sigma)+R_{6,i}(\sigma),\end{aligned}$$

where $|\zeta_i(\sigma)|\le \alpha K_i(\sigma)/2$ and $\widehat{U}_i^\alpha(\sigma)$ denotes the term without $(*)$ in the RHS of the above equation. Then, by some calculations, it can be shown that

$$\begin{aligned}&|R_{4,i}(\sigma)|+|\partial_\sigma R_{4,i}(\sigma)|\lesssim(1+|X_{i-1}|)^C\big(h_n Z_i^2+|\partial_\sigma K_i(\sigma)|+|\partial_\sigma^2 K_i(\sigma)|\big), \\ &\big|\partial_\sigma R_{5,i}(\sigma)\big|\lesssim(1+|X_{i-1}|)^C\big(h_n+|R_{4,i}(\sigma)|+|\partial_\sigma R_{4,i}(\sigma)|\big)\big(1+Z_i^2+|\partial_\sigma K_i(\sigma)|\big)\big(1+|K_i(\sigma)|\big)e^{|K_i(\sigma)|}\end{aligned}$$

and

$$|R_{6,i}(\sigma)| \;\lesssim\; (1+|X_{i-1}|)^C\big(1+Z_i^2\big)\big(1+Z_i^2+|\partial_\sigma K_i(\sigma)|\big)|K_i(\sigma)|e^{|K_i(\sigma)|}+|\partial_\sigma R_{5,i}(\sigma)|.$$

Furthermore, in view of Lemma 2, one can check that

$$\max_{i\le n}\sup_{\sigma\in\Theta}|R_{6,i}(\sigma)|=o(1)\quad a.s..$$

Therefore, it follows from Lemmas 3 and 4 that

$$\frac{1}{n}\sum_{i=1}^{n}\partial_\sigma^2 V_{n,i}^\alpha(\sigma)$$
$$\xrightarrow{a.s.} \frac{1}{2}\int\left[-\frac{\alpha}{\sqrt{1+\alpha}}+(1+\alpha)\Big\{1+\alpha\frac{c(x,\sigma_0)}{c(x,\sigma)}\Big\}^{-\frac{1}{2}}-(1+\alpha)\Big\{1+\alpha\frac{c(x,\sigma_0)}{c(x,\sigma)}\Big\}^{-\frac{3}{2}}\right]\frac{\partial_\sigma^2 c(x,\sigma)}{c(x,\sigma)^{\frac{\alpha}{2}+1}}d\mu_0(x)$$
$$+\frac{1}{4}\int\left[\frac{\alpha(2+\alpha)}{\sqrt{1+\alpha}}-(1+\alpha)(2+\alpha)\Big\{1+\alpha\frac{c(x,\sigma_0)}{c(x,\sigma)}\Big\}^{-\frac{1}{2}}+(1+\alpha)(4+\alpha)\Big\{1+\alpha\frac{c(x,\sigma_0)}{c(x,\sigma)}\Big\}^{-\frac{3}{2}}\right]$$
$$\times\frac{(\partial_\sigma c(x,\sigma))^2}{c(x,\sigma)^{\frac{\alpha}{2}+2}}d\mu_0(x)$$
$$+\frac{\alpha(1+\alpha)}{4}\int\Big\{1+\alpha\frac{c(x,\sigma_0)}{c(x,\sigma)}\Big\}^{-\frac{3}{2}}\left[\frac{c(x,\sigma_0)(\partial_\sigma c(x,\sigma))^2}{c(x,\sigma)^{\frac{\alpha}{2}+3}}-3\Big\{1+\alpha\frac{c(x,\sigma_0)}{c(x,\sigma)}\Big\}^{-1}\right.$$
$$\left.\times\frac{c(x,\sigma_0)^2(\partial_\sigma c(x,\sigma))^2}{c(x,\sigma)^{\frac{\alpha}{2}+4}}\right]d\mu_0(x)\qquad\text{unformly in }\sigma,$$

which yields the lemma. □

Proof of Theorem 2.2. Let $l_n^\alpha(\sigma)=n^{-1}\sum_{i=1}^n V_{n,i}^\alpha(\sigma)$. Using Taylor's theorem, we have that

$$0=\sqrt{n}\partial_\sigma l_n^\alpha(\sigma_0)+\sqrt{n}\,(\hat\sigma_n^\alpha-\sigma_0)\int_0^1\partial_\sigma^2 l_n^\alpha\big(\sigma_0+u(\hat\sigma_n^\alpha-\sigma_0)\big)du.$$

Note that by Lemmas 6 and 7,

$$\sqrt{n}\partial_\sigma l_n^\alpha(\sigma_0)\xrightarrow{d} N(0,\Sigma_\alpha^*),\tag{S2.16}$$

where Σ_α^* is the one given in (S2.11). Moreover, since the limit of $\partial_\sigma^2 l_n^\alpha(\sigma)$ is continuous in σ by **A2**, **A4** and **A6(ii)**, it follows from Lemma 8 that

$$\int_0^1\partial_\sigma^2 l_n^\alpha\big(\sigma_0+u(\hat\sigma_n^\alpha-\sigma_0)\big)du\xrightarrow{a.s.}\frac{\alpha^2+2}{4(1+\alpha)^{\frac{3}{2}}}\int\frac{(\partial_\sigma c(x,\sigma_0))^2}{c(x,\sigma_0)^{\frac{\alpha}{2}+2}}d\mu_0(x).\tag{S2.17}$$

Therefore, the theorem is established from (S2.16) and (S2.17). □

S3 Proof of Theorem 2.3

Proof of Theorem 2.3. First, we note that Lemmas 1–4 also hold for the diffusion process of the form $dX_t=a(X_t,\theta)dt+b(X_t,\sigma)dW_t$. In this case, $X_{t_i^n}$ can be written as

$$X_{t_i^n}=X_{t_{i-1}^n}+a(X_{t_{i-1}^n},\theta_0)h_n+b_{i-1}(\sigma_0)Z_i\sqrt{h_n}+\Delta_i,$$

and thus, we have

$$\begin{aligned}X_{t_i^n}-\mathbf{r}_{k_0}(h_n,X_{t_{i-1}^n},\theta,\sigma)&=X_{t_i^n}-X_{t_{i-1}^n}-a(X_{t_{i-1}^n},\theta_0)h_n-\sum_{j=2}^{k_0}\frac{h_n^j}{j!}L_{\theta,\sigma}^i X_{t_{i-1}^n}\\&=b_{i-1}(\sigma_0)Z_i\sqrt{h_n}+\{a(X_{t_{i-1}^n},\theta_0)-a(X_{t_{i-1}^n},\theta)\}h_n+\Delta_i-\sum_{j=2}^{k_0}\frac{h_n^j}{j!}L_{\theta,\sigma}^i X_{t_{i-1}^n}\\&:=b_{i-1}(\sigma_0)Z_i\sqrt{h_n}+J_i(\theta,\sigma),\end{aligned}$$

where

$$\Delta_i = \int_{t_{i-1}^n}^{t_i^n} \big\{a(X_s,\theta_0) - a(X_{t_{i-1}^n},\theta_0)\big\}ds + \int_{t_{i-1}^n}^{t_i^n} \big\{b(X_s,\sigma_0) - b(X_{t_{i-1}^n},\sigma_0)\big\}dW_s.$$

Since $J_i(\theta,\sigma)$ is $o(h^{1-\kappa})$ almost surely for any $\kappa > 0$, it can be seen that (S1.12) also holds, which in turn yields Lemma 5 for the general diffusion case. That is,

$$\sup_{(\theta,\sigma)\in\Theta} \Big| \frac{1}{n}\sum_{i=1}^{n} V_{n,i}^{\alpha}(\theta,\sigma) - V(\sigma)\Big| = o(1) \quad a.s.,$$

which establishes the theorem. □