Panel Data Partially Linear Varying-Coefficient Model with Errors Correlated in Space and Time

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Supplementary Material

As we mentioned in the mainbody of the paper, we will present the proofs of the theorems and some additional simulation results here due to the page number restriction.

S1 Details of proving theorems

To present the proofs of the main results we first introduce and prove two lemmas. The first lemma is due to Kapoor, Kelejian, and Prucha (2007).

Lemma 1 *(i) Let $\mathbf{R}_N$ be a (sequence of) $N \times N$ matrices whose row and column sums are bounded uniformly in absolute value, and let $\mathbf{S}$ be some $k \times k$ matrix (with $k \geq 1$ fixed). Then the row and column sums of $\mathbf{S} \otimes \mathbf{R}_N$ are bounded uniformly in absolute value.*

(ii) If $\mathbf{A}_N$ and $\mathbf{B}_N$ are (sequences of) $kN \times kN$ matrices (with $k \geq 1$ fixed), whose row and column sums are bounded uniformly in absolute value, then so are the row and column sums of $\mathbf{A}_N\mathbf{B}_N$ and $\mathbf{A}_N + \mathbf{B}_N$. If $\mathbf{Z}_N$ is a (sequence of) $kN \times p$ matrices whose elements are uniformly bounded in absolute value, then so are the elements of $\mathbf{A}_N\mathbf{Z}_N$ and $(kN)^{-1}\mathbf{Z}_N^\tau\mathbf{A}_N\mathbf{Z}_N$.

Denote

$$\tilde{\boldsymbol{\beta}}_N^w = (\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{X}_N)^{-1}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{Y}_N \text{ and } \tilde{\boldsymbol{\theta}}_N^w = (\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N^{-1}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N^{-1}(\mathbf{Y}_N - \mathbf{X}_N\tilde{\boldsymbol{\beta}}_N^w)$$

and $\tilde{\mathbf{m}}_N^w(u) = (\tilde{m}_{1,N}^w(u), \cdots, \tilde{m}_{q,N}^w(u))^\tau = \boldsymbol{\zeta}_N^*(u)\tilde{\boldsymbol{\theta}}_N^w$, where $\boldsymbol{\zeta}_N^*(u)$ is defined in Section 2.

For $\tilde{\boldsymbol{\beta}}_N^w$ and $(\tilde{m}_{1,N}^w(\cdot), \cdots, \tilde{m}_{j,N}^w(\cdot))^\tau$, we have the following asymptotic properties.

Lemma 2 *Under Assumptions A to F, the following statements (i) and (ii) hold.*

(i) $\sqrt{NT}(\tilde{\boldsymbol{\beta}}_N^w - \boldsymbol{\beta}) \to_D N(0, \boldsymbol{\Omega}^{w-1})$ *as* $N \to \infty$*, where* $\boldsymbol{\Omega}^w$ *is defined in Assumption E.*

(ii) $\max_{1\le j\le q} \|\tilde{m}_{j,N}^w - m_j\|_{L_2}^2 = O_p\left(\kappa_N N^{-1} + \varphi_N^2\right) = O_p(\max_{1\le j\le q} \kappa_N N^{-1} + \kappa_N^{-4})$.

Proof. According to the model (1), we have

$$\mathbf{Y}_N = \mathbf{X}_N\boldsymbol{\beta} + \mathbf{Z}_N \odot \mathbf{M}_N(\mathbf{U}_N) + \boldsymbol{\varepsilon}_N = \mathbf{X}_N\boldsymbol{\beta} + \mathbf{Z}_N^*\boldsymbol{\theta}_N + \{\mathbf{Z}_N \odot \mathbf{M}_N(\mathbf{U}_N) - \mathbf{Z}_N^*\boldsymbol{\theta}_N\} + \boldsymbol{\varepsilon}_N,$$

where $\mathbf{M}_N(\mathbf{U}_N) = (M_N^\tau(\mathbf{U}_N(1)), \cdots, M_N^\tau(\mathbf{U}_N(T)))^\tau$. Therefore,

$$\begin{aligned}\sqrt{NT}(\tilde{\boldsymbol{\beta}}_N^w - \boldsymbol{\beta}) &= \sqrt{NT}\left\{(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{X}_N)^{-1}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{Y}_N - \boldsymbol{\beta}\right\} \\ =&\sqrt{NT}\mathbf{W}_N^{\Sigma_N}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \boldsymbol{\varepsilon}_N + \sqrt{NT}\mathbf{W}_N^{\Sigma_N}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} (\mathbf{Z}_N \odot \mathbf{M}_N(\mathbf{U}_N) - \mathbf{Z}_N^*\boldsymbol{\theta}_N),\end{aligned}$$

where $\mathbf{W}_N^{\Sigma_N} = \left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{X}_N\right)^{-1}$. By Assumption A, the item $\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{X}_N$ can be decomposed as

$$\begin{aligned}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{X}_N =& \mathbf{A}(\mathbf{U}_N)^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{A}(\mathbf{U}_N) + \mathbf{A}(\mathbf{U}_N)^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} + \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{A}(\mathbf{U}_N) \\ &+ \boldsymbol{\Pi}_N^\tau \boldsymbol{\Sigma}_N^{-1}\boldsymbol{\Pi}_N - \boldsymbol{\Pi}_N^\tau \boldsymbol{\Sigma}_N^{-1}\mathbf{Z}_N^*(\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N^{-1}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\Pi}_N \\ =& J_{1,N} + \cdots + J_{5,N},\end{aligned}$$

where $\mathbf{A}(\mathbf{U}_N) = ((\mathbf{Z}_N \odot \boldsymbol{\varphi}_{1N}(\mathbf{U}_N), \cdots, \mathbf{Z}_N \odot \boldsymbol{\varphi}_{pN}(\mathbf{U}_N)) - \mathbf{Z}_N^*\boldsymbol{\gamma}_N)$ and $\boldsymbol{\varphi}_{jN}(\mathbf{U}_N) = (\boldsymbol{\varphi}_j(U_{11}), \cdots, \boldsymbol{\varphi}_j(U_{N1}), \cdots, \boldsymbol{\varphi}_j(U_{NT}))^\tau$ with $\boldsymbol{\varphi}_j(\cdot) = (\varphi_{j1}(\cdot), \cdots, \varphi_{jq}(\cdot))^\tau$. Applying the polynomial spline properties it is easy to see that

$$\frac{1}{NT}J_{1,N} \le \frac{1}{NT}A(\mathbf{U}_N)^\tau A(\mathbf{U}_N) = O(\kappa_N^4) = o(N^{-\frac{1}{2}}).$$

Applying Assumption F yields $(NT)^{-1}J_{2,N} = O\left(\kappa_N^{-2}N^{-1/2}\right) = o(N^{-1/2})$ and $(NT)^{-1}J_{3,N} = O\left(\kappa_N^{-2}N^{-1/2}\right) = o(N^{-1/2})$. In addition,

$$\frac{1}{NT}J_{5,N} \le O(1)\cdot\frac{1}{NT}\boldsymbol{\Pi}_N^\tau \boldsymbol{\Sigma}_N^{-1}\mathbf{Z}_N^*(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\Pi}_N = O\left(\kappa_N^2 N^{-1}\right) = o(N^{-\frac{1}{2}}).$$

Therefore,

$$\frac{1}{NT}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{X}_N = \frac{1}{NT}\boldsymbol{\Pi}_N^\tau \boldsymbol{\Sigma}_N^{-1}\boldsymbol{\Pi}_N + o(1) = \boldsymbol{\Omega} + o(1).$$

By the same arguments, we obtain

$$\frac{1}{NT}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{A}(\mathbf{U}_N) = O(\kappa_N^4) + O\left(\kappa_N^2 N^{-1}\right) = o(N^{-\frac{1}{2}}).$$

It follows that

$$\sqrt{NT}(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{X}_N)^{-1}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\boldsymbol{\Sigma}_N^{-1}} \mathbf{A}(\mathbf{U}_N) = o(1).$$

For $\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}} \boldsymbol{\varepsilon}_N$, it can be decomposed into

$$\frac{1}{\sqrt{NT}} \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}} \boldsymbol{\varepsilon}_N = J_{6,N} - J_{7,N} + \frac{1}{\sqrt{NT}} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \boldsymbol{\varepsilon}_N,$$

where $J_{6,N} = (NT)^{-1/2} \mathbf{A}(\mathbf{U}_N)^\tau \left\{ \mathbf{\Sigma}_N^{-1} - \mathbf{\Sigma}_N^{-1} \mathbf{Z}_N^* (\mathbf{Z}_N^{*\tau} \mathbf{\Sigma}_N^{-1} \mathbf{Z}_N^*)^{-1} \mathbf{Z}_N^{*\tau} \mathbf{\Sigma}_N^{-1} \right\} \boldsymbol{\varepsilon}_N$ and

$$J_{7,N} = \frac{1}{\sqrt{NT}} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \mathbf{Z}_N^* (\mathbf{Z}_N^{*\tau} \mathbf{\Sigma}_N^{-1} \mathbf{Z}_N^*)^{-1} \mathbf{Z}_N^{*\tau} \mathbf{\Sigma}_N^{-1} \boldsymbol{\varepsilon}_N + \frac{1}{\sqrt{NT}} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \boldsymbol{\varepsilon}_N.$$

For $J_{6,N}$, we have

$$E(J_{6,N} J_{6,N}^\tau) = O(1) \frac{1}{NT} \mathbf{A}(\mathbf{U}_N)^\tau \mathbf{A}(\mathbf{U}_N) = O(\kappa_N^{-4}) = o(1),$$

implying that $J_{6,N} = o_p(1)$. For $J_{7,N}$, we have

$$E(J_{7,N} J_{7,N}^\tau) = \frac{1}{NT} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \mathbf{Z}_N^* (\mathbf{Z}_N^{*\tau} \mathbf{\Sigma}_N^{-1} \mathbf{Z}_N^*)^{-1} \mathbf{Z}_N^{*\tau} \mathbf{\Sigma}_N^{-1} \mathbf{\Pi}_N = O\left(\kappa_N^2 N^{-1}\right),$$

implying that $J_{7,N} = o_p(1)$. As a result,

$$\begin{aligned} \sqrt{NT}(\tilde{\boldsymbol{\beta}}_N^w - \boldsymbol{\beta}) =& \sqrt{NT} \mathbf{W}_N^{\Sigma_N} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \boldsymbol{\varepsilon}_N + o_p(1) \\ =& \sqrt{NT} \mathbf{W}_N^{\Sigma_N} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \left\{ \mathbf{I}_T \otimes (\mathbf{I}_N - \lambda \mathbf{W}_N)^{-1} \right\} (\mathbf{I}_N \otimes \mathbf{1}_N) \boldsymbol{\mu}_N \\ &+ \sqrt{NT} \mathbf{W}_N^{\Sigma_N} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \left\{ \mathbf{I}_T \otimes (\mathbf{I}_N - \lambda \mathbf{W}_N)^{-1} \right\} \boldsymbol{\nu}_N + o_p(1). \end{aligned}$$

It follows from Assumption A, C and Lemma 1 that the elements of $\mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \left\{ \mathbf{I}_T \otimes (\mathbf{I}_N - \lambda \mathbf{W}_N)^{-1} \right\}$ $(\mathbf{1}_N \otimes \mathbf{I}_T)$ and $\mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \left\{ \mathbf{I}_T \otimes (\mathbf{I}_N - \lambda \mathbf{W}_N)^{-1} \right\}$ are uniformly bounded in absolute value, combining Theorem 30 in Pötscher and Prucha (2001), we obtain

$$\frac{1}{\sqrt{NT}} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \left\{ \mathbf{I}_T \otimes (\mathbf{I}_N - \lambda \mathbf{W}_N)^{-1} \right\} (\mathbf{1}_T \otimes \mathbf{I}_N) \boldsymbol{\mu}_N \to_D N(0, \mathbf{\Omega}_1^w)$$

and

$$\frac{1}{\sqrt{NT}} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \left\{ \mathbf{I}_T \otimes (\mathbf{I}_N - \lambda \mathbf{W}_N)^{-1} \right\} \boldsymbol{\nu}_N \to_D N(0, \mathbf{\Omega}_2^w) \quad \text{as } N \to \infty,$$

where

$$\mathbf{\Omega}_1^w = \lim_{N \to \infty} \frac{1}{NT} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \left\{ \sigma_\mu^2 \mathbf{1}_N \mathbf{1}_N^\tau \otimes (\mathbf{I}_N - \lambda \mathbf{W}_N)^{-1} (\mathbf{I}_N - \lambda \mathbf{W}_N^\tau)^{-1} \right\} \mathbf{\Sigma}_N^{-1} \mathbf{\Pi}_N$$

and

$$\mathbf{\Omega}_2^w = \lim_{N \to \infty} \frac{1}{NT} \mathbf{\Pi}_N^\tau \mathbf{\Sigma}_N^{-1} \left\{ \mathbf{\Gamma}_N \otimes (\mathbf{I}_N - \lambda \mathbf{W}_N)^{-1} (\mathbf{I}_N - \lambda \mathbf{W}_N^\tau)^{-1} \right\} \mathbf{\Sigma}_N^{-1} \mathbf{\Pi}_N.$$

Independence between two processes $\{\mu_i\}_{i=1}^N$ and $\{\nu_{it}\}_{i=1}^N$ and the equality $\mathbf{\Omega}_1^w + \mathbf{\Omega}_2^w = \mathbf{\Omega}^w$ lead to complete the proof of the results (i).

We now prove (ii). Define $\boldsymbol{\iota}_j = \mathrm{diag}(\underbrace{0,\cdots,0}_{j-1},1,0\cdots,0)\otimes \mathbf{I}_{\kappa_N}$. According to the definition of $\widehat{m}_{j,N}(\cdot)$, it can be expressed as

$$\begin{aligned}\widehat{m}_{j,N}(\cdot) =&\boldsymbol{\zeta}_N^*(\cdot)\boldsymbol{\iota}_j(\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N{}^{-1}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\boldsymbol{\varepsilon}_N + \boldsymbol{\zeta}_N^*(\cdot)\boldsymbol{\iota}_j(\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N{}^{-1}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\mathbf{X}_N(\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}_N^w)\\ &+ \boldsymbol{\zeta}_N^*(\cdot)\boldsymbol{\iota}_j(\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N{}^{-1}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}(\mathbf{Z}_N\odot\mathbf{M}_N(\mathbf{U}_N)-\mathbf{Z}_N^*\boldsymbol{\theta}_N) + \boldsymbol{\zeta}_N^*(\cdot)\boldsymbol{\iota}_j\boldsymbol{\theta}_N.\end{aligned}$$

Then

$$\begin{aligned}&\|\tilde{m}_{j,N}^w - m_j\|_{L_2}^2 = \int_{\mathcal{U}}(\tilde{m}_{j,N}^w(u)-m_j(u))^2du\\ \le& 2\boldsymbol{\varepsilon}_N^\tau\mathbf{Z}_N^*(\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N{}^{-1}\mathbf{Z}_N^*)^{-1}\boldsymbol{\iota}_j^\tau\left\{\int_{\mathcal{U}}\boldsymbol{\zeta}_N^{*\tau}(u)\boldsymbol{\zeta}_N^*(u)du\right\}\boldsymbol{\iota}_j(\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N{}^{-1}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\boldsymbol{\varepsilon}_N\\ &+ 2(\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}_N^w)^\tau\mathbf{X}_N^\tau\mathbf{Z}_N^*(\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N{}^{-1}\mathbf{Z}_N^*)^{-1}\boldsymbol{\iota}_j^\tau\left\{\int_{\mathcal{U}}\boldsymbol{\zeta}_N^{*\tau}(u)\boldsymbol{\zeta}_N^*(u)du\right\}\boldsymbol{\iota}_j(\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N^{-1}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\mathbf{X}_N(\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}_N^w)\\ &+ 2\mathbf{B}(\mathbf{U}_N)^\tau\mathbf{Z}_N^*(\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N^{-1}\mathbf{Z}_N^*)^{-1}\boldsymbol{\iota}_j^\tau\left\{\int_{\mathcal{U}}\boldsymbol{\zeta}_N^{*\tau}(u)\boldsymbol{\zeta}_N^*(u)du\right\}\boldsymbol{\iota}_j(\mathbf{Z}_N^{*\tau}\boldsymbol{\Sigma}_N^{-1}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\mathbf{B}(\mathbf{U}_N)\\ &+ 2\int_{\mathcal{U}}(\boldsymbol{\zeta}_N^*(u)\boldsymbol{\iota}_j\boldsymbol{\theta}_N - m_j(u))^2du = J_{8,N}+\cdots+J_{11,N},\end{aligned}$$

where $\mathbf{B}(\mathbf{U}_N) = \mathbf{Z}_N\odot\mathbf{M}_N(\mathbf{U}_N)-\mathbf{Z}_N^*\boldsymbol{\theta}_N$. Applying Assumption A and the polynomial spline properties leads to $J_{8,N} = O(N^{-2})\mathrm{E}(\boldsymbol{\varepsilon}_N^\tau\mathbf{Z}_N^*\mathbf{Z}_N^{*\tau}\boldsymbol{\varepsilon}_N) = O(\kappa_N N^{-1})$ and $J_{10,N} = O(\kappa_N^{-4})$. Applying Assumption A, Lemma 1 and the polynomial spline properties leads to $J_{9,N} = O_p(N^{-1}) = o(\kappa_N N^{-1})$. Applying the polynomial spline properties causes $J_{11,N} = O(\kappa_N^{-4})$. As a consequence, we have proved the result (ii).

Proof of Theorem 1. Theorem 1 can be proved in the same way as we prove Lemma 2. We here omit its details.

Proof of Theorem 2. The proof of the consistency of $\widehat{\lambda}_N$ follows the same argument as that in Kelejian and Prucha (2010). Applying Theorem 1, one can readily check the conditions of Lemma 3.1 in Pötscher and Prucha (1997) for our problem. We now establish the root-N consistency of $\widehat{\lambda}_N$. According to the definition of $\hbar(\lambda)$, $\hbar(\lambda)$ can be written as

$$\hbar(\lambda) = \widehat{\boldsymbol{\psi}}_N - \widehat{\boldsymbol{\Psi}}_N\begin{pmatrix}\lambda\\ \lambda^2\end{pmatrix} = \begin{pmatrix}N^{-1}\widehat{\widehat{\boldsymbol{\eta}}}_N^\tau\mathbf{C}_{1,N}(\lambda)\widehat{\widehat{\boldsymbol{\eta}}}_N\\ N^{-1}\widehat{\widehat{\boldsymbol{\eta}}}_N^\tau\mathbf{C}_{2,N}(\lambda)\widehat{\widehat{\boldsymbol{\eta}}}_N\end{pmatrix},$$

where $\mathbf{C}_{j,N}(\lambda) = (\mathbf{I}_N-\lambda\mathbf{W}_N^\tau)\mathbf{A}_{j,N}(\mathbf{I}_N-\lambda\mathbf{W}_N), j = 1,2$. By Assumption C and Lemma 1, the rows and column sums of $\mathbf{C}_{j,N}(\lambda)$ are uniformly bounded in absolute value. Minimizing $Q_N = (\hbar(\lambda))^\tau\boldsymbol{\Upsilon}_N\hbar(\lambda)$ with resect to λ yields the first order condition $\left(\partial\hbar(\widehat{\lambda}_N)/\partial\lambda\right)^\tau\boldsymbol{\Upsilon}_N\hbar(\widehat{\lambda}_N) = 0$. Expanding only $\hbar(\widehat{\lambda}_N)$ about λ in the first order condition and reorganizing terms yield

$$\left(\frac{\partial\hbar(\widehat{\lambda}_N)}{\partial\lambda}\right)^\tau\boldsymbol{\Upsilon}_N\frac{\partial\hbar(\bar{\lambda}_N)}{\partial\lambda}\sqrt{N}(\widehat{\lambda}_n-\lambda) = -\left(\frac{\partial\hbar(\widehat{\lambda}_N)}{\partial\lambda}\right)^\tau\boldsymbol{\Upsilon}_N\hbar(\lambda),$$

where $\bar{\lambda}_N$ lies between $\widehat{\lambda}_N$ and λ and $\bar{\lambda}_N - \lambda = o_p(1)$ by the consistency of $\widehat{\lambda}_N$. Due to the fact that $\partial\hbar(\widehat{\lambda})/\partial\lambda = -\widehat{\mathbf{\Psi}}_N \begin{pmatrix}1 & 2\lambda\end{pmatrix}^\tau$, we have

$$\left(\frac{\partial\hbar(\widehat{\lambda}_N)}{\partial\lambda}\right)^\tau \mathbf{\Upsilon}_N \frac{\partial\hbar(\bar{\lambda}_N)}{\partial\lambda} = \begin{pmatrix}1\\ 2\widehat{\lambda}_N\end{pmatrix}^\tau \widehat{\mathbf{\Psi}}_N^\tau \mathbf{\Upsilon}_N \widehat{\mathbf{\Psi}}_N \begin{pmatrix}1\\ 2\widehat{\lambda}_N\end{pmatrix} \equiv \widehat{\mathbf{\Xi}}_N.$$

Let $\mathbf{\Xi}_N = \begin{pmatrix}1 & 2\lambda\end{pmatrix} \mathbf{\Psi}_N^\tau \mathbf{\Upsilon}_N \mathbf{\Psi}_N \begin{pmatrix}1 & 2\lambda\end{pmatrix}^\tau$. Then we will prove $\widehat{\mathbf{\Psi}}_N = \mathbf{\Psi}_N + o_p(1), \mathbf{\Psi}_N = O(1)$, $\widehat{\mathbf{\Xi}}_N^{-1} - \mathbf{\Xi}_N^{-1} = o_p(1)$ and $\sqrt{N}\hbar(\lambda) = \mathbf{\Phi}_N^{\frac{1}{2}}\xi_N + o_p(1)$, where $\mathbf{\Phi}_N = \left(2N^{-1}\mathrm{tr}(\mathbf{A}_{j_1,N}\mathbf{A}_{j_2,N})\right)_{2\times 2}$, $\xi_N = \mathbf{\Phi}_N^{1/2}\mathbf{v}_N$ with $\mathbf{v}_N = \begin{pmatrix}N^{-1/2}\bar{\boldsymbol{\eta}}_N^\tau \mathbf{A}_{1,N}\bar{\boldsymbol{\eta}}_N & N^{-1/2}\bar{\boldsymbol{\eta}}_N^\tau \mathbf{A}_{2,N}\bar{\boldsymbol{\eta}}_N\end{pmatrix}^\tau$ and $\xi_N \to_D N(0, \mathbf{I}_2)$. With

$$\widehat{\varepsilon}_{it,N} = \varepsilon_{it} + \mathbf{X}_{it}^\tau(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_N) + \sum_{j=1}^q Z_{itj}(m_j(U_{it}) - \widehat{m}_{j,N}(U_{it})), \quad i = 1, \cdots, N,\ t = 1, \cdots, T,$$

and

$$\begin{aligned}\widehat{\bar{\varepsilon}}_{it,N} =& \frac{1}{T}\sum_{t=1}^T \varepsilon_{it} + \frac{1}{T}\sum_{t=1}^T \mathbf{X}_{it}^\tau(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_N) + \frac{1}{T}\sum_{t=1}^T\sum_{j=1}^q Z_{itj}(m_j(U_{it}) - \widehat{m}_{j,N}(U_{it}))\\ =& \frac{1}{T}\sum_{t=1}^T \varepsilon_{it} + \Lambda_{i,N}^{(1)} + \Lambda_{i,N}^{(2)}, \quad i = 1, \cdots, N,\end{aligned}$$

according to the definition of $\widehat{\psi}_{11,N}$, we decomposed it into

$$\begin{aligned}\widehat{\psi}_{11,N} =& \frac{2}{N}\Bigg(\bar{\boldsymbol{\varepsilon}}_N^\tau \mathbf{W}_N^\tau \mathbf{A}_{1,N}\bar{\boldsymbol{\varepsilon}}_N + 2\sum_{i=1}^N\sum_{i_1=1}^N \Lambda_{i_1,N}^{(1)} W_{ii_1}\sum_{i_2=1}^N A_{1ii_2}\Lambda_{i_2,N}^{(1)}\\ &+ 2\sum_{i=1}^N\sum_{i_1=1}^N \Lambda_{i_1,N}^{(2)} W_{ii_1}\sum_{i_2=1}^N A_{1ii_2}\Lambda_{i_2,N}^{(2)} + 2\sum_{i=1}^N\sum_{i_1=1}^N \Lambda_{i_1,N}^{(1)} W_{ii_1}\sum_{i_2=1}^N A_{1ii_2}\Lambda_{i_2,N}^{(2)}\\ &+ 2\sum_{i=1}^N\sum_{i_1=1}^N \Lambda_{i_1,N}^{(2)} W_{ii_1}\sum_{i_2=1}^N A_{1ii_2}\Lambda_{i_2,N}^{(1)} + 2\sum_{i=1}^N\sum_{i_1=1}^N \frac{1}{T}\sum_{t=1}^T \varepsilon_{i_1t} W_{ii_1}\sum_{i_2=1}^N A_{1ii_2}\Lambda_{i_2,N}^{(1)}\\ &+ 2\sum_{i=1}^N\sum_{i_1=1}^N \Lambda_{i_1,N}^{(1)} W_{ii_1}\sum_{i_2=1}^N A_{1ii_2}\frac{1}{T}\sum_{t=1}^T \varepsilon_{i_2t} + 2\sum_{i=1}^N\sum_{i_1=1}^N \frac{1}{T}\sum_{t=1}^T \varepsilon_{i_1t} W_{ii_1}\sum_{i_2=1}^N A_{1ii_2}\Lambda_{i_2,N}^{(2)}\\ &+ 2\sum_{i=1}^N\sum_{i_1=1}^N \Lambda_{i_1,N}^{(2)} W_{ii_1}\sum_{i_2=1}^N A_{1ii_2}\frac{1}{T}\sum_{t=1}^T \varepsilon_{i_2t}\Bigg)\\ =& J_{1,N} + \cdots + J_{9,N}.\end{aligned}$$

It follows from According to Assumption C , Lemma 1 and Lemma C.1 (a) of Su (2012) that

$$J_{1,N} = 2N^{-1}\mathrm{E}\bar{\boldsymbol{\varepsilon}}_N^\tau \mathbf{W}_N^\tau \mathbf{A}_{1,N}\bar{\boldsymbol{\varepsilon}}_N + O_p(N^{-\frac{1}{2}}).$$

Combining Assumption A1, C, Lemma 1 and Theorem 1 (i) causes

$$J_{2,N} = \frac{2}{N}\sum_{i=1}^{N}\sum_{i_1=1}^{N}\left\{\frac{1}{T}\sum_{t=1}^{T}\mathbf{X}_{i_1t}^{\tau}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_N)\right\}W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\left\{\frac{1}{T}\sum_{t=1}^{T}\mathbf{X}_{i_2t}^{\tau}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_N)\right\} = O_p(N^{-\frac{1}{2}}).$$

Similarly, by Assumption A1, C, Lemma 1 and Theorem 1 (ii), we have

$$J_{3,N} = O_p\left\{\left(\max_{1\le j\le q}\kappa_N N^{-1}+\varphi_N^2\right)^2\right\}.$$

Using Cauchy-Schwarz inequality yields $J_{4,N} = O_p\left\{\left(\max_{1\le j\le q}\kappa_N N^{-1}+\varphi_N^2\right)N^{-1/2}\right\}$ and $J_{5,N} = O_p\left\{\left(\max_{1\le j\le q}\kappa_N N^{-1}+\varphi_N^2\right)N^{-1/2}\right\}$. For $J_{6,N}$, we have

$$J_{6,N} = 2N^{-1}\sum_{j=1}^{p}\left\{\sum_{i=1}^{N}\sum_{i_1=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{i_1t}\right)W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\left(\frac{1}{T}\sum_{t=1}^{T}X_{i_2tj}\right)\right\}(\beta_j-\widehat{\beta}_{j,N}).$$

It is easy to see that

$$\mathrm{E}\left\{\sum_{i=1}^{N}\sum_{i_1=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{i_1t}\right)W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\left(\frac{1}{T}\sum_{t=1}^{T}X_{i_2tj}\right)\right\} = 0.$$

According to Assumption C and Lemma 1, for all $j=1,\cdots,p$, there exists a constant c such that

$$\left|\sum_{i=1}^{N}W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\left(\frac{1}{T}\sum_{t=1}^{T}X_{i_2tj}\right)\right| \le c.$$

Therefore, with Lemma 2, it holds that

$$\mathrm{Var}\left\{\sum_{i=1}^{N}\sum_{i_1=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{i_1t}\right)W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\left(\frac{1}{T}\sum_{t=1}^{T}X_{i_2tj}\right)\right\} = O(N).$$

This implies that

$$\sum_{i=1}^{N}\sum_{i_1=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{i_1t}\right)W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\left(\frac{1}{T}\sum_{t=1}^{T}X_{i_2tj}\right) = O_p(N^{\frac{1}{2}}).$$

So $J_{6,N} = O_p(N^{-1})$. Following the same argument, we can show that $J_{7,N} = O_p(N^{-1})$ as well. In addition, according to the definition of $\Lambda_{i,N}^{(2)}$, we have

$$\begin{aligned}\Lambda_{i,N}^{(2)} =& \frac{1}{T}\sum_{t=1}^{T}\sum_{j=1}^{q} Z_{itj}\left\{m_j(U_{it}) - (\boldsymbol{\zeta}(U_{it}))^{\tau}(\mathbf{0}_{\kappa_N\times(j-1)\kappa_N}, \mathbf{I}_{\kappa_N}, \mathbf{0}_{\kappa_N\times(q-j-1)\kappa_N})\boldsymbol{\theta}_N\right\}\\ &- \frac{1}{T}\sum_{t=1}^{T}\sum_{j=1}^{q} Z_{itj}(\boldsymbol{\zeta}(U_{it}))^{\tau}(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^{*})^{-1}\mathbf{Z}_N^{*\tau}\left\{\mathbf{Z}_N \odot (\mathbf{M}_N(\mathbf{U}_N) - \boldsymbol{\zeta}_N\boldsymbol{\theta}_N)\right\}\\ &- \frac{1}{T}\sum_{t=1}^{T}\sum_{j=1}^{q} Z_{itj}(\boldsymbol{\zeta}(U_{it}))^{\tau}(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^{*})^{-1}\mathbf{Z}_N^{*\tau}\mathbf{X}_N(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_N)\\ &- \frac{1}{T}\sum_{t=1}^{T}\sum_{j=1}^{q} Z_{itj}(\boldsymbol{\zeta}(U_{it}))^{\tau}(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^{*})^{-1}\mathbf{Z}_N^{*\tau}\boldsymbol{\varepsilon}_N = \Lambda_{i,N}^{(2)1} + \cdots + \Lambda_{i,N}^{(2)4}.\end{aligned}$$

According to the properties of polynomial spline, it is easy to see that

$$2N^{-1}\sum_{i=1}^{N}\sum_{i_1=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{i_1t}W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}(\Lambda_{i_2,N}^{(2)1} + \Lambda_{i_2,N}^{(2)2}) = O_p(N^{-\frac{1}{2}})\cdot O(\kappa_N^{-2}).$$

Same as for $J_{6,N}$, we can show that

$$2N^{-1}\sum_{i=1}^{N}\sum_{i_1=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{i_1t}W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\Lambda_{i_2,N}^{(2)3} = O_p(N^{-1}).$$

In a matrix form, we have

$$\begin{aligned}&\frac{2}{N}\sum_{i=1}^{N}\sum_{i_1=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{i_1t}W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\Lambda_{i_2,N}^{(2)4}\\ =&\frac{2}{N}\boldsymbol{\varepsilon}_N^{\tau}(T^{-1}\mathbf{1}_T\otimes\mathbf{I}_N)\mathbf{W}_N^{\tau}\mathbf{A}_{1,N}\bar{\mathbf{Z}}_N^{*}(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^{*})^{-1}\mathbf{Z}_N^{*\tau}\boldsymbol{\varepsilon}_N\\ =&\frac{2}{N}\boldsymbol{\eta}_N^{\tau}(\mathbf{I}_T\otimes\mathbf{W}_N^{\tau})(T^{-1}\mathbf{1}_T\otimes\mathbf{I}_N)\mathbf{W}_N^{\tau}\mathbf{A}_{1,N}\bar{\mathbf{Z}}_N^{*}(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^{*})^{-1}\mathbf{Z}_N^{*\tau}(\mathbf{I}_T\otimes\mathbf{W}_N)\boldsymbol{\varepsilon}_N\end{aligned}$$

with $\bar{\mathbf{Z}}_N^{*} = \mathbf{Z}_N^{*}(T^{-1}\mathbf{1}_T\otimes\mathbf{I}_N)$. Denote

$(\mathbf{I}_T\otimes\mathbf{W}_N^{\tau})(T^{-1}\mathbf{1}_T\otimes\mathbf{I}_N)\mathbf{W}_N^{\tau}\mathbf{A}_{1,N}\bar{\mathbf{Z}}_N^{*}(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^{*})^{-1}\mathbf{Z}_N^{*\tau}(\mathbf{I}_T\otimes\mathbf{W}_N) = (\varpi_{i_1+(t_1-1)N,i_2+(t_2-1)N})$

with $t_1, t_2 = 1, \ldots, T$ and $i_1, i_2 = 1, \cdots, N$. According to Assumption C, Lemma 1 and the properties of polynomial spline, we have $\max_{1\le i_1,i_2\le N,1\le t_1,t_2\le T}\varpi_{i_1+(t_1-1)N,i_2+(t_2-1)N} = O(\kappa_N N^{-1})$. Obviously,

$$\begin{aligned}&\mathrm{E}\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{i_1=1}^{N}\sum_{t=1}^{T}\varepsilon_{i_1t}W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\Lambda_{i_2,N}^{(2)4}\right)\\ &= \frac{1}{N}\sum_{i=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2\neq t_1}^{T}\varpi_{i+(t_1-1)N,i+(t_2-1)N}\sigma_{\nu}^2 + \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\varpi_{i+(t-1)N,i+(t-1)N}(\sigma_{\mu}^2+\sigma_{\nu}^2) = O\left(\kappa_N N^{-1}\right)\end{aligned}$$

and

$$
\begin{aligned}
&\mathrm{E}\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{i_1=1}^{N}\sum_{t=1}^{T}\varepsilon_{i_1t}W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\Lambda_{i_2,N}^{(2)4}\right)^2\\
=&\frac{4}{N^2}\sum_{i=1}^{N}\mathrm{E}\left\{\left(\sum_{t_1=1}^{T}\eta_{it_1}\sum_{t_2=1}^{T}\eta_{it_2}\right)\varpi_{i+(t_1-1)N,i+(t_2-1)N}\left(\sum_{t_3=1}^{T}\eta_{it_3}\sum_{t_4=1}^{T}\eta_{it_4}\right)\varpi_{i+(t_4-1)N,i+(t_4-1)N}\right\}\\
&+\frac{1}{N}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\left(\sum_{t_1=1}^{T}\eta_{i_1t_1}\right)\left(\sum_{t_2=1}^{T}\eta_{i_1t_2}\right)\left(\sum_{t_3=1}^{T}\eta_{i_2t_3}\right)\left(\sum_{t_4=1}^{T}\eta_{i_2t_4}\right)(\omega_{i_1i_2,N}^1+\omega_{i_1i_2,N}^2+\omega_{i_1i_2,N}^3)\\
=&O(N^{-3}\kappa_N^2)+O(N^{-2}\kappa_N^2)=o(N^{-1}),
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_{i_1i_2,N}^1 =&(\varpi_{i_1+(t_1-1)N,i_2+(t_2-1)N}\varpi_{i_1+(t_3-1)N,i_2+(t_4-1)N}\\
\omega_{i_1i_2,N}^2 =&\varpi_{i_1+(t_1-1)N,i_1+(t_2-1)N}\varpi_{i_2+(t_3-1)N,i_2+(t_4-1)N}\\
\omega_{i_1i_2,N}^3 =&\varpi_{i_2+(t_1-1)N,i_1+(t_2-1)N}\varpi_{i_2+(t_3-1)N,i_1+(t_4-1)N}).
\end{aligned}
$$

This implies $2N^{-1}\sum_{i=1}^{N}\sum_{i_1=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{i_1t}W_{ii_1}\sum_{i_2=1}^{N}A_{1ii_2}\Lambda_{i_2,N}^{(2)4}=o_p(N^{-1/2})$. So $J_{8,N}=o_p(N^{-1/2})$. Similarly, it hods that $J_{9,N}=o_p(N^{-1/2})$. As a result, we see that $\widehat{\psi}_{11,N}=\psi_{11,N}+o_p(N^{-1/2})$. For $\widehat{\psi}_{12,N}$, $\widehat{\psi}_{21,N}$ and $\widehat{\psi}_{22,N}$ we can also prove that $\widehat{\psi}_{12,N}=\psi_{12,N}+o_p(N^{-1/2})$, $\widehat{\psi}_{21,N}=\psi_{21,N}+o_p(N^{-1/2})$ and $\widehat{\psi}_{22,N}=\psi_{22,N}+o_p(N^{-1/2})$. So $\widehat{\mathbf{\Psi}}_N=\mathbf{\Psi}_N+o_p(N^{-1/2})=\mathbf{\Psi}_N+o_p(1)$, $\mathbf{\Psi}_N=O(1)$ is obvious. $\widehat{\mathbf{\Xi}}_N-\mathbf{\Xi}_N=o_p(1)$ follows from the consistency of $\widehat{\lambda}_N$, $\bar{\lambda}_N$, $\widehat{\mathbf{\Psi}}_N$ and the Slutsky lemma. According to the proof of $\widehat{\mathbf{\Psi}}_N=\mathbf{\Psi}_N+o_p(N^{-1/2})$, we can show that $N^{1/2}\hbar(\lambda)=N^{1/2}(N^{-1}\bar{\boldsymbol{\eta}}_N^\tau\mathbf{C}_{1,N}(\lambda)\bar{\boldsymbol{\eta}}_N,N^{-1}\bar{\boldsymbol{\eta}}_N^\tau\mathbf{C}_{2,N}(\lambda)\bar{\boldsymbol{\eta}}_N)^\tau+o_p(1)$. Applying Theorem A.1 in Su (2012) gets $N^{1/2}(N^{-1}\bar{\boldsymbol{\eta}}_N^\tau\mathbf{C}_{1,N}(\lambda)\bar{\boldsymbol{\eta}}_N,N^{-1}\bar{\boldsymbol{\eta}}_N^\tau\mathbf{C}_{2,N}(\lambda)\bar{\boldsymbol{\eta}}_N)^\tau=\mathbf{\Phi}_N^{1/2}\xi_N+o_p(1)$. Above all, we obtain

$$
\sqrt{N}(\widehat{\lambda}_N-\lambda)=\widehat{\mathbf{\Xi}}_N^{-1}\begin{pmatrix}1 & 2\widehat{\lambda}_N\end{pmatrix}\widehat{\mathbf{\Psi}}_N^\tau\mathbf{\Upsilon}_N\left(\mathbf{\Phi}_N^{\frac{1}{2}}\xi_N+o_p(1)\right)=(\mathbf{J}_N^\tau\mathbf{\Upsilon}_N\mathbf{J}_N)^{-1}\mathbf{J}_N^\tau\mathbf{\Upsilon}_N\mathbf{\Phi}_N^{\frac{1}{2}}\xi_N+o_p(1).
$$

Thus, the proof of Theorem 2 is complete.

Proof of Theorem 3. (i) Let $J(N,T)=(NT_s)^{-1}$, where $T_s=T-(s+1)$, and $t_s=t+(s-1)$, then

$$
\mathbf{Q}_{0,N}=J(N,T)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau(\eta_{it_s},\cdots,\eta_{it})
$$

and

$$
\Delta_{it,N}=\mathbf{X}_{it}^\tau(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_N)+Z_{it1}(m_1(U_{it})-\widehat{m}_{1,N}(U_{it}))+\cdots+Z_{itq}(m_q(U_{it})-\widehat{m}_{q,N}(U_{it})).
$$

Since $\widehat{\varepsilon}_{it,N}=Y_{it}-\mathbf{X}_{it}^\tau\widehat{\boldsymbol{\beta}}_N-Z_{it1}\widehat{m}_{1,N}(U_{it})-\cdots-Z_{itq}\widehat{m}_{q,N}(U_{it})=\varepsilon_{it}+\Delta_{it,N}$ and $\eta_{it}=\varepsilon_{it}-\lambda\sum_{i_1=1}^{N}W_{ii_1}\varepsilon_{i_1t}$, $\widehat{\eta}_{it,N}$ can be expressed as

$$
\widehat{\eta}_{it,N}=\eta_{it}+\left(\Delta_{it,N}-\lambda\sum_{i_1=1}^{N}W_{ii_1}\Delta_{i_1t,N}\right)+(\lambda-\widehat{\lambda}_N)\sum_{i_1=1}^{N}W_{ii_1}(\varepsilon_{i_1t}+\Delta_{i_1t,N})=\eta_{it}+\Delta_{it,N}^*.
$$

As a result, we have

$$\begin{aligned}\widehat{\mathbf{Q}}_{0,N} =&\mathbf{Q}_{0,N} + J(N,T)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\Delta^*_{it_s,N},\cdots,\Delta^*_{it,N})^\tau(\Delta^*_{it_s,N},\cdots,\Delta^*_{it,N})\\ &+ J(N,T)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau(\Delta^*_{it_s,N},\cdots,\Delta^*_{it,N})\\ &+ J(N,T)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\Delta^*_{it_s,N},\cdots,\Delta^*_{it,N})^\tau(\eta_{it_s},\cdots,\eta_{it})\\ =&\mathbf{Q}_{0,N} + J_{1,N} + J_{2,N} + J_{3,N}.\end{aligned}$$

Define

$$\begin{aligned}\Delta^*_{it,N} =&\left\{\Delta_{it,N} - \lambda\sum_{i_1=1}^{N}W_{ii_1}\Delta_{i_1t,N} + (\lambda-\widehat{\lambda}_N)\sum_{i_1=1}^{N}W_{ii_1}\Delta_{i_1t,N}\right\} + (\lambda-\widehat{\lambda}_N)\sum_{i_1=1}^{N}W_{ii_1}\varepsilon_{i_1t}\\ =&\Delta^{*(1)}_{it,N} + \Delta^{*(2)}_{it,N}.\end{aligned}$$

Then

$$\begin{aligned}J_{1,N} \leq&2J(N,T)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\Delta^{*(1)}_{it_s,N},\cdots,\Delta^{*(1)}_{it,N})^\tau(\Delta^{*(1)}_{it_s,N},\cdots,\Delta^{*(1)}_{it,N})\\ &+ 2J(N,T)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\Delta^{*(2)}_{it_s,N},\cdots,\Delta^{*(2)}_{it,N})^\tau(\Delta^{*(2)}_{it_s,N},\cdots,\Delta^{*(2)}_{it,N})\\ =&J^{(1)}_{1,N} + J^{(2)}_{1,N}.\end{aligned}$$

According to Theorems 1-2 and Assumption C, $\Delta^{*(1)}_{it,N} = O_p\left(N^{-1/2}\right) + O_p\left(\sqrt{\kappa_N N^{-1} + \varphi_N^2}\right)$ and then $J^{(1)}_{1,N} = O_p\left(N^{-1}\right) + O_p\left(\kappa_N N^{-1} + \varphi_N^2\right) = o_p\left(N^{-1/2}\right)$. In addition,

$$J^{(2)}_{1,N} = 2(\lambda-\widehat{\lambda}_N)^2 J(N,T)\sum_{t=1}^{T_s}\begin{pmatrix}\boldsymbol{\varepsilon}^\tau_N(t_s)\mathbf{W}^\tau_N\mathbf{W}_N\boldsymbol{\varepsilon}_N(t_s) & \cdots & \boldsymbol{\varepsilon}^\tau_N(t_s)\mathbf{W}^\tau_N\mathbf{W}_N\boldsymbol{\varepsilon}_N(t)\\ \boldsymbol{\varepsilon}^\tau_N(t+s)\mathbf{W}^\tau_N\mathbf{W}_N\boldsymbol{\varepsilon}_N(t_s) & \cdots & \boldsymbol{\varepsilon}^\tau_N(t+s)\mathbf{W}^\tau_N\mathbf{W}_N\boldsymbol{\varepsilon}_N(t)\\ \vdots & \vdots & \vdots\\ \boldsymbol{\varepsilon}^\tau_N(t)\mathbf{W}^\tau_N\mathbf{W}_N\boldsymbol{\varepsilon}_N(t_s) & \cdots & \boldsymbol{\varepsilon}^\tau_N(t)\mathbf{W}^\tau_N\mathbf{W}_N\boldsymbol{\varepsilon}_N(t)\end{pmatrix}.$$

According to Theorem 2, Lemmas 1 and 2, $N^{-1}\boldsymbol{\varepsilon}^\tau_N(t+(s-l_1))\mathbf{W}^\tau_N\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+(s-l_2)) = O_p(1)$ for $l_1, l_2 = 1,\cdots,s$. So, $J^{(2)}_{1,N} = O_p(N^{-1}) = o_p(N^{-1/2})$. Therefore, $J_{1,N} = o_p(N^{-1/2})$.

For $J_{2,N}$ we have

$$
\begin{aligned}
NT_sJ_{2,N} =& \sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau(\Delta_{it_s,N},\cdots,\Delta_{it,N}) \\
&- \sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau(\lambda\sum_{i_1=1}^{N}W_{ii_1}\Delta_{i_1t_s,N},\cdots,\lambda\sum_{i_1=1}^{N}W_{ii_1}\Delta_{i_1t,N}) \\
&+ (\lambda-\widehat{\lambda}_N)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau(\sum_{i_1=1}^{N}W_{ii_1}\varepsilon_{i_1t_s},\cdots,\sum_{i_1=1}^{N}W_{ii_1}\varepsilon_{i_1t}) \\
&+ (\lambda-\widehat{\lambda}_N)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau(\sum_{i_1=1}^{N}W_{ii_1}\Delta_{i_1t_s,N},\cdots,\sum_{i_1=1}^{N}W_{ii_1}\Delta_{i_1t,N}) \\
=& J_{2,N}^{(1)}+J_{2,N}^{(2)}+J_{2,N}^{(3)}+J_{3,N}^{(4)}.
\end{aligned}
$$

According to the definition of $\Delta_{it,N}$, it holds that

$$
\begin{aligned}
J_{2,N}^{(1)} =& \sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau(\mathbf{X}_{it_s}^\tau(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_N),\cdots,\mathbf{X}_{it}^\tau(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_N)) \\
&+ \sum_{i,t}(\eta_{it_s},\cdots,\eta_{it})^\tau(\sum_{j=1}^{q}Z_{it_s,j}(m_j(U_{it_s})-\widehat{m}_{j,N}(U_{it_s})),\cdots,\sum_{j=1}^{q}Z_{itj}(m_j(U_{it})-\widehat{m}_{j,N}(U_{it}))) \\
=& J_{2,N}^{(1)*}+J_{2,N}^{(1)**}.
\end{aligned}
$$

Following the same routine as the proof of Theorem 1 yields

$$
J(N,T)\sum_{i=1}^{N}\sum_{t=1}^{T_s}\mathbf{X}_{i,t+m}(\eta_{it_s},\cdots,\eta_{it}) = O_p(N^{-\frac{1}{2}}) \text{ for } 0\le m\le s-1.
$$

Therefore, Theorem 1(i) leads to $J_{2,N}^{(1)*}=O_p(1)=o(N^{1/2})$. Based on the definition of $\widehat{m}_{j,N}(U_{it})$, we have

$$
\begin{aligned}
&\sum_{j=1}^{q}Z_{itj}(m_j(U_{it})-\widehat{m}_{j,N}(U_{it})) \\
=& \sum_{j=1}^{q}Z_{itj}\left\{m_j(U_{it})-(\boldsymbol{\zeta}(\mathbf{U}_{it,N}))^\tau(\mathbf{0}_{\kappa_N\times(j-1)\kappa_N},\mathbf{I}_{\kappa_N},\mathbf{0}_{\kappa_N\times(q-j-1)\kappa_N})(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}(\mathbf{Z}_N\odot\mathbf{M}_N)\right\} \\
&- \sum_{j=1}^{q}Z_{itj}(\boldsymbol{\zeta}(\mathbf{U}_{it,N}))^\tau(\mathbf{0}_{\kappa_N\times(j-1)\kappa_N},\mathbf{I}_{\kappa_N},\mathbf{0}_{\kappa_N\times(q-j-1)\kappa_N})(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\boldsymbol{\varepsilon}_N \\
&- \sum_{j=1}^{q}Z_{itj}(\boldsymbol{\zeta}(\mathbf{U}_{it,N}))^\tau(\mathbf{0}_{\kappa_N\times(j-1)\kappa_N},\mathbf{I}_{\kappa_N},\mathbf{0}_{\kappa_N\times(q-j-1)\kappa_N})(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\mathbf{X}_N(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_N) \\
=& \wp_1+\wp_2+\wp_3.
\end{aligned}
$$

By the properties of polynomial spline we can show that

$$\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau\wp_1 = O_p(N^{\frac{1}{2}})\cdot O_p\left(\sqrt{\kappa_N N^{-1}+\varphi_N^2}\right) = o_p(N^{\frac{1}{2}}).$$

Further,

$$\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau\wp_2 = O_p(N^{-1}) + \frac{1}{N}\boldsymbol{\varepsilon}_N^\tau\mathbf{Z}_N^*(\mathbf{Z}_N^{*\tau}\mathbf{Z}_N^*)^{-1}\mathbf{Z}_N^{*\tau}\boldsymbol{\varepsilon}_N = O_p(\kappa_N) = o_p(N^{\frac{1}{2}}).$$

The $\sqrt{N}$ consistency of $\widehat{\boldsymbol{\beta}}_N$ and the same argument for $\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau\wp_2$ lead to $\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau\wp_3 = O_p(\kappa_N) = o_p(N^{1/2})$ as well. Together we have $J_{2,N}^{(1)**} = o_p(N^{1/2})$. As a result, $J_{2,N}^{(1)} = o_p(N^{1/2})$. Following the same line, we can show that $J_{2,N}^{(2)} = o_p(N^{1/2})$ and $J_{2,N}^{(4)} = o_p(N^{1/2})$. Hence

$$\begin{aligned}
J_{2,N} =& J(N,T)(\lambda-\widehat{\lambda}_N)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s},\cdots,\eta_{it})^\tau(\sum_{i_1=1}^{N}W_{ii_1}\varepsilon_{i_1t_s},\cdots,\sum_{i_1=1}^{N}W_{ii_1}\varepsilon_{i_1t}) + o_p(N^{-\frac{1}{2}})\\
=&\frac{\lambda-\widehat{\lambda}_N}{T_s}\cdot\sum_{t=1}^{T_s}\begin{pmatrix} E\boldsymbol{\eta}_N^\tau(t_s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t_s) & \cdots & E\boldsymbol{\eta}_N^\tau(t_s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t)\\ E\boldsymbol{\eta}_N^\tau(t+s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t_s) & \cdots & E\boldsymbol{\eta}_N^\tau(t+s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t)\\ \vdots & \vdots & \vdots\\ E\boldsymbol{\eta}_N^\tau(t)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t_s) & \cdots & E\boldsymbol{\eta}_N^\tau(t)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t)\end{pmatrix} + o_p(N^{-\frac{1}{2}})\\
=&(\lambda-\widehat{\lambda}_N)\Im_{01} + o_p(N^{-\frac{1}{2}}).
\end{aligned}$$

Similarly, we have

$$J_{3,N} = J(N,T)(\lambda-\widehat{\lambda}_N)\sum_{i=1}^{N}\sum_{t=1}^{T_s}\mathbf{C}(\mathbf{W}_N,\boldsymbol{\varepsilon}_N)^\tau\mathbf{C}(\mathbf{W}_N,\boldsymbol{\varepsilon}_N)+o_p(N^{-\frac{1}{2}}) = (\lambda-\widehat{\lambda}_N)\Im_{01,N}^\tau+o_p(N^{-\frac{1}{2}}),$$

where $\mathbf{C}(\mathbf{W}_N,\boldsymbol{\varepsilon}_N) = (\sum_{i_1=1}^{N}W_{ii_1}\varepsilon_{i_1t_s},\cdots,\sum_{i_1=1}^{N}W_{ii_1}\varepsilon_{i_1t})$. This implies that $\widehat{\mathbf{Q}}_{0,N} = \mathbf{Q}_{0,N}+(\lambda-\widehat{\lambda}_N)(\Im_{01,N}+\Im_{01,N}^\tau)+o_p(N^{-1/2}) = \mathbf{Q}_{0,N}+(\lambda-\widehat{\lambda}_N)\Im_{0,N}+o_p(N^{-1/2})$, where $\Im_{0,N} = \Im_{01,N}+\Im_{01,N}^\tau$.

By the same argument, we can show that $\widehat{\mathbf{Q}}_{1,N} = \mathbf{Q}_{1,N}+(\lambda-\widehat{\lambda}_N)\Im_{1,N}+o_p(N^{-1/2})$, where $\Im_{1,N} = \Im_{11,N}+\Im_{12,N}^\tau$ with

$$\Im_{11,N} = \frac{1}{T_s}\sum_{t=1}^{T_s}\begin{pmatrix} E\boldsymbol{\eta}_N^\tau(t_s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+s) & \cdots & E\boldsymbol{\eta}_N^\tau(t_s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+1)\\ E\boldsymbol{\eta}_N^\tau(t+s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+s) & \cdots & E\boldsymbol{\eta}_N^\tau(t+s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+1)\\ \vdots & \vdots & \vdots\\ E\boldsymbol{\eta}_N^\tau(t)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+s) & \cdots & E\boldsymbol{\eta}_N^\tau(t)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+1),\end{pmatrix}$$

and $\Im_{12,N}$ has the same definition as $\Im_{11,N}$ except for switching $\boldsymbol{\varepsilon}(t)$ with $\boldsymbol{\eta}(t)$,

$$\widehat{\mathbf{Q}}_{2,N} = \mathbf{Q}_{2,N}+(\lambda-\widehat{\lambda}_N)\Im_{2,N}+o_p(N^{-\frac{1}{2}}),$$

where $\Im_{2,N} = \Im_{21,N} + \Im_{22,N}$ with

$$\Im_{21,N} = \frac{1}{T_s}\sum_{t=1}^{T_s}\begin{pmatrix} E\boldsymbol{\eta}_N^\tau(t+s-1)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+s) \\ E\boldsymbol{\eta}_N^\tau(t+s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+s) \\ \vdots \\ E\boldsymbol{\eta}_N^\tau(t)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+s) \end{pmatrix}$$

and $\Im_{22,N}$ has the same definition as $\Im_{21,N}$ except for switching $\boldsymbol{\varepsilon}(t)$ with $\boldsymbol{\eta}(t)$, and

$$\widehat{\mathbf{Q}}_{3,N} = \mathbf{Q}_{3,N} + (\lambda - \widehat{\lambda}_N)\Im_{3,N} + o_p(N^{-\frac{1}{2}}),$$

where $\Im_{3,N} = \Im_{31,N} + \Im_{32,N}$ with

$$\Im_{31,N} = \frac{1}{T_s}\sum_{t=1}^{T_s}\begin{pmatrix} E\boldsymbol{\eta}_N^\tau(t_s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+s+1) \\ E\boldsymbol{\eta}_N^\tau(t+s)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+s+1) \\ \vdots \\ E\boldsymbol{\eta}_N^\tau(t)\mathbf{W}_N\boldsymbol{\varepsilon}_N(t+s+1) \end{pmatrix}$$

and $\Im_{32,N}$ has the same definition as $\Im_{31,N}$ except for switching $\boldsymbol{\varepsilon}(t)$ with $\boldsymbol{\eta}(t)$. As a result, for $\widehat{\boldsymbol{\rho}}_N$,

$$\sqrt{NT}(\widehat{\boldsymbol{\rho}}_N - \boldsymbol{\rho}) = \sqrt{NT}\left\{\left(\widehat{\mathbf{Q}}_{0,N} - \widehat{\mathbf{Q}}_{1,N}\right)^{-1}\left(\widehat{\mathbf{Q}}_{2,N} - \widehat{\mathbf{Q}}_{3,N}\right) - \boldsymbol{\rho}\right\}$$

$$= \frac{\sqrt{NT}}{\mathbf{Q}_{0,N} - \mathbf{Q}_{1,N}}[(\mathbf{Q}_{2,N} - \mathbf{Q}_{3,N}) - (\mathbf{Q}_{0,N} - \mathbf{Q}_{1,N})\boldsymbol{\rho} + (\lambda - \widehat{\lambda}_N)((\Im_{0,N} - \Im_{1,N}) - (\Im_{2,N} - \Im_{3,N})\boldsymbol{\rho})] + o_p(1).$$

According to the definition of $\mathbf{Q}_{0,N}$ and $\mathbf{Q}_{1,N}$, we conclude that the statistic

$$\mathbf{Q}_{0,N} - \mathbf{Q}_{1,N} = J(N,T)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(\eta_{it_s}, \cdots, \eta_{it})^\tau\{(\eta_{it_s}, \cdots, \eta_{it}) - (\eta_{i(t+s)}, \cdots, \eta_{i(t+1)})\}$$

converges in probability to $\mathrm{E}\{(\eta_{it_s}, \cdots, \eta_{it})^\tau[(\eta_{it_s}, \cdots, \eta_{it}) - (\eta_{i(t+s)}, \cdots, \eta_{i(t+1)})]\}$, e-quivalently, $\mathrm{E}\{(\nu_{it_s}, \cdots, \nu_{it})^\tau[(\nu_{it_s}, \cdots, \nu_{it}) - (\nu_{i(t+s)}, \cdots, \nu_{i(t+1)})\}$ as $N \to \infty$. According to error model (2), we see that $\eta_{i(t+s)} = \rho_1\eta_{it_s} + \cdots + \rho_s\eta_{it} + (1-\rho_1-\cdots-\rho_s)\mu_i + e_{i(t+s)}$ and $\eta_{i(t+(s+1))} = \rho_1\eta_{i(t+s)} + \cdots + \rho_s\eta_{i(t+1)} + (1-\rho_1-\cdots-\rho_s)\mu_i + e_{i(t+(s+1))}$. Therefore,

$$\sqrt{NT_s}\left\{\mathbf{Q}_{2,N} - \mathbf{Q}_{3,N} - (\mathbf{Q}_{0,N} - \mathbf{Q}_{1,N})\,\boldsymbol{\rho}\right\}$$

$$= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{\sqrt{T_s}}\sum_{t=1}^{T_s}\left\{(\nu_{it_s}+\mu_i, \cdots, \nu_{it}+\mu_i)^\tau e_{i(t+s)} - (\nu_{it_s}+\mu_i, \cdots, \nu_{it}+\mu_i)^\tau e_{i(t+(s+1))}\right\}\right]$$

$$= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{\chi}_{i,N},$$

where $\{\boldsymbol{\chi}_i\}_{i=1}^N$ is an i.i.d. random vector sequence with mean 0 and covariance matrix

$$\mathrm{Cov}(\boldsymbol{\chi}_{i,N}) = \frac{1}{T_s}\mathrm{Cov}\sum_{t=1}^{T_s}\left\{(\nu_{it_s}+\mu_i, \cdots, \nu_{it}+\mu_i)^\tau(e_{i(t+s)} - e_{i(t+(s+1))})\right\} = 2\sigma_e^2\left(\sigma_\mu^2\mathbf{1}_s\mathbf{1}_s^\tau + \boldsymbol{\Gamma}_s\right),$$

where $\gamma_\nu(j) = \mathrm{E}(\nu_{it,N}\nu_{i(t+j),N})$. On the other hand, by Theorem 2, we have the conclusion that $\sqrt{NT}(\lambda - \widehat{\lambda}_N)((\Im_{0,N} - \Im_{1,N}) - (\Im_{2,N} - \Im_{3,N})\,\boldsymbol{\rho}) = O_p(1)$. In addition, it follows from Theorem 2 that $\sqrt{NT}(\lambda - \widehat{\lambda}_N) = (\mathbf{J}_N^\tau \boldsymbol{\Upsilon}_N \mathbf{J}_N)\mathbf{J}_N^\tau \boldsymbol{\Upsilon}_N \boldsymbol{\Phi}_N \mathbf{v}_N + o_p(1)$. By simple calculations, we have $\mathrm{E}(\mathbf{v}_N \sum_{i=1}^N \boldsymbol{\chi}_{i,N}) = 0$. Therefore, Theorem 3 (i) holds.

Next, we prove (ii). Since $\widehat{\ell}_{it,N} = \widehat{\eta}_{it+s,N} - \sum_{k=1}^s \widehat{\rho}_{k,N}\widehat{\eta}_{i(t+s-k),N}$, it can expressed as

$$\begin{aligned}\widehat{\ell}_{it,N} =& e_{i(t+s)} + \left\{(1-\sum_{k=1}^{s}\rho_k)\mu_i\right\} + \{(\rho_1 - \widehat{\rho}_{1,N})\eta_{it_s} - \cdots - (\rho_s - \widehat{\rho}_{s,N})\eta_{it}\} \\ &+ \left\{\Delta^*_{i(t+s)} - \widehat{\rho}_{1,N}\Delta^*_{it_s,N} - \cdots - \widehat{\rho}_{s,N}\Delta^*_{it,N}\right\} \\ =& e_{i(t+s)} + J_{4,N} + J_{5,N} + J_{6,N}.\end{aligned}$$

As a result, we have

$$\begin{aligned}\widehat{L}_N \equiv \sum_{i=1}^N\sum_{t=1}^{T_s}\widehat{\ell}^2_{it,N} =& \sum_{i=1}^N\sum_{t=1}^{T_s}\left(e^2_{i(t+s)} + J^2_{4,N} + J^2_{5,N} + J^2_{6,N} + 2e_{i(t+s)}J_{4,N} + 2e_{i(t+s)}J_{5,N}\right) \\ &+ 2\sum_{i=1}^N\sum_{t=1}^{T_s}\left(e_{i(t+s)}J_{6,N} + J_{4,N}J_{5,N} + J_{4,N}J_{6,N} + J_{5,N}J_{6,N}\right).\end{aligned}$$

Applying Theorem 3 (i) yields

$$J(N,T)\sum_{i=1}^N\sum_{t=1}^{T_s}J^2_{5,N} = O_p(N^{-1}) \quad \text{and} \quad 2J(N,T)\sum_{i=1}^N\sum_{t=1}^{T_s}e_{i(t+s)}J_{5,N} = O_p(N^{-1}).$$

It follows from Theorem 1, 2 and Lemma 2 that $J(N,T)\sum_{i=1}^N\sum_{t=1}^{T_s}J^2_{6,N} = O_p(N^{-1}) = o_p(N^{-1/2})$. The Cauchy-Schwarz inequality tells us that $2J(N,T)\sum_{i=1}^N\sum_{t=1}^{T_s}J_{5,N}J_{6,N} = O_p(N^{-1}) = o_p(N^{-1/2})$. In addition, it is easy to see that

$$2J(N,T)\sum_{i=1}^N\sum_{t=1}^{T_s}J_{4,N}J_{5,N} = -2J(N,T)\sum_{i=1}^N\sum_{t=1}^{T_s}J_{4,N}\{(\rho_1 - \widehat{\rho}_{1,N}) - \cdots - (\rho_s - \widehat{\rho}_{s,N})\}\mu_i.$$

Following the same argument as proving for (i) we have

$$2J(N,T)\sum_{i=1}^N\sum_{t=1}^{T_s}e_{i(t+s)}J_{6,N} = 2(\lambda - \widehat{\lambda}_N)J(N,T)\sum_{i=1}^N\sum_{t=1}^{T_s}e_{i(t+s)}\sum_{i_1=1}^N W_{ii_1}\varepsilon_{i_1(t+s)}$$

and

$$2J(N,T)\sum_{i=1}^N\sum_{t=1}^{T_s}J_{4,N}J_{6,N} = 2(\lambda - \widehat{\lambda}_N)J(N,T)\sum_{i=1}^N\sum_{t=1}^{T_s}(1-\sum_{k=1}^s\rho_k)\mu_i\sum_{i_1=1}^N W_{ii_1}\varepsilon_{i_1(t+s)}.$$

So we obtain

$$\begin{aligned}\widehat{L}_N =& \sum_{i=1}^{N}\sum_{t=1}^{T_s} e_{i(t+s)}^2 + \frac{1}{NJ(N,T)}\sum_{i=1}^{N}(1-\sum_{k=1}^{s}\rho_k)^2\mu_i^2 \\ &+ 2\sum_{i=1}^{N}\sum_{t=1}^{T_s} e_{i(t+s)}(1-\sum_{k=1}^{s}\rho_k)\mu_i + 2(\lambda-\widehat{\lambda}_N)\sum_{i=1}^{N}\sum_{t=1}^{T_s} e_{i(t+s)}\sum_{i_1=1}^{N} W_{ii_1}\varepsilon_{i_1(t+s)} \\ &- \frac{2}{NJ(N,T)}\sum_{i=1}^{N}(1-\sum_{k=1}^{s}\rho_k)\{(\rho_1-\widehat{\rho}_{1,N})-\cdots-(\rho_s-\widehat{\rho}_{s,N})\}\mu_i^2 \\ &+ 2(\lambda-\widehat{\lambda}_N)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(1-\sum_{k=1}^{s}\rho_k)\mu_i\sum_{i_1=1}^{N} W_{ii_1}\varepsilon_{i_1(t+s)}.\end{aligned}$$

By the same argument, we compute the cross-product item $\sum_{i=1}^{N}\sum_{t=1}^{T_s}\widehat{\ell}_{it,N}\widehat{\ell}_{i(t+1),N}$, denoted by $\widehat{CL}_N$, as follow:

$$\begin{aligned}\widehat{CL}_N =& \sum_{i=1}^{N}\sum_{t=1}^{T_s} e_{i(t+s)}e_{i,t+s+1} + \frac{1}{NJ(N,T)}\sum_{i=1}^{N}(1-\sum_{k=1}^{s}\rho_k)^2\mu_i^2 \\ &+ \sum_{i=1}^{N}\sum_{t=1}^{T_s} e_{i(t+s)}(1-\sum_{k=1}^{s}\rho_k)\mu_i + \sum_{i=1}^{N}\sum_{t=1}^{T_s} e_{i,t+s+1}(1-\sum_{k=1}^{s}\rho_k)\mu_i \\ &+ (\lambda-\widehat{\lambda}_N)\sum_{i=1}^{N}\sum_{t=1}^{T_s} e_{i(t+s+1)}\sum_{i_1=1}^{N} W_{ii_1}\varepsilon_{i_1(t+s)} + (\lambda-\widehat{\lambda}_N)\sum_{i=1}^{N}\sum_{t=1}^{T_s+1} e_{i(t+s)}\sum_{i_1=1}^{N} W_{ii_1}\varepsilon_{i_1(t+s+1)} \\ &- \frac{2}{NJ(N,T)}\sum_{i=1}^{N}(1-\sum_{k=1}^{s}\rho_k)\{(\rho_1-\widehat{\rho}_{1,N})-\cdots-(\rho_s-\widehat{\rho}_{s,N})\}\mu_i^2 \\ &+ (\lambda-\widehat{\lambda}_N)\sum_{i=1}^{N}\sum_{t=1}^{T_s}(1-\sum_{k=1}^{s}\rho_k)\mu_i\left(\sum_{i_1=1}^{N} W_{ii_1}\varepsilon_{i_1(t+s)} + \sum_{i_1=1}^{N} W_{ii_1}\varepsilon_{i_1(t+s+1)}\right).\end{aligned}$$

It implies that

$$\begin{aligned}(\widehat{L}_N - \widehat{CL}_N)) =& \sum_{i=1}^{N}\sum_{t=1}^{T_s}\widehat{\ell}_{it,N}^2 - \sum_{i=1}^{N}\sum_{t=1}^{T_s}\widehat{\ell}_{it,N}\widehat{\ell}_{i,t+1,N} \\ =& \sum_{i=1}^{N}\sum_{t=1}^{T_s}[e_{i(t+s)}^2 - e_{i(t+s)}e_{i(t+s+1)} + (e_{i(t+s)}-e_{i(t+s+1)})(1-\sum_{k=1}^{s}\rho_k)\mu_i] \\ &+ \frac{\lambda-\widehat{\lambda}_N}{J(N,T)}\Im_{4,N} + o_p\left(J(N,T)^{-1}\right).\end{aligned}$$

As a consequence, we obtain

$$\sqrt{NT}(\widehat{\sigma}_{e,N}^2 - \sigma_e^2) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\varsigma_{i,N} + \sqrt{NT}(\lambda-\widehat{\lambda}_N)\Im_{4,N} + o_p(1),$$

where $\varsigma_{i,N} = \sqrt{T}T_s^{-1}\sum_{t=1}^{T_s}\left\{e_{i(t+s)}^2 - \sigma_e^2 - e_{i(t+s)}e_{i(t+s+1)} - (e_{i(t+s)} - e_{i(t+s+1)})(1-\sum_{k=1}^{s}\rho_k)\mu_i\right\}$. It isn't hard to see that $\varsigma_{i,N}$ is an i.i.d. random vector sequence with mean 0 and variance

$$\mathrm{Var}(\varsigma_{i,N}) = \frac{T}{T_s}\left\{\mathrm{Var}(e_{it}^2) + 2(1-\sum_{k=1}^{s}\rho_k)^2\sigma_\mu^2\sigma_e^2 + \sigma_e^4\right\}.$$

On the other hand, by Theorem 2, we have the conclusion that $\sqrt{NT}(\lambda - \widehat{\lambda}_N)\Im_{4,N} = O_p(1)$. In addition, it follows from the proof of Theorem 2 that $\sqrt{NT}(\lambda - \widehat{\lambda}_N) = (\mathbf{J}_N^\tau \mathbf{\Upsilon}_N \mathbf{J}_N)\mathbf{J}_N^\tau \mathbf{\Upsilon}_N \mathbf{\Phi}_N \mathbf{v}_N + o_p(1)$. By a simple calculation, we have $\mathrm{E}(\mathbf{v}_N\sum_{i=1}^{N}\varsigma_{i,N}) = 0$. Therefore, we have proved the part (ii) of Theorem 3.

Proof of Theorem 4. From to the definitions of $\widehat{\boldsymbol{\beta}}_N^w$ and $\tilde{\boldsymbol{\beta}}_N^w$ it holds that

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_N^w - \boldsymbol{\beta} =& \tilde{\boldsymbol{\beta}}_N^w - \boldsymbol{\beta}_N + \left\{\mathbf{W}_N^{\widehat{\Sigma}_N} - \mathbf{W}_N^{\Sigma_N}\right\}\left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}} - \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\right)(\mathbf{Z}_N \odot \mathbf{M}_N)\\
&+ \left\{\mathbf{W}_N^{\widehat{\Sigma}_N} - \mathbf{W}_N^{\Sigma_N}\right\}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}}(\mathbf{Z}_N \odot \mathbf{M}_N) + \mathbf{W}_N^{\Sigma_N}\left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}} - \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\right)(\mathbf{Z}_N \odot \mathbf{M}_N)\\
&+ \left\{\mathbf{W}_N^{\widehat{\Sigma}_N} - \mathbf{W}_N^{\Sigma_N}\right\}\left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}} - \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\right)\boldsymbol{\varepsilon}_N + \left\{\mathbf{W}_N^{\widehat{\Sigma}_N} - \mathbf{W}_N^{\Sigma_N}\right\}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\boldsymbol{\varepsilon}_N\\
&+ \mathbf{W}_N^{\Sigma_N}\left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}} - \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}}\right)\boldsymbol{\varepsilon}_N,\end{aligned}$$

where $\mathbf{W}_N^{\widehat{\Sigma}_N} = (\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}}\mathbf{X}_N)^{-1}$. By Lemma 2 (i) and the fact that $(\mathbf{A} + a\mathbf{B})^{-1} = \mathbf{A}^{-1} - a\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} + O(a^2)$ as $a \to 0$, to prove Theorem 4 we only need to prove the following equalities:

$$\frac{1}{NT}\left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}}\mathbf{X}_N - \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\mathbf{X}_N\right) = O_p(N^{-\frac{1}{2}}), \tag{A1}$$

$$\frac{1}{NT}\left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}} - \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\right)(\mathbf{Z}_N \odot \mathbf{M}_N) = o_p(N^{-\frac{1}{2}}), \tag{A2}$$

$$\frac{1}{NT}\left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}}\mathbf{X}_N - \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\mathbf{X}_N\right)\boldsymbol{\varepsilon}_N = o_p(N^{-\frac{1}{2}}), \tag{A3}$$

$$\frac{1}{NT}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\mathbf{X}_N = O_p(1) \tag{A4}$$

and

$$\frac{1}{NT}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}}(\mathbf{Z}_N \odot \mathbf{M}_N) = o_p(N^{-\frac{1}{2}}),\ \frac{1}{NT}\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}}\boldsymbol{\varepsilon}_N = O_p(N^{-\frac{1}{2}}). \tag{A5}$$

According to the proof of Lemma 2(i) and the fact that $\widehat{\mathbf{\Sigma}}_N^{-1} = (\widehat{\sigma}_{\mu,N}^2\mathbf{1}_T\mathbf{1}_T^\tau + \widehat{\mathbf{\Gamma}}_N)^{-1} \otimes \{\mathbf{F}(\widehat{\lambda}_N)\mathbf{F}(\widehat{\lambda}_N)^\tau\}$, where $\mathbf{F}(\widehat{\lambda}_N) = \mathbf{I}_N - \widehat{\lambda}_N\mathbf{W}_N$, the difference item $(NT)^{-1}(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}}\mathbf{X}_N -$

$\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}} \mathbf{X}_N)$, denoted by $(NT)^{-1}\mathbf{D}_N$, can be expressed as

$$\begin{aligned}\frac{1}{NT}\mathbf{D}_N =& \frac{1}{NT}\left(\mathbf{\Pi}^\tau \widehat{\mathbf{\Sigma}}_N^{-1}\mathbf{\Pi} - \mathbf{\Pi}^\tau \mathbf{\Sigma}_N^{-1}\mathbf{\Pi}\right) + o_p(N^{-\frac{1}{2}})\\ =& \frac{1}{NT}\mathbf{\Pi}^\tau (\widehat{\sigma}^2_{\mu,N}\mathbf{1}_T\mathbf{1}_T^\tau + \widehat{\mathbf{\Gamma}}_N)^{-1} \otimes \{\mathbf{F}(\widehat{\lambda}_N)\mathbf{F}(\widehat{\lambda}_N)^\tau - \mathbf{F}(\lambda)\mathbf{F}(\lambda)^\tau\}\mathbf{\Pi}\\ &+ \frac{1}{NT}\mathbf{\Pi}^\tau \left\{(\widehat{\sigma}^2_{\mu,N}\mathbf{1}_T\mathbf{1}_T^\tau + \widehat{\mathbf{\Gamma}}_N)^{-1} - (\sigma^2_{\mu}\mathbf{1}_T\mathbf{1}_T^\tau + \mathbf{\Gamma})^{-1}\right\} \otimes \{\mathbf{F}(\lambda)\mathbf{F}(\lambda)^\tau\}\mathbf{\Pi} + o_p(N^{-\frac{1}{2}}).\end{aligned}$$

Since $\mathbf{F}(\widehat{\lambda}_N)\mathbf{F}(\widehat{\lambda}_N) - \mathbf{F}(\lambda)\mathbf{F}(\lambda) = (\widehat{\lambda}_N^2 - \lambda^2)\mathbf{W}_N\mathbf{W}_N^\tau - (\widehat{\lambda}_N - \lambda)(\mathbf{W}_N + \mathbf{W}_N^\tau)$ and $(\widehat{\sigma}^2_{\mu,N}\mathbf{1}_T\mathbf{1}_T^\tau + \widehat{\mathbf{\Gamma}}_N)^{-1} - (\sigma^2_{\mu}\mathbf{1}_T\mathbf{1}_T^\tau + \mathbf{\Gamma})^{-1} = O_p(N^{-1/2})$ from Theorem 3, we have $(NT)^{-1}\mathbf{D}_N = O_p(N^{-1/2})$. This implies that (A1) holds. Applying the polynomial spline properties and

$$\frac{1}{NT}\left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}} - \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\right)(\mathbf{Z}_N \odot \mathbf{M}_N) = \frac{1}{NT}\left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}} - \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\right)(\mathbf{Z}_N \odot \mathbf{M}_N - \mathbf{Z}_N^*\boldsymbol{\theta}_N),$$

by the same argument as that for (A1), we can show that (A2) holds. Moreover, from the proof of Lemma 2 we know that (A4) and (A5) hold as well. Therefore, to complete the proof we only need to prove (A3). It is easy to see that

$$\frac{1}{NT}\left(\mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\widehat{\mathbf{\Sigma}}_N^{-1}}\boldsymbol{\varepsilon} - \mathbf{X}_N^\tau \mathbf{M}_{\mathbf{Z}_N^*}^{\mathbf{\Sigma}_N^{-1}}\boldsymbol{\varepsilon}\right) = o_p(N^{-\frac{1}{2}}) + J_{1,N} + J_{2,N} + J_{3,N},$$

where

$$\begin{aligned}J_{1,N} =& \frac{1}{NT}\mathbf{\Pi}^\tau \left\{\mathbf{G}(\widehat{\sigma}^2_\mu, \widehat{\Gamma}) - \mathbf{G}(\sigma^2_\mu, \Gamma)\right\} \otimes \left\{(\widehat{\lambda}_N^2 - \lambda^2)\mathbf{W}_N\mathbf{W}_N^\tau - (\widehat{\lambda}_N - \lambda)(\mathbf{W}_N + \mathbf{W}_N^\tau)\right\}\boldsymbol{\varepsilon}_N,\\ J_{2,N} =& \frac{1}{NT}\mathbf{\Pi}^\tau \left[\mathbf{G}(\sigma^2_\mu, \Gamma) \otimes \left\{(\widehat{\lambda}_N^2 - \lambda^2)\mathbf{W}_N\mathbf{W}_N^\tau - (\widehat{\lambda}_N - \lambda)(\mathbf{W}_N + \mathbf{W}_N^\tau)\right\}\right]\boldsymbol{\varepsilon}_N \text{ and}\\ J_{2,N} =& \frac{1}{NT}\mathbf{\Pi}^\tau \left\{\mathbf{G}(\widehat{\sigma}^2_\mu, \widehat{\Gamma}) - \mathbf{G}(\sigma^2_\mu, \Gamma)\right\} \otimes \{(\mathbf{I}_N - \lambda\mathbf{W}_N)(\mathbf{I}_N - \lambda\mathbf{W}_N^\tau)\}\boldsymbol{\varepsilon}_N\end{aligned}$$

with $\mathbf{G}(\sigma^2_\mu, \Gamma) = (\sigma^2_\mu \mathbf{1}\mathbf{1}^\tau + \Gamma)^{-1}$. It is easy to see that $\mathrm{E}\{\mathbf{\Pi}^\tau(t_1)\mathbf{W}_N\mathbf{W}_N^\tau\boldsymbol{\varepsilon}_N(t_2)\} = 0$ and $\mathrm{Cov}\{\mathbf{\Pi}^\tau(t_1)\mathbf{W}_N\mathbf{W}_N^\tau\boldsymbol{\varepsilon}_N(t_2)\} = O(N)$. Therefore, $\mathbf{\Pi}^\tau(t_1)\mathbf{W}_N\mathbf{W}_N^\tau\boldsymbol{\varepsilon}_N(t_2) = O_p(N^{1/2})$ for $1 \le t_1, t_2 \le T$.

Let $\mathbf{G}^* = \mathbf{G}(\widehat{\sigma}^2_\mu, \widehat{\Gamma}) - \mathbf{G}(\sigma^2_\mu, \Gamma) = (g^*_{t_1,t_2})$ for $1 \le t_1, t_2 \le T$. Then

$$\begin{aligned}&\frac{1}{NT}\mathbf{\Pi}^\tau \left\{\mathbf{G}(\widehat{\sigma}^2_\mu, \widehat{\Gamma}) - \mathbf{G}(\sigma^2_\mu, \Gamma)\right\} \otimes \left\{(\widehat{\lambda}_N^2 - \lambda^2)\mathbf{W}_N\mathbf{W}_N^\tau\right\}\boldsymbol{\varepsilon}_N\\ =& \frac{1}{NT}\sum_{t_1=1}^T\sum_{t_2=1}^T(\widehat{\lambda}_N^2 - \lambda^2)g^*_{t_1t_2}\mathbf{\Pi}^\tau(t_1)\mathbf{W}_N\mathbf{W}_N^\tau\boldsymbol{\varepsilon}_N(t_2) = O(N^{-1}) \cdot O_p(N^{-\frac{1}{2}}) \cdot O_p(N^{-\frac{1}{2}}) \cdot O_p(N^{\frac{1}{2}}).\end{aligned}$$

Similarly, we can show that $\frac{1}{NT}\mathbf{\Pi}^\tau \left\{\mathbf{G}(\widehat{\sigma}^2_\mu, \widehat{\Gamma}) - \mathbf{G}(\sigma^2_\mu, \Gamma)\right\} \otimes \left\{(\widehat{\lambda}_N - \lambda)(\mathbf{W}_N + \mathbf{W}_N^\tau)\right\}\boldsymbol{\varepsilon}_N = O_p(N^{-3/2})$. Namely, $J_{1,N} = o_p(N^{-1/2})$. Following the same arguments, we can show that $J_{2,N} = O_p(N^{-1}) = o_p(N^{-1/2})$ and $J_{3,N} = O_p(N^{-1}) = o_p(N^{-1/2})$. This shows that (A3) holds. Thus, we complete the proof of the desired results.

S2 Simulation results for those cases with $T = 10$

Table 1. Finite sample performances of the proposed WSLSE of regression coefficient β and WPSSE of nonprarmetric function $m(\cdot)$ under $T = 10$

		$N = 100$		$N = 200$		$N = 300$	
	(λ, ρ)	$(0.3, 0.3)$	$(0.6, 0.6)$	$(0.3, 0.3)$	$(0.6, 0.6)$	$(0.3, 0.3)$	$(0.6, 0.6)$
$\widehat{\beta}_{1,N}$	est	1.0005	1.0025	0.9986	0.9993	1.0008	1.0030
	std	0.0440	0.0856	0.0356	0.0658	0.0314	0.0578
	estd	0.0438	0.0937	0.0355	0.0725	0.0309	0.0600
	cp	0.9490	0.9400	0.9490	0.9450	0.9450	0.9520
$\widehat{\beta}_{2,N}$	est	-1.4994	-1.5000	-1.4991	-1.5004	-1.5008	-1.5013
	std	0.0271	0.0444	0.0223	0.0384	0.0213	0.0318
	estd	0.0266	0.1434	0.0227	0.0411	0.0208	0.0326
	cp	0.9450	0.9570	0.9480	0.9500	0.9390	0.9470
$\widehat{\beta}^w_{1,N}$	est	1.0024	1.0017	0.9986	0.9983	1.0011	0.9999
	std	0.0202	0.0181	0.0163	0.0141	0.0144	0.0125
	estd	0.0199	0.0182	0.0165	0.0147	0.0144	0.0127
	cp	0.9450	0.9540	0.9580	0.9600	0.9510	0.9580
$\widehat{\beta}^w_{2,N}$	est	-1.5013	-1.4981	-1.4988	-1.5017	-1.4998	-1.5004
	std	0.0148	0.0126	0.0125	0.0104	0.0117	0.0090
	estd	0.0148	0.0134	0.0125	0.0108	0.0114	0.0094
	cp	0.9570	0.9600	0.9480	0.9570	0.9420	0.9520
$\tilde{\beta}^w_{1,N}$	est	1.0024	1.0017	0.9986	0.9982	1.0010	0.9999
	std	0.0199	0.0179	0.0162	0.0141	0.0144	0.0124
	estd	0.0197	0.0175	0.0164	0.0143	0.0143	0.0124
	cp	0.9400	0.9420	0.9550	0.9500	0.9530	0.9530
$\tilde{\beta}^w_{2,N}$	est	-1.5013	-1.4982	-1.4988	-1.5017	-1.4999	-1.5004
	std	0.0146	0.0125	0.0125	0.0104	0.0117	0.0090
	estd	0.0147	0.0129	0.0124	0.0104	0.0113	0.0091
	cp	0.9560	0.9510	0.9480	0.9510	0.9410	0.9520
$\widehat{\beta}^*_{1,N}$	est	0.9998	1.0017	1.0019	0.9984	1.0026	0.9997
	std	0.0301	0.0319	0.0228	0.0216	0.0185	0.0179
$\widehat{\beta}^*_{2,N}$	est	-1.5022	-1.4958	-1.4999	-1.4967	-1.4993	-1.5009
	std	0.0242	0.0244	0.0169	0.0158	0.0128	0.0138
$\widehat{m}_{1,N}(\cdot)$	sm(RASE)	0.1916	0.2574	0.1712	0.2124	0.1487	0.1943
	std(RASE)	0.0429	0.0659	0.0345	0.0512	0.0269	0.0416
$\widehat{m}_{2,N}(\cdot)$	sm(RASE)	0.1937	0.2373	0.1762	0.2140	0.1694	0.1967
	std(RASE)	0.0297	0.0463	0.0207	0.0362	0.0179	0.0294
$\widehat{m}^w_{1,N}(\cdot)$	sm(RASE)	0.1422	0.1311	0.1288	0.1173	0.1173	0.1113
	std(RASE)	0.0243	0.0188	0.0181	0.0129	0.0138	0.0106
$\widehat{m}^w_{2,N}(\cdot)$	sm(RASE)	0.1598	0.1535	0.1527	0.1485	0.1487	0.1445
	std(RASE)	0.0138	0.0103	0.0098	0.0077	0.0077	0.0062
$\tilde{m}^w_{1,N}(\cdot)$	sm(RASE)	0.1417	0.1304	0.1285	0.1170	0.1171	0.1111
	std(RASE)	0.0242	0.0186	0.0178	0.0127	0.0137	0.0105
$\tilde{m}^w_{2,N}(\cdot)$	sm(RASE)	0.1594	0.1532	0.1525	0.1484	0.1486	0.1444
	std(RASE)	0.0136	0.0102	0.0097	0.0077	0.0077	0.0061
$\widehat{m}^*_{1,N}(\cdot)$	sm(RASE)	0.1909	0.1853	0.1460	0.1444	0.1311	0.1297
	std(RASE)	0.0422	0.0397	0.0248	0.0250	0.0187	0.0188
$\widehat{m}^*_{2,N}(\cdot)$	sm(RASE)	0.1933	0.1841	0.1632	0.1647	0.1551	0.1547
	std(RASE)	0.0297	0.0244	0.0163	0.0163	0.0116	0.0112

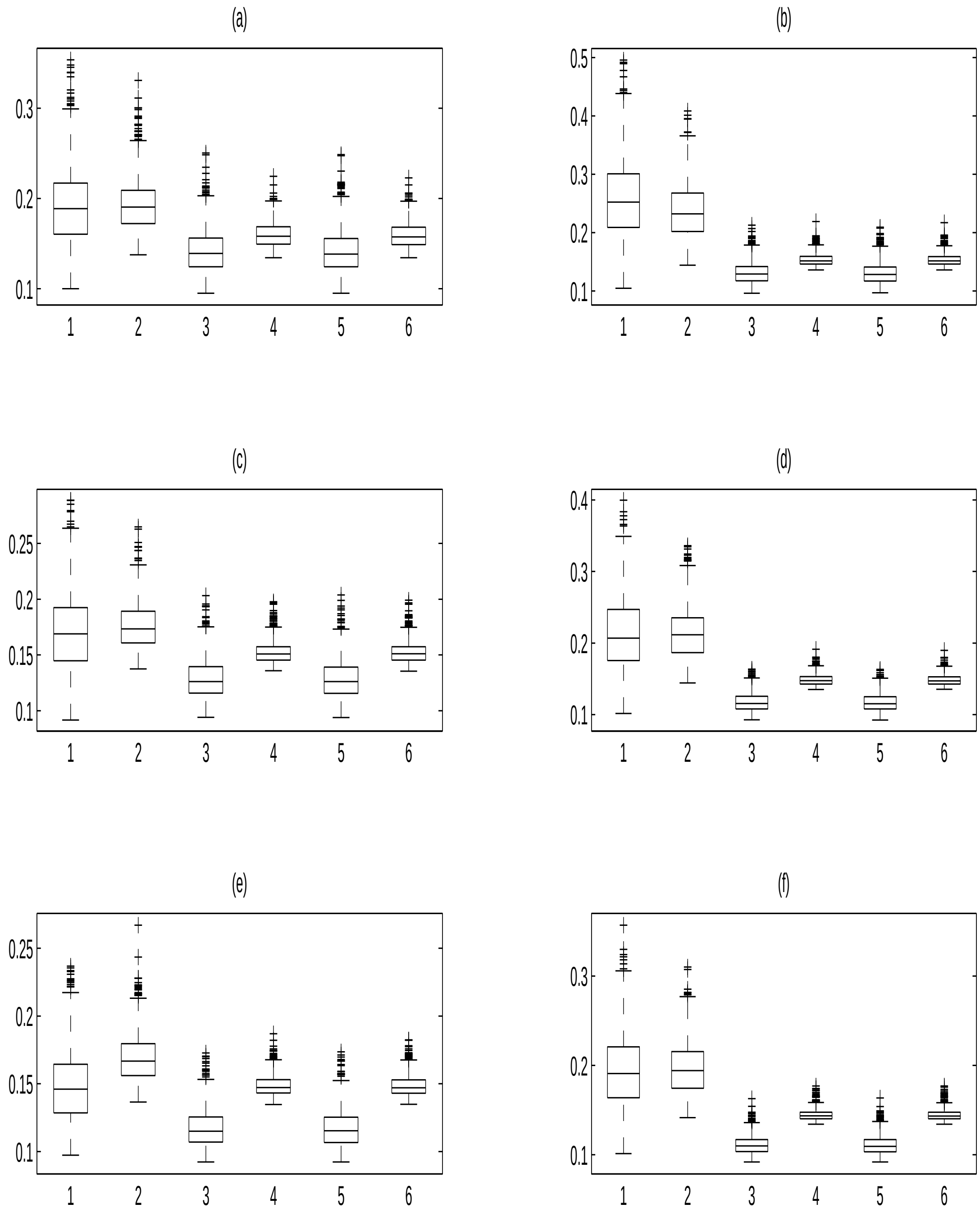

Figure 1: Box plots of the RASE values for the three nonparametric function estimators $(\widehat{m}_{1,N}(u), \widehat{m}_{2,N}(u))^{\tau}$, $(\widehat{m}^{w}_{1,N}(u), \widehat{m}^{w}_{2,N}(u))^{\tau}$ and $(\tilde{m}^{w}_{1,N}(u), \tilde{m}^{w}_{2,N}(u))^{\tau}$ with $T = 10$. Each boxplot is based on the 1,000 RASE values for a particular combination. Indices 1, 2, 3, 4, 5 and 6 are for $\widehat{m}_{1,N}(u), \widehat{m}_{2,N}(u), \widehat{m}^{w}_{1,N}(u), \widehat{m}^{w}_{2,N}(u), \tilde{m}^{w}_{1,N}(u)$ and $\tilde{m}^{w}_{2,N}(u)$, respectively. $N = 100, (\lambda, \rho) = (0.3, 0.3)$ in plot (a); $N = 100, (\lambda, \rho) = (0.6, 0.6)$ in plot (b); $N = 200, (\lambda, \rho) = (0.3, 0.3)$ in plot (c); $N = 200, (\lambda, \rho) = (0.6, 0.6)$ in plot (d); $N = 300, (\lambda, \rho) = (0.3, 0.3)$ in plot (e); And $N = 300, (\lambda, \rho) = (0.6, 0.6)$ in plot (f).