LIKELIHOOD APPROXIMATIONS FOR BIG NONSTATIONARY SPATIAL TEMPORAL LATTICE DATA

Joseph Guinness and Montserrat Fuentes

NC State University, Department of Statistics

Supplementary Material

S1 Proof of Matrix Approximation

Theorem 1. *If $A(\boldsymbol{u},\boldsymbol{\omega})$ is $d+1$ times continuously differentiable in $\boldsymbol{\omega}$ for every $\boldsymbol{u}$, then*

$$\|\Delta_{\boldsymbol{n}}(A) - K_{\boldsymbol{n}}(A)\|_F^2 = O(n^{1-1/d}).$$

Proof. The covariance matrix $\Delta_{\boldsymbol{n}}(A)$ has entries

$$\Delta_{\boldsymbol{n}}(\boldsymbol{x},\boldsymbol{y}) = \frac{(2\pi)^d}{n}\sum_{\boldsymbol{j}\in\mathbb{Z}_{\boldsymbol{n}}} A(\boldsymbol{x}/\boldsymbol{n},\boldsymbol{\omega_j})A^*(\boldsymbol{y}/\boldsymbol{n},\boldsymbol{\omega_j})\exp(i\boldsymbol{\omega}_{\boldsymbol{j}}'(\boldsymbol{x}-\boldsymbol{y})).$$

Define $f_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{\omega}) = A(\boldsymbol{x}/\boldsymbol{n},\boldsymbol{\omega})A^*(\boldsymbol{y}/\boldsymbol{n},\boldsymbol{\omega})$ (the dependence of f on $\boldsymbol{n}$ is suppressed). Since $A(\boldsymbol{u},\boldsymbol{\omega})$ is $d+1$ times continuously differentiable in $\boldsymbol{\omega}$ for every $\boldsymbol{u}$, we can write

$$f_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{\omega}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{k})\exp(i\boldsymbol{\omega}'\boldsymbol{k})$$

with $c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{k}) \le T(\max_j |k_j|)^{-d-1}$ for $\boldsymbol{k}\neq\boldsymbol{0}$ for some $T<\infty$ (Körner, 1989). Then

$$K_{\boldsymbol{n}}(\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbb{T}^d}\sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{k})\exp(i\boldsymbol{\omega}'(\boldsymbol{k}+\boldsymbol{x}-\boldsymbol{y}))d\boldsymbol{\omega} = c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}).$$

Similarly,

$$\Delta_{\boldsymbol{n}}(\boldsymbol{x},\boldsymbol{y}) = \frac{(2\pi)^d}{n}\sum_{\boldsymbol{j}\in\mathbb{Z}_{\boldsymbol{n}}}\sum_{\boldsymbol{k}\in\mathbb{Z}^d} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{k})\exp(i\boldsymbol{\omega}_{\boldsymbol{j}}'(\boldsymbol{k}+\boldsymbol{x}-\boldsymbol{y})) = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{\ell}\circ\boldsymbol{n}),$$

where $\boldsymbol{\ell}\circ\boldsymbol{n} = (\ell_1 n_1,\ldots,\ell_d n_d)$. Therefore

$$\Delta_{\boldsymbol{n}}(\boldsymbol{x},\boldsymbol{y}) - K_{\boldsymbol{n}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d\setminus\boldsymbol{0}} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{\ell}\circ\boldsymbol{n}).$$

The difference in Frobenius norm is

$$\|\Delta_{\boldsymbol{n}} - K_{\boldsymbol{n}}\|_F^2 = \sum_{\boldsymbol{x},\boldsymbol{y}} \left(\Delta_{\boldsymbol{n}}(\boldsymbol{x},\boldsymbol{y}) - K_{\boldsymbol{n}}(\boldsymbol{x},\boldsymbol{y})\right)^2 = \sum_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d\backslash\boldsymbol{0}} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{\ell}\circ\boldsymbol{n}) \right)^2,$$

where $\sum_{\boldsymbol{x},\boldsymbol{y}}$ denotes $\sum_{\boldsymbol{x}\in\mathbb{Z}_{\boldsymbol{n}}}\sum_{\boldsymbol{y}\in\mathbb{Z}_{\boldsymbol{n}}}$. We partition the inner sum into two parts,

$$\begin{aligned}\mathbb{Z}^d\backslash\boldsymbol{0} &= \{\boldsymbol{\ell} : |\ell_j| \le 1\,\forall\, j\}\backslash\boldsymbol{0} \cup \{\boldsymbol{\ell} : |\ell_j| > 1 \text{ for some } j\} \\ &= S_1 \cup S_2\end{aligned}$$

Then we rewrite

$$\begin{aligned}\|\Delta_{\boldsymbol{n}} - K_{\boldsymbol{n}}\|_F^2 &= \sum_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{S_1} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{\ell}\circ\boldsymbol{n}) + \sum_{S_2} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{\ell}\circ\boldsymbol{n}) \right)^2 \\ &\le 2\sum_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{S_1} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{\ell}\circ\boldsymbol{n}) \right)^2 + 2\sum_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{S_2} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{\ell}\circ\boldsymbol{n}) \right)^2 \\ &= 2R_1 + 2R_2\end{aligned}$$

Treating R_1 and R_2 separately,

$$\begin{aligned}R_1 &= \sum_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{\boldsymbol{\ell}\in S_1} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{\ell}\circ\boldsymbol{n}) \right)^2 \\ &\le (3^d-1)\sum_{\boldsymbol{x},\boldsymbol{y}}\sum_{\boldsymbol{\ell}\in S_1} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{\ell}\circ\boldsymbol{n})^2,\end{aligned}$$

since S_1 contains 3^d-1 terms. Therefore,

$$R_1 \le (3^d-1)\sum_{\boldsymbol{\ell}\in S_1}\sum_{\boldsymbol{x},\boldsymbol{y}} \frac{T^2}{\max_j |y_j - x_j + \ell_j n_j|^{2d+2}}.$$

Recall that for every $\boldsymbol{\ell}\in S_1$, $|l_j| = 1$ for at least one j, call this $j_{\boldsymbol{\ell}}$.

$$\begin{aligned}R_1 &\le (3^d-1)\sum_{\boldsymbol{\ell}\in S_1}\sum_{\boldsymbol{x},\boldsymbol{y}} \frac{T^2}{|y_{j_{\boldsymbol{\ell}}} - x_{j_{\boldsymbol{\ell}}} + n_{j_{\boldsymbol{\ell}}}|^{2d+2}} \\ &= (3^d-1)\sum_{\boldsymbol{\ell}\in S_1}\frac{n}{n_{j_{\boldsymbol{\ell}}}}\sum_{y=1}^{n_{j_{\boldsymbol{\ell}}}}\sum_{x=1}^{n_{j_{\boldsymbol{\ell}}}} \frac{T^2}{|y - x + n_{j_{\boldsymbol{\ell}}}|^{2d+2}} \\ &= (3^d-1)\sum_{\boldsymbol{\ell}\in S_1}\frac{n}{n_{j_{\boldsymbol{\ell}}}}\sum_{h=-n_{j_{\boldsymbol{\ell}}}+1}^{n_{j_{\boldsymbol{\ell}}}-1} (n_{j_{\boldsymbol{\ell}}} - |h|)\frac{T^2}{|h + n_{j_{\boldsymbol{\ell}}}|^{2d+2}} \\ &\le (3^d-1)\sum_{\boldsymbol{\ell}\in S_1}\frac{n}{n_{j_{\boldsymbol{\ell}}}}\sum_{h=-n_{j_{\boldsymbol{\ell}}}+1}^{n_{j_{\boldsymbol{\ell}}}-1} \frac{T^2}{(n_{j_{\boldsymbol{\ell}}} - |h|)^{2d+1}},\end{aligned}$$

where the last inequality follows because $n_j - |h| \le |n_j + h|$.

$$R_1 \le (3^d - 1) \sum_{\boldsymbol{\ell} \in S_1} \frac{n}{n_{j_{\boldsymbol{\ell}}}} \left[\frac{T^2}{n_{j_{\boldsymbol{\ell}}}} + 2 \sum_{h=1}^{n_{j_{\boldsymbol{\ell}}}-1} \frac{T^2}{(n_{j_{\boldsymbol{\ell}}} - h)^{2d+1}} \right].$$

Reversing the order of the second sum gives

$$\begin{aligned} R_1 &\le (3^d - 1) \sum_{\boldsymbol{\ell} \in S_1} \frac{n}{n_{j_{\boldsymbol{\ell}}}} \left[\frac{T^2}{n_{j_{\boldsymbol{\ell}}}} + 2 \sum_{h=1}^{n_{j_{\boldsymbol{\ell}}}-1} \frac{T^2}{h^{2d+1}} \right] \\ &\le (3^d - 1) \sum_{\boldsymbol{\ell} \in S_1} \frac{n}{n_{j_{\boldsymbol{\ell}}}} \left[\frac{T^2}{n_{j_{\boldsymbol{\ell}}}} + 2 \sum_{h=1}^{\infty} \frac{T^2}{h^{2d+1}} \right] \\ &\le \frac{n}{n_1} (3^d - 1) \sum_{\boldsymbol{\ell} \in S_1} \left[\frac{T^2}{n_1} + 2 \sum_{h=1}^{\infty} \frac{T^2}{h^{2d+1}} \right]. \end{aligned}$$

The second sum in the last expression converges for every $\boldsymbol{\ell} \in S_1$, and S_1 has a finite number of elements, so $R_1 = O(n/n_1) = O(n^{1-1/d})$.

Proceeding with R_2,

$$\begin{aligned} R_2 &= \sum_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{\boldsymbol{\ell} \in S_2} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y} - \boldsymbol{x} + \boldsymbol{\ell} \circ \boldsymbol{n}) \right)^2 \\ &\le \sum_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{\boldsymbol{\ell} \in S_2} \frac{T}{\max_j |y_j - x_j + \ell_j n_j|^{d+1}} \right)^2. \end{aligned}$$

Express the set $S_2 = \cup_{k=2}^{\infty} B_k$, where $B_k = \{\boldsymbol{\ell} : \max_j |\ell_j| = k\}$. The set B_k contains

$$|B_k| = \sum_{j=1}^{d} \binom{d}{j} \left[(2k)^{d-j} - (-1)^j (2k)^{d-j} \right]$$

elements, which is a polynomial in k of degree $d-1$, which we call $P_{d-1}(k)$. Then

$$\begin{aligned} R_2 &\le \sum_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{k=2}^{\infty} \sum_{\boldsymbol{\ell} \in B_k} \frac{T}{\max_j |y_j - x_j + \ell_j n_j|^{d+1}} \right)^2 \\ &\le \sum_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{k=2}^{\infty} \sum_{\boldsymbol{\ell} \in B_k} \frac{T}{((k-1) n_1)^{d+1}} \right)^2, \end{aligned}$$

since $|y_j - x_j| < n_j$ and $n_1 \le n_j$ for every j. Finally,

$$R_2 \le n_1^{-2d-2} \sum_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{k=1}^{\infty} \frac{T P_{d-1}(k)}{k^{d+1}} \right)^2.$$

The sum over k converges, and the sum over $\boldsymbol{x}$ and $\boldsymbol{y}$ contains n^2 terms, so $R_2 = O(n^2 n_1^{-2d-2}) = O(n^{-2/d})$.

Therefore $\|\Delta_{\boldsymbol{n}} - K_{\boldsymbol{n}}\|_F^2 = O(n^{1-1/d})$, the size of R_1. □

S2 Nearly Exact Simulation

Due to the integral representation of the locally stationary processes we study, exact simulation is not possible, but we describe how to efficiently produce nearly exact simulations. Let J denote an integer and define $\omega_{1j} = 2\pi j/(Jn_1)$ and $\omega_{2k} = 2\pi k/(Jn_2)$. If $Z(\omega_{1j}, \omega_{2k})$ are uncorrelated complex normal random variables, and J is sufficiently large, then

$$Y_{\boldsymbol{n},J}(\boldsymbol{x}) = \sqrt{\frac{2\pi}{J^2 n_1 n_2}} \sum_{j=1}^{Jn_1} \sum_{k=1}^{Jn_2} A(\boldsymbol{x}/\boldsymbol{n}, \omega_{1j}, \omega_{2k}) \exp(i(\omega_{1j}, \omega_{2k})'\boldsymbol{x}) Z(\omega_{1j}, \omega_{2k})$$

is a good approximation to $Y_{\boldsymbol{n}}(\boldsymbol{x})$, and can be computed efficiently in $O(Jn\log(Jn))$ floating point operations with a two-dimensional fast Fourier transform. We have found $J = 8$ to be sufficiently large in all of our simulations.

S3 Histograms for Square Root Transformation

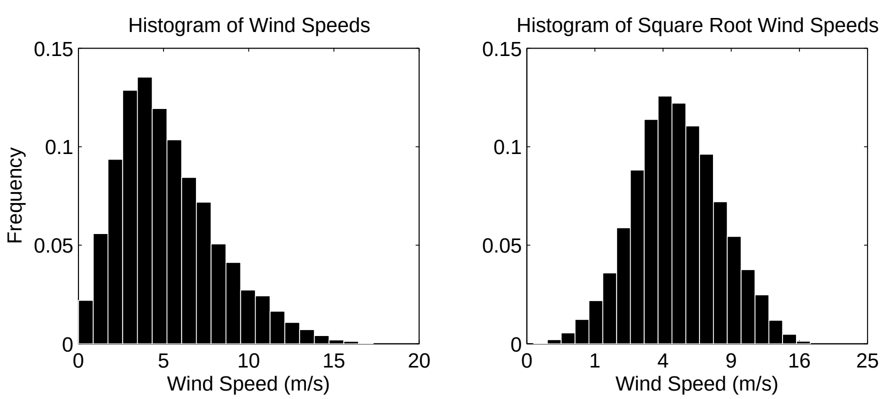

Figure 1: Histograms of wind speeds and square root wind speeds.

References

Körner, T. (1989). *Fourier Analysis.* Cambridge University Press.