Non-stationary Multivariate Spatial Covariance Estimation via Low-Rank Regularization

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Supplementary Material

The supplementary material contains proofs of Proposition 1 and Corollary 1 in Section 2.2.

Proof of Proposition 1

Let $\tilde{\mathbf{S}} \equiv \mathbf{S} - \sigma^2 \mathbf{I}_n$. With simple algebra, the objective function can be rewritten as

$$\begin{aligned}\phi(\mathbf{M}, v^2) = {} & \frac{1}{2} \left\| \mathbf{F}\mathbf{M}\mathbf{F}' + v^2 \mathbf{I}_n - \mathbf{H}_F \tilde{\mathbf{S}} \mathbf{H}_F \right\|_F^2 + \frac{1}{2} \left\| \mathbf{H}_F \tilde{\mathbf{S}} \mathbf{H}_F - \tilde{\mathbf{S}} \right\|_F^2 \\ & + v^2 \mathrm{trace}(\mathbf{H}_F \tilde{\mathbf{S}} \mathbf{H}_F - \tilde{\mathbf{S}}) + \tau \left\| \mathbf{F}\mathbf{M}\mathbf{F}' \right\|_* . \end{aligned} \tag{S.1}$$

We find the minimizer $\hat{\mathbf{M}}_\tau(v^2)$ of $\phi(\mathbf{M}, v^2)$ with respect to $\mathbf{M}$ as a function of v^2. Since the second and the third terms of (S.1) do not involve $\mathbf{M}$, this is equivalent to finding the minimizer of function

$$\phi^*(\mathbf{M}) = \frac{1}{2} \left\| \mathbf{F}\mathbf{M}\mathbf{F}' + v^2 \mathbf{I}_n - \mathbf{H}_F \tilde{\mathbf{S}} \mathbf{H}_F \right\|_F^2 + \tau \left\| \mathbf{F}\mathbf{M}\mathbf{F}' \right\|_* .$$

Let $\tilde{\mathbf{Q}}\tilde{\mathbf{D}}\tilde{\mathbf{Q}}'$ be the eigen-decomposition of $\mathbf{FMF}'$, where $\tilde{\mathbf{D}} = \mathrm{diag}(\tilde{d}_1, \ldots, \tilde{d}_n)$. Then

$$\begin{aligned}
\phi^*(\mathbf{M}) &= \frac{1}{2}\left\|\mathbf{Q}(\mathbf{D}-v^2\mathbf{I}_n)\mathbf{Q}' - \tilde{\mathbf{Q}}\tilde{\mathbf{D}}\tilde{\mathbf{Q}}'\right\|_F^2 + \tau\left\|\tilde{\mathbf{Q}}\tilde{\mathbf{D}}\tilde{\mathbf{Q}}'\right\|_* \\
&= \frac{1}{2}\mathrm{trace}((\mathbf{D}-v^2\mathbf{I}_n)^2) + \frac{1}{2}\mathrm{trace}(\tilde{\mathbf{D}}^2) - \mathrm{trace}(\mathbf{Q}(\mathbf{D}-v^2\mathbf{I}_n)\mathbf{Q}'\tilde{\mathbf{Q}}\tilde{\mathbf{D}}\tilde{\mathbf{Q}}') + \tau\,\mathrm{trace}(\tilde{\mathbf{D}}) \\
&\geq \frac{1}{2}\mathrm{trace}((\mathbf{D}-v^2\mathbf{I}_n)^2) + \frac{1}{2}\mathrm{trace}(\tilde{\mathbf{D}}^2) - \mathrm{trace}((\mathbf{D}-v^2\mathbf{I}_n)\tilde{\mathbf{D}}) + \tau\,\mathrm{trace}(\tilde{\mathbf{D}}) \\
&= \frac{1}{2}\mathrm{trace}\big((\mathbf{D}-v^2\mathbf{I}_n-\tilde{\mathbf{D}})^2\big) + \tau\,\mathrm{trace}(\tilde{\mathbf{D}}),
\end{aligned} \tag{S.2}$$

where the inequality follows from Theobald's trace inequality (Theobald (1975)) with equality if $\tilde{\mathbf{Q}} = \mathbf{Q}$. Since $\tilde{\mathbf{D}} \succeq \mathbf{0}$, the minimizer of $\phi^*(\mathbf{M})$ is obtained when $\tilde{\mathbf{D}} = \mathrm{diag}\big((d_1-\tau-v^2)_+, \ldots, (d_n-\tau-v^2)_+\big)$ in (S.2). It follows that

$$\begin{aligned}
\mathbf{F}\hat{\mathbf{M}}_\tau(v^2)\mathbf{F}' &= \mathbf{Q}\,\mathrm{diag}\big((d_1-\tau-v^2)_+, \ldots, (d_n-\tau-v^2)_+\big)\,\mathbf{Q}' \\
&= \mathbf{Q}_{K^*}\,\mathrm{diag}\big((d_1-\tau-v^2)_+, \ldots, (d_{K^*}-\tau-v^2)_+\big)\,\mathbf{Q}'_{K^*}.
\end{aligned}$$

Using $\mathbf{H}_F\mathbf{F} = \mathbf{F}$ and $\mathbf{H}_F\mathbf{Q}_{K^*} = \mathbf{Q}_{K^*}$, we obtain

$$\hat{\mathbf{M}}_\tau(v^2) = (\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'\mathbf{Q}_{K^*}\,\mathrm{diag}\big((d_1-\tau-v^2)_+, \ldots, (d_{K^*}-\tau-v^2)_+\big)\,\mathbf{Q}'_{K^*}\mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}.$$

It remains to prove (6). Plugging $\hat{\mathbf{M}}_\tau(v^2)$ into (S.1) and applying (S.2), we have

$$\begin{aligned}
\phi\big(\hat{\mathbf{M}}_\tau(v^2), v^2\big) &= \phi^*\big(\hat{\mathbf{M}}_\tau(v^2)\big) + v^2\mathrm{trace}(\mathbf{H}_F\tilde{\mathbf{S}}\mathbf{H}_F - \tilde{\mathbf{S}}) + \frac{1}{2}\left\|\mathbf{H}_F\tilde{\mathbf{S}}\mathbf{H}_F - \tilde{\mathbf{S}}\right\|_F^2 \\
&= \frac{1}{2}\sum_{k=1}^{K^*}\big(d_k - v^2 - (d_k-v^2-\tau)_+\big)^2 + \frac{1}{2}v^4(n-K^*) + \tau\sum_{k=1}^{K^*}(d_k-v^2-\tau)_+ \\
&\quad + v^2\mathrm{trace}(\mathbf{H}_F\tilde{\mathbf{S}}\mathbf{H}_F - \tilde{\mathbf{S}}) + \frac{1}{2}\left\|\mathbf{H}_F\tilde{\mathbf{S}}\mathbf{H}_F - \tilde{\mathbf{S}}\right\|_F^2 \\
&= \frac{1}{2}\sum_{k=1}^{K^*}\Big\{\big(d_k - v^2 - (d_k-v^2-\tau)_+\big)^2 + 2\tau(d_k-v^2-\tau)_+\Big\} + \frac{1}{2}v^4(n-K^*) \\
&\quad + v^2\mathrm{trace}(\mathbf{D} - \tilde{\mathbf{S}}) + \frac{1}{2}\left\|\mathbf{H}_F\tilde{\mathbf{S}}\mathbf{H}_F - \tilde{\mathbf{S}}\right\|_F^2 \\
&= \frac{1}{2}\sum_{k=1}^{K^*}\big\{(d_k-v^2)^2 - (d_k-v^2-\tau)_+^2\big\} + \frac{1}{2}v^4(n-K^*) + v^2\mathrm{trace}(\mathbf{D} - \tilde{\mathbf{S}}) \\
&\quad + \frac{1}{2}\left\|\mathbf{H}_F\tilde{\mathbf{S}}\mathbf{H}_F - \tilde{\mathbf{S}}\right\|_F^2 \\
&= \frac{1}{2}\sum_{k=1}^{K^*}\big\{d_k^2 - (d_k-v^2-\tau)_+^2\big\} + \frac{1}{2}v^4 n - v^2\mathrm{trace}(\tilde{\mathbf{S}}) + \frac{1}{2}\left\|\mathbf{H}_F\tilde{\mathbf{S}}\mathbf{H}_F - \tilde{\mathbf{S}}\right\|_F^2,
\end{aligned}$$

which gives (6). □

Proof of Corollary 1

From the definition of $\mathbf{L}$, $\mathbf{L}\mathbf{L}' = \mathbf{H}_F$ and $\mathbf{L}'\mathbf{L} = \mathbf{I}_K$. Since from (5), $\big(\mathbf{L}'\mathbf{Q}_{K^*}\big)'\big(\mathbf{L}'\mathbf{Q}_{K^*}\big) = \mathbf{Q}_{K^*}'\mathbf{H}_F\mathbf{Q}_{K^*} = \mathbf{Q}_{K^*}'\mathbf{Q}_{K^*} = \mathbf{I}_{K^*}$, we let $\mathbf{P}$ be a $K \times K$ orthogonal matrix having $\mathbf{P}_{K^*} = \mathbf{L}'\mathbf{Q}_{K^*}$ as its first K^* columns. It follows that

$$\mathbf{P}\mathbf{D}_K\mathbf{P}' = \mathbf{L}'\mathbf{Q}_{K^*}\mathbf{D}_{K^*}\mathbf{Q}_{K^*}'\mathbf{L} = \mathbf{L}'\mathbf{H}_F(\mathbf{S} - \sigma^2\mathbf{I}_n)\mathbf{H}_F\mathbf{L} = \mathbf{L}'(\mathbf{S} - \sigma^2\mathbf{I}_n)\mathbf{L}.$$

where $\mathbf{D}_K = \mathrm{diag}(d_1, \ldots, d_K)$, with $d_{K^*+1} = \cdots = d_K = 0$. That is, $\mathbf{P}\mathbf{D}_K\mathbf{P}'$ is an eigen-decomposition of $\mathbf{L}'(\mathbf{S} - \sigma^2\mathbf{I}_n)\mathbf{L}$. Since $\mathbf{Q}_{K^*} = \mathbf{L}\mathbf{L}'\mathbf{Q}_{K^*} = \mathbf{L}\mathbf{P}_{K^*} = \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1/2}\mathbf{P}_{K^*}$ this, together with (7), gives (8) and completes the proof. □

References

Theobald, C. (1975). An inequality for the trace of the product of two symmetric matrices, *Mathematical Proceedings of the Cambridge Philosophical Society* **77**: 265–267.