SUPPLEMENTARY MATERIAL FOR THE PAPER "EXACT MODERATE AND LARGE DEVIATIONS FOR LINEAR PROCESSES"

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This supplementary material contains proofs of the theorems of the main paper "Exact Moderate and Large Deviations for Linear Processes". We refer the main paper for references and equation numbers.

4 Proofs

4.1 Preliminary approximations

Let $(X_i)_{1\leq i\leq n}$ be independent random variables. We shall approximate the tail distribution of partial sums by the tail of the sums of truncated random variables and a term involving the tail probabilities of individual summands. We use the following notations:

$$S_n=\sum_{i=1}^{n}X_i,\ S(j)=\sum_{i\neq j}^{n}X_i$$

and for $x>0$ and $\varepsilon>0$ we set

$$X_i^{(\varepsilon x)}=X_iI(X_i<\varepsilon x),\ S_n^{(\varepsilon x)}=\sum_{i=1}^{n}X_i^{(\varepsilon x)}\text{ and }S_n^{(\varepsilon x)}(j)=\sum_{i\neq j}^{n}X_i^{(\varepsilon x)}. \tag{26}$$

We shall prove the following key lemma that will be further exploited to approximate the tail distribution of $\mathbb{P}(S_n\geq x)$ in terms of the sum of the truncated random variables and the tail distributions of the individual summands.

Lemma 4.1 *For any $0<\eta<1$, and $\varepsilon>0$ such that $1-\eta>\varepsilon$ we have*

$$|\mathbb{P}(S_n\geq x)-\mathbb{P}(S_n^{(\varepsilon x)}\geq x)-\sum_{j=1}^{n}\mathbb{P}(X_j\geq(1-\eta)x)|\leq$$
$$4(\sum_{j=1}^{n}\mathbb{P}(X_j\geq\varepsilon x))^2+3\sum_{j=1}^{n}\mathbb{P}(X_j\geq\varepsilon x)(\mathbb{P}(|S_n(j)|>\eta x)$$
$$+\sum_{j=1}^{n}\mathbb{P}((1-\eta)x\leq X_j<(1+\eta)x).$$

Proof. We decompose the event $\{S_n\geq x\}$ according to $\max_{i\neq j}X_i<\varepsilon x$ or $\max_{i\neq j}X_i\geq\varepsilon x$, and the last one can happen if exactly one of the variables is larger than εx or at least two variables exceed εx. Formally,

$$\mathbb{P}(S_n \geq x) = \sum_{j=1}^{n} \mathbb{P}(S_n \geq x,\ X_j \geq \varepsilon x,\ \max_{i \neq j} X_i < \varepsilon x)$$
$$+\mathbb{P}(\bigcup_{1 \leq i \leq n-1} \bigcup_{i+1 \leq j \leq n} \{S_n \geq x,\ X_j \geq \varepsilon x,\ X_i \geq \varepsilon x\})$$
$$+\mathbb{P}(S_n \geq x,\ \max_{1 \leq i \leq n} X_i < \varepsilon x) = A + B + C = \sum_{j=1}^{n} A_j + B + C.$$

The term B can be easily majorated by

$$B \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{P}(X_j \geq \varepsilon x)\mathbb{P}(X_i \geq \varepsilon x) \leq (\sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x))^2.$$

We analyze now the first term. We introduce a new parameter $\eta > 0$. Since for any two events A and B we have $|P(A) - P(B)| \leq P(AB') + P(A'B)$, (here the prime stays for the complement), for each j we have

$$|A_j - \mathbb{P}(X_j \geq (1-\eta)x)| \leq \mathbb{P}(S_n \geq x,\ X_j \geq \varepsilon x,\ X_j < (1-\eta)x)$$
$$+\mathbb{P}(X_j \geq (1-\eta)x,\ S_n < x) + \mathbb{P}(X_j \geq (1-\eta)x,\ X_j < \varepsilon x)$$
$$+\mathbb{P}(X_j \geq (1-\eta)x,\ \max_{i \neq j} X_i \geq \varepsilon x) = I + II + III + IV.$$

We treat each term separately. By independence and since $S_n \geq x$ and $X_j < (1-\eta)x$ imply $S_n(j) \geq \eta x$, we derive

$$I \leq \mathbb{P}(X_j \geq \varepsilon x)\mathbb{P}(S_n(j) \geq \eta x).$$

The second term is treated in the following way:

$$II \leq \mathbb{P}((1-\eta)x \leq X_j < (1+\eta)x) + \mathbb{P}(X_j \geq (1+\eta)x,\ S_n < x)$$
$$\leq \mathbb{P}((1-\eta)x \leq X_j < (1+\eta)x) + \mathbb{P}(X_j \geq (1+\eta)x)\mathbb{P}(-S_n(j) \geq \eta x).$$

Since $1 - \eta > \varepsilon$ the third term is: $III = 0$. By independence, the forth term is

$$IV = \mathbb{P}(X_j \geq (1-\eta)x)\mathbb{P}(\max_{i \neq j} X_i \geq \varepsilon x).$$

Overall, by the previous estimates and because $1 - \eta > \varepsilon$, we obtain

$$|A - \sum_{j=1}^{n} \mathbb{P}(X_j \geq (1-\eta)x)| \leq 2 \sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x)(\mathbb{P}(|S_n(j)| > \eta x)$$
$$+(\sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x))^2 + \sum_{j=1}^{n} \mathbb{P}((1-\eta)x \leq X_j < (1+\eta)x).$$

It remains to analyze the last term, C. Notice that

$$|C - \mathbb{P}(S_n^{(\varepsilon x)} \geq x)| = \mathbb{P}(S_n^{(\varepsilon x)} \geq x) - \mathbb{P}(S_n^{(\varepsilon x)} \geq x, \ \max_{1 \leq i \leq n} X_i < \varepsilon x)$$
$$= \mathbb{P}(S_n^{(\varepsilon x)} \geq x, \max_{1 \leq i \leq n} X_i \geq \varepsilon x).$$

Now we treat this term by the same arguments we have already used, by dividing the maximum in two parts:

$$\mathbb{P}(S_n^{(\varepsilon x)} \geq x, \max_{1 \leq i \leq n} X_i \geq \varepsilon x) = \sum_{j=1}^{n} \mathbb{P}(S_n^{(\varepsilon x)} \geq x, \ X_j \geq \varepsilon x, \ \max_{i \neq j} X_i < \varepsilon x)$$
$$+\mathbb{P}(\bigcup_{1 \leq i \leq n-1} \bigcup_{i+1 \leq j \leq n} \{S_n^{(\varepsilon x)} \geq x, \ X_j \geq \varepsilon x, \ X_{ni} \geq \varepsilon x\}) = \sum_{j=1}^{n} F_j + G.$$

The last term, G is majorated exactly as B. As for the first term, we notice that because $X_j \geq \varepsilon x$ the term $X_j^{(\varepsilon x)}$ does not appear in the sum, and by independence we obtain

$$F_j = \mathbb{P}(S_n^{(\varepsilon x)}(j) \geq x, \ X_j \geq \varepsilon x, \ \max_{i \neq j} X_i < \varepsilon x)$$
$$\leq \mathbb{P}(S_n^{(\varepsilon x)}(j) \geq x)\mathbb{P}(X_j \geq \varepsilon x).$$

Now, clearly we have

$$\mathbb{P}(S_n^{(\varepsilon x)}(j) \geq x) \leq \mathbb{P}(\max_i X_i \geq \varepsilon x) + \mathbb{P}(S_n^{(\varepsilon x)}(j) \geq x, \ \max_i X_i < \varepsilon x)$$
$$= \mathbb{P}(\max_i X_i \geq \varepsilon x) + \mathbb{P}(S_n(j) \geq x, \ \max_i X_i < \varepsilon x),$$

implying that

$$\sum_{j=1}^{n} F_j \leq \sum_{j=1}^{n} \mathbb{P}(X_{nj} \geq \varepsilon x)(\mathbb{P}(\max_i X_i \geq \varepsilon x) + \mathbb{P}(S_n(j) \geq x)).$$

Overall,

$$|C - \mathbb{P}(S_n^{(\varepsilon x)} \geq x)| \leq 2(\sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x))^2 + \sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x)\mathbb{P}(S_n(j) \geq x).$$

By gathering all the information above and taking into account that

$$|\mathbb{P}(S_n \geq x) - \mathbb{P}(S_n^{(\varepsilon x)} \geq x) - \sum_{j=1}^{n} \mathbb{P}(X_j \geq (1-\eta)x)| \leq$$
$$|A - \sum_{j=1}^{n} \mathbb{P}(X_j \geq (1-\eta)x)| + |C - \mathbb{P}(S_n^{(\varepsilon x)} \geq x)| + |B|,$$

the lemma is established. $\diamondsuit$

The following similar lemma is for the sum of infinite many terms.

Lemma 4.2 *Let $1-\eta > \varepsilon > 0$ and $x > 0$; let $X_1, X_2, \cdots,$ be independent random variables. Assume that the sum $S = \sum_{i=1}^{\infty} X_i$ exists almost surely. Let $S_{(j)} = S - X_j,\ X_i^{(\varepsilon x)} = X_i I(X_i < \varepsilon x)$. Then $S^{(\varepsilon x)} = \sum_{i=1}^{\infty} X_i^{(\varepsilon x)}$ exists almost surely and*

$$|\mathbb{P}(S \geq x) - \mathbb{P}(S^{(\varepsilon x)} \geq x) - \sum_{j=1}^{\infty} \mathbb{P}(X_j \geq (1-\eta)x)| \leq$$

$$4(\sum_{j=1}^{\infty} \mathbb{P}(X_j \geq \varepsilon x))^2 + 3\sum_{j=1}^{\infty} \mathbb{P}(X_j \geq \varepsilon x)(\mathbb{P}(|S(j)| > \eta x)$$

$$+\sum_{j=1}^{\infty} \mathbb{P}((1-\eta)x \leq X_j < (1+\eta)x).$$

Proof. By Kolmogorov's three-series theorem, $S^{(\varepsilon x)} = \sum_{i=1}^{\infty} X_i^{(\varepsilon x)}$ converges almost surely. Let $\Omega_0 \in \Omega$ with $\mathbb{P}(\Omega_0) = 1$ be the set that both $\sum_{i=1}^{\infty} X_i$ and $\sum_{i=1}^{\infty} X_i^{(\varepsilon x)}$ converge. Hence on Ω_0, we understand $S(\omega)$ as just the sum $\sum_{i=1}^{\infty} X_i(\omega)$. Then following the proof of Lemma 4.1, we have Lemma 4.2. $\diamondsuit$

If S_n is stochastically bounded, i.e., $\lim_{K\to\infty} \sup_n \mathbb{P}(|S_n| > K) = 0$, the approximation in Lemma 4.1 has a simple asymptotic form.

Proposition 4.1 *Assume that S_n is stochastically bounded, the variables are centered and $x_n \to \infty$. Then for any $0 < \eta < 1$, and $\varepsilon > 0$ such that $1-\eta > \varepsilon$, we have*

$$|\mathbb{P}(S_n \geq x_n) - \mathbb{P}(S_n^{(\varepsilon x_n)} \geq x_n) - \sum_{j=1}^{n} \mathbb{P}(X_j \geq (1-\eta)x_n)| \leq \tag{27}$$

$$o(1)\sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x_n) + \sum_{j=1}^{n} \mathbb{P}((1-\eta)x_n \leq X_j < (1+\eta)x_n),$$

where $o(1)$ depends on the sequence x_n, η and ε and converges to 0 as $n \to \infty$.

Proof. We just notice that for independent centered random variables, if S_n is stochastically bounded, by Lévy inequality (Inequality 1.1.3 in de la Peña and Giné 1999), we have $\max_{1\leq i\leq n} |X_i|$ is stochastically bounded too. By taking into account that $|S_n(j)| \leq |S_n| + \max_{1\leq i\leq n} |X_i|$, and using the fact that $x_n \to \infty$ as $n \to \infty$ we obtain

$$\sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x_n)\mathbb{P}(|S_n(j)| \geq \eta x_n) \leq \max_{1\leq j\leq n} \mathbb{P}(|S_n(j)| \geq \eta x_n) \sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x_n)$$

$$\leq \left(\mathbb{P}(|S_n| \geq \eta x_n/2) + \mathbb{P}(\max_{1\leq i\leq n} |X_i| \geq \eta x_n/2)\right) \sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x_n)$$

$$= o(1) \sum_{j=1}^{n} \mathbb{P}(X_j \geq \varepsilon x_n) \text{ as } n \to \infty.$$

Then, by independence

$$
\begin{aligned}
&\mathbb{P}(\max_{1\le j\le n}|X_j|\ge \varepsilon x_n)\\
&\qquad = \mathbb{P}(|X_1|\ge \varepsilon x_n)+\sum_{k=2}^{n}\mathbb{P}(\max_{1\le j\le k-1}|X_j|<\varepsilon x_n)\mathbb{P}(|X_k|\ge \varepsilon x_n)\\
&\qquad \ge \mathbb{P}(\max_{1\le j\le n}|X_j|<\varepsilon x_n)\sum_{k=1}^{n}\mathbb{P}(|X_j|\ge \varepsilon x_n),
\end{aligned}
$$

which gives

$$
\begin{aligned}
(\sum_{j=1}^{n}\mathbb{P}(|X_j|\ge \varepsilon x_n))^2 &\le \frac{\mathbb{P}(\max_{1\le j\le n}|X_j|\ge \varepsilon x_n)}{\mathbb{P}(\max_{1\le j\le n}|X_j|<\varepsilon x_n)}\sum_{j=1}^{n}\mathbb{P}(|X_j|\ge \varepsilon x_n)\\
&= o(1)\sum_{j=1}^{n}\mathbb{P}(|X_j|\ge \varepsilon x_n) \text{ as } n\to\infty,
\end{aligned}
$$

since $x_n\to\infty$ as $n\to\infty$ and $\max_{1\le j\le n}|X_j|$ is stochastically bounded. $\diamondsuit$

Remark 4.1 *Based on Lemma 4.2, it is easy to verify that Proposition 4.1 is still valid if we extend the sums up to infinity.*

4.2 Proof of Theorem 2.2

It is convenient to normalize by the variance of partial sum and we shall consider without restricting the generality that

$$
\mathbb{E}\xi_0^2=1,\ \sum_{i=1}^{k_n}c_{ni}^2=1 \text{ and } \max_{1\le i\le k_n}c_{ni}^2\to 0. \tag{28}
$$

Then we have $\sum_{i=1}^{k_n}c_{ni}^t\le \max_{1\le i\le k_n}c_{ni}^{t-2}\to 0$ implying that $D_{nt}^{-1}\to\infty$. Moreover, the sequence $\sum_{i=1}^{k_n}c_{ni}\xi_i$ is stochastically bounded and we analyze the two terms of the right side and the last term of the left side in Proposition 4.1. Let $x_n\to\infty$ as $n\to\infty$. In order to ease the notation we shall denote $x=x_n$, but we keep in mind that x depends on n and tends to infinite with n. By taking into account that $x/c_{ni}\ge x\to\infty$ and h is a slowly varying function we notice first that for any $a>0$

$$
\lim_{x\to\infty}\max_{1\le i\le k_n}|\frac{h(ax/c_{ni})}{h(x/c_{ni})}-1|=0.
$$

We derive for any $|\gamma|<1$ fixed

$$
\begin{aligned}
&|\sum_{i=1}^{k_n}c_{ni}^t(h(\frac{x}{c_{ni}})-h((1+\gamma)\frac{x}{c_{ni}}))|\le\\
&\sum_{i=1}^{k_n}c_{ni}^t h(\frac{x}{c_{ni}})|1-\frac{h((1+\gamma)x/c_{ni})}{h(x/c_{ni})}|=o(1)\sum_{i=1}^{k_n}c_{ni}^t h(\frac{x}{c_{ni}}),\ \text{ as } n\to\infty,
\end{aligned}
$$

implying that

$$\frac{\sum_{i=1}^{k_n}\mathbb{P}(c_{ni}\xi_i \geq (1\pm\eta)x)}{\sum_{i=1}^{k_n}\mathbb{P}(c_{ni}\xi_i \geq x)} = \frac{\sum_{i=1}^{k_n} c_{ni}^t h((1\pm\eta)x/c_{ni})}{(1\pm\eta)^t \sum_{i=1}^{k_n} c_{ni}^t h(x/c_{ni})} \to 1$$
$$\text{when } n\to\infty \text{ followed by } \eta\to 0.$$

Then, we also have

$$\frac{\sum_{i=1}^{k_n}\mathbb{P}((1-\eta)x \leq c_{ni}\xi_i < (1+\eta)x)}{\sum_{i=1}^{k_n}\mathbb{P}(c_{ni}\xi_i \geq x)} \to 0 \text{ as } n\to\infty \text{ and } \eta\to 0.$$

Similarly, for every $\varepsilon > 0$ fixed we have that

$$\frac{\sum_{i=1}^{k_n}\mathbb{P}(c_{ni}\xi_i \geq \varepsilon x)}{\sum_{i=1}^{k_n}\mathbb{P}(c_{ni}\xi_i \geq x)} = \frac{\sum_{i=1}^{k_n} c_{ni}^t h(\varepsilon x/c_{ni})}{\varepsilon^t \sum_{i=1}^{k_n} c_{ni}^t h(x/c_{ni})} \to \frac{1}{\varepsilon^t} \text{ as } n\to\infty,$$

and then,

$$\sum_{i=1}^{k_n}\mathbb{P}(c_{ni}\xi_i \geq \varepsilon x) \ll \sum_{i=1}^{k_n}\mathbb{P}(c_{ni}\xi_i \geq x) \text{ as } n\to\infty.$$

So far, for any $\varepsilon > 0$ fixed, by letting $n\to\infty$ first and after that, passing with η to 0, we deduce by the above consideration combined with Proposition 4.1 that

$$\mathbb{P}(S_n \geq x) = \sum_{i=1}^{k_n}\mathbb{P}(c_{ni}\xi_i \geq x)(1+o(1)) + \mathbb{P}(S_n^{(\varepsilon x)} \geq x) \quad \text{as } n\to\infty. \tag{29}$$

It remains to study the term $\mathbb{P}(S_n^{(\varepsilon x)} \geq x)$. We shall base this part of the proof on Corollary 1.7 in S. Nagaev (1979), given in the Appendix, which we apply with $m > t$, that will be selected later. Because we assume $\mathbb{E}(\xi_0^2) = 1$ and $\sum_{i=1}^{k_n} c_{ni}^2 = 1$, we have for all y, $B_n^2(-\infty, y) \leq 1$, and therefore, Theorem 5.1 implies:

$$\mathbb{P}(S_n^{(\varepsilon x)} \geq x) \leq \exp(-\alpha^2 x^2/2e^m) + (A_n(m;0,\varepsilon x)/(\beta\varepsilon^{m-1}x^m))^{\beta/\varepsilon}\ .$$

with $\alpha = 1-\beta = 2/(m+2)$. Then, obviously, it is enough to show that for $x = x_n$ as in Theorem 2.2 we can select $\varepsilon > 0$ such that

$$\exp\left(-\frac{\alpha^2 x^2}{2e^m}\right) + \left(\frac{A_n(m;0,\varepsilon x)}{\beta\varepsilon^{m-1}x^m}\right)^{\beta/\varepsilon} = o(1)\sum_{i=1}^{k_n}\frac{c_{ni}^t}{x^t}h\left(\frac{x}{c_{ni}}\right) \text{ as } n\to\infty. \tag{30}$$

Let $x = x_n \geq C[\ln(D_{nt}^{-1})]^{1/2}$ where $C > e^{m/2}(m+2)/\sqrt{2}$. As we mentioned at the beginning of the proof, we clearly have $x_n \to \infty$.

We shall estimate each term in the left hand side of (30) separately. Because, by the definition of α we have $C > e^{m/2}\alpha^{-1}\sqrt{2}$, we can select $0 < \eta < 1$ such that $C^2\alpha^2/2e^m = (1-\eta)^{-2}$.

Taking into account the fact that for any $c > 0$ and $d > 0$ we have $y^d \exp(-cy) = o(\exp(-c(1-\eta)y)$ as $y \to \infty$, by the definition on x and η, we obtain:

$$x^{(t-2\eta)/(1-\eta)} \exp(-\frac{\alpha^2 x^2}{2e^m}) = o(1) \exp(-\frac{\alpha^2 x^2}{2e^m}(1-\eta))$$
$$= o(1)(\sum_{i=1}^{k_n} c_{ni}^t)^{C^2\alpha^2(1-\eta)/2e^m} = o(1)(\sum_{i=1}^{k_n} c_{ni}^t)^{(1-\eta)^{-1}}.$$

Applying now the Hölder inequality we clearly have,

$$\sum_{i=1}^{k_n} c_{ni}^t = \sum_{i=1}^{k_n} c_{ni}^{2\eta} c_{ni}^{t-2\eta} \leq (\sum_{i=1}^{k_n} c_{ni}^2)^\eta (\sum_{i=1}^{k_n} c_{ni}^{(t-2\eta)/(1-\eta)})^{1-\eta}. \tag{31}$$

Taking into account that $\sum_{i=1}^{k_n} c_{ni}^2 = 1$, we obtain overall

$$\exp(-\frac{\alpha^2 x^2}{2e^m}) = o(1) x^{-(t-2\eta)/(1-\eta)} \sum_{i=1}^{k_n} c_{ni}^{(t-2\eta)/(1-\eta)}.$$

Since $t > 2$, $(t-2\eta)/(1-\eta) > t$. Then, by combining this observation with the properties of slowly varying functions we have

$$\exp(-\frac{\alpha^2 x^2}{2e^m}) = o(1) \sum_{i=1}^{k_n} \frac{c_{ni}^t}{x^t} h(\frac{x}{c_{ni}}).$$

We select ε by analyzing the second term in the left hand side of (30). Notice that by integration by parts formula, for every $z > y > 0$,

$$\mathbb{E}\xi_0^m I(0 \leq \xi_0 < z) =$$
$$-z^m \mathbb{P}(\xi_0 \geq z) + m \int_0^z u^{m-1} \mathbb{P}(\xi_0 \geq u) du \leq y^m + m \int_y^z u^{m-1} \mathbb{P}(\xi_0 \geq u) du.$$

Replacing $z = \varepsilon x / c_{ni}$, taking into account condition (4), the properties of slowly varying functions, and the facts that $x/c_{ni} \to \infty$ and $m > t$, we have

$$\mathbb{E}\xi_0^m I(0 \leq c_{ni}\xi_0 < \varepsilon x) \leq y^m + 2m \int_y^{\frac{\varepsilon x}{c_{ni}}} u^{m-t-1} h(u) du = O((\frac{x}{c_{ni}})^{m-t} h(\frac{x}{c_{ni}}))$$

for y sufficiently large. It follows that

$$A_n(m; 0, \varepsilon x) = \sum_{i=1}^{k_n} c_{ni}^m \mathbb{E}\xi_0^m I(0 \leq c_{ni}\xi_0 < \varepsilon x)$$
$$\ll \sum_{i=1}^{k_n} c_{ni}^m (\frac{x}{c_{ni}})^{m-t} h(\frac{x}{c_{ni}}) = x^{m-t} \sum_{i=1}^{k_n} c_{ni}^t h(\frac{x}{c_{ni}}).$$

Choose ε with $0 < \varepsilon < \beta$. Then the second term has the order

$$\left(\frac{A_n(m;0,\varepsilon x)}{\beta\varepsilon^{m-1}x^m}\right)^{\beta/\varepsilon} \ll \left(\frac{x^{m-t}}{x^m}\sum_{i=1}^{k_n} c_{ni}^t h(\frac{x}{c_{ni}})\right)^{\beta/\varepsilon} = o\left(\sum_{i=1}^{k_n} \frac{c_{ni}^t}{x^t} h(\frac{x}{c_{ni}})\right).$$

Overall we obtain for any $x \geq C(\ln(\sum_{i=1}^{k_n} c_{ni}^t)^{-1})^{1/2}$ with $C > e^{m/2}(m+2)/\sqrt{2}$,

$$\mathbb{P}(S_n \geq x) = (1+o(1))\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_0 \geq x) \text{ as } n \to \infty,$$

where $m > t$. Since $C_t > e^{t/2}(t+2)/\sqrt{2}$ we can select and fix $m > t$ such that $C_t > e^{m/2}(m+2)/\sqrt{2}$. $\Diamond$

4.3 Proof of Theorem 2.3

For simplicity we normalize by the variance of S_n and assume (28). This result easily follows from Theorem 1.1 in Frolov (2005) when moments strictly larger than 2 are available. This theorem is given for convenience in the Appendix (Theorem 5.2). Because we assume the existence of moments of order $p > 2$, we have

$$\Lambda_n(u,s,\epsilon) \leq u\sum_{j=1}^{k_n} c_{nj}^2 \mathbb{E}\xi_0^2 I(|c_{nj}\xi_0| > \epsilon/s) \leq \epsilon^{2-p}us^{p-2}D_{np}\mathbb{E}|\xi_0|^p.$$

where $D_{np} = \sum_{j=1}^{k_n}|c_{nj}|^p$. Then, for $x^2 \leq 2\ln(1/D_{np})$,

$$\Lambda_n(x^4,x^5,\epsilon) \leq \epsilon^{2-p}x^{4+5(p-2)}D_{np}\mathbb{E}|\xi_0|^p \leq \epsilon^{2-p}D_{np}(2\ln(1/D_{np}))^{(5p-6)/2}\mathbb{E}|\xi_0|^p,$$

which converges to 0 since $D_{np} \leq \max_{1\leq j\leq k_n}|c_{nj}|^{p-2} \to 0$ by (10). Notice also that the L_{np} in Theorem 5.2 satisfies $L_{np} \leq D_{np}\mathbb{E}|\xi_0|^p \to 0$. The latter implies $x^2 - 2\ln(L_{np}^{-1}) - (p-1)\ln\ln(L_{np}^{-1}) \to -\infty$ provided $x^2 \leq 2\ln(D_{np}^{-1})$. Then the result is immediate from Theorem 5.2. $\Diamond$

4.4 Proof of Theorem 2.1

Again for simplicity we normalize by the variance and assume (28). Without loss of generality we may assume $2 < p < t$. This is so because if $p \geq t$ with $\mathbb{E}(|\xi_0|^p) < \infty$ then we can find a p' such that $2 < p' < t$ and $\mathbb{E}(|\xi_0'|^p) < \infty$. We shall consider a sequence x_n which converges to ∞. So, let $x = x_n \to \infty$.

Starting from the relation (29) and applying Proposition 5.1 to the second term in the right hand side we obtain for any $\varepsilon > 0$ and $x^2 \leq c_\varepsilon \ln(D_{np}^{-1})$ with $c_\varepsilon < 1/\varepsilon$ and for all n sufficiently large $\mathbb{P}(S_n^{(\varepsilon x)} \geq x) = (1-\Phi(x))(1+o(1))$. We notice now that by (31) applied with $\eta = (t-p)/(t-2)$ and simple considerations,

$$D_{nt} \ll D_{np} \ll (D_{nt})^{(p-2)/(t-2)}. \qquad (32)$$

So far, by using this last relation, we showed by (29) and the above considerations that (11) holds for $0 < x \le C[\ln(D_{nt}^{-1})]^{1/2}$ with C an arbitrary positive number. On the other hand, because $1 - \Phi(x) \le (2\pi)^{-1/2} x^{-1} \exp(-x^2/2)$, by Theorem 2.2 and by the arguments leading to the proof of relation (30), there is a constant $c_1 > 0$ such that for $x > c_1 [\ln(D_{nt}^{-1})]^{1/2}$, we simultaneously have

$$\mathbb{P}\left(S_n \ge x\right) = (1 + o(1)) \sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_0 \ge x)$$

and

$$1 - \Phi(x) = o(\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_0 \ge x)).$$

Then (11) holds for all $x > 0$ since C is arbitrarily large and can be selected such that $c_1 < C$.

Now if the sequence x_n is bounded we apply first Theorem 2.3 and obtain the moderate deviation result in (13). Then, because $x_n \ge c > 0$ we notice that, by the arguments leading to the proof of relation (30), the second part in the right hand side of (11) is dominant, so the first part is negligible as $n \to \infty$. $\Diamond$

4.5 Proof of Corollary 2.1

Again without loss of generality we normalize by the variance and assume (28). The ideas involved in the proof of this corollary already appeared in the previous proofs, so we shall mention only the changes. We start from (11). To prove (12) we have to show that

$$1 - \Phi(x) = o(\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_0 \ge x))$$

for $x \ge a(\ln D_{nt}^{-1})^{1/2}$ with $a > 2^{1/2}$. First we shall use the relation $1 - \Phi(x) \le (2\pi)^{-1/2} x^{-1} \exp(-x^2/2)$. Then, we adapt the proof we used to establish the first part of (30), when we compared $\exp(-\alpha^2 x^2/2e^m)$ to $\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_0 \ge x)$. The main difference is that now we take $m = 0$ and $\alpha = 1$.

For the proof of (13), we use the inequality $1 - \Phi(x) \ge (2\pi)^{-1/2}(1 + x)^{-1} \exp(-x^2/2)$. By (4) and (32) we have for every $0 < \varepsilon < t - 2$,

$$\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_0 \ge x) \ll \sum_{i=1}^{k_n} \frac{c_{ni}^{t-\varepsilon}}{x^{t-\varepsilon}} \ll \frac{1}{x^{t-\varepsilon}} (D_{nt})^{(t-2-\varepsilon)/(t-2)}.$$

Then, it is easy to see that, because ε can be made arbitrarily small, for $1 < x \le b(\ln D_{nt}^{-1})^{1/2}$ with $b < 2^{1/2}$ we have

$$\sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_0 \ge x) = o(1 - \Phi(x)).$$

When $0 < x \le 1$ we apply Theorem 2.3. $\Diamond$

4.6 Proof of Corollary 2.3

As in the other proofs, for simplicity we assume $\mathbb{E}\xi_0^2 = 1$.
Proof of part (ii). Because the Fuk-Nagaev inequality (Theorem 5.1) and the inequalities in Lemma 4.1 and Proposition 4.1 are still valid for the case $k_n = \infty$ (see Remark 5.1 in the Appendix, Lemma 4.2 and Remark 4.1 in Subsection 4.1), all the arguments in the proof of Theorem 2.2 hold under the conditions of this corollary.

Proof of part (iii). The result (iii) in this corollary is obtained on the same lines as of Theorem 2.3. The modification of the proof is rather standard but computationally intensive. There are several ideas behind this proof. The infinite series is decomposed as a sum up to k_n and the rest R_n. The sequence k_n is selected independently of x_n such that the rest of the series R_n is negligible for the moderate deviation result. This is possible because the coefficients b_{ni}, defined as $b_{ni} = a_{1-i} + ... + a_{n-i}$ with $\sum_{i\in\mathbb{Z}} a_i^2 < \infty$, have some regularity properties. For instance by the Hölder's inequality,

$$b_{ni}^2 \leq n(a_{1-i}^2 + ... + a_{n-i}^2)$$

and so, for any $k > n$

$$\sum\nolimits_{|i|\geq k} b_{ni}^2 \leq n^2 \sum\nolimits_{|i|\geq k-n-1} a_{1-i}^2. \tag{33}$$

We then note that the existence of moments of order $p > 2$ for ξ_0 and $\sum_{i\in\mathbb{Z}} a_i^2 < \infty$ imply that X_0 also has finite moments of order p. Indeed, by Rosenthal inequality (see for instance Theorem 1.5.13 in de la Peña and Giné, 1999), there is a constant C_p such that

$$\mathbb{E}|\sum_{j=n}^{m} a_j\xi_j|^p \leq C_p[(\sum_{j=n}^{m} a_j^2)^{p/2} + \mathbb{E}|\xi_0|^p \sum_{j=n}^{m} |a_j|^p]$$

which implies that $\mathbb{E}|\sum_{j=n}^{m} a_j\xi_j|^p \to 0$ as $m \geq n \to \infty$, and therefore X_0 exists in $\mathbb{L}_p$.

For k_n a sequence of integers, denote $R_n = \sum_{|i|>k_n} b_{ni}\xi_i$ and note that R_n is also well defined in $\mathbb{L}_p$. Again by Rosenthal inequality we obtain

$$\mathbb{E}|R_n|^p \leq C_p[(\sum_{|i|>k_n} b_{ni}^2)^{p/2} + \mathbb{E}|\xi_0|^p \sum_{|i|>k_n} |b_{ni}|^p]. \tag{34}$$

We select now k_n large enough such that

$$\sum_{|i|>k_n} b_{ni}^2 \leq ||\xi_0||_p^2 (\sum_j |b_{nj}|^p)^{2/p}.$$

This is possible by relation (33) and the fact that $\sum_i a_i^2 < \infty$. With this selection we obtain

$$\mathbb{E}|R_n|^p \leq 2C_p\mathbb{E}|\xi_0|^p \sum_i |b_{ni}|^p. \tag{35}$$

Write now

$$S_n = \sum_{|i|\le k_n} b_{ni}\xi_i + R_n.$$

We view S_n as the sum of $k_n + 1$ independent random variables and then apply Theorem 5.2 as in the proof of Theorem 2.3. By taking into account (35), the term L_{np} from Theorem 5.2 is

$$\begin{aligned} L_{np} &= \frac{1}{\sigma_n^p}[\sum_{|i|\le k_n} |b_{ni}|^p \mathbb{E}(\xi_0^p I(\xi_0 > 0) + \mathbb{E}(R_n^p I(R_n > 0)] \\ &\le \frac{2C_p + 1}{\sigma_n^p} \sum_i |b_{ni}|^p \mathbb{E}|\xi_0|^p = (2C_p + 1) U_{np} \mathbb{E}|\xi_0|^p = L'_{np}. \end{aligned}$$

Because we assume the existence of moments of order p, by (35) we have

$$\begin{aligned} \Lambda_n(u, s, \epsilon) &\le \frac{u}{\sigma_n^2}[\sum_{|j|\le k_n} b_{nj}^2 \mathbb{E}\xi_0^2 I(|b_{nj}\xi_0| > \epsilon\sigma_n/s) + \mathbb{E}R_n^2 I(|R_n| > \epsilon\sigma_n/s)] \\ &\le \frac{us^{p-2}}{\sigma_n^p \epsilon^{p-2}}[\sum_{|j|\le k_n} |b_{nj}|^p \mathbb{E}|\xi_0|^p + \mathbb{E}|R_n|^p] \le \frac{us^{p-2}}{\epsilon^{p-2}} L'_{np}. \end{aligned}$$

Therefore, for $x^2 \le 2\ln(1/L'_{np}) \le 2\ln(1/L_{np})$,

$$\Lambda_n(x^4, x^5, \epsilon) \le \epsilon^{2-p} x^{4+5(p-2)} L'_{np} \ \le \epsilon^{2-p}(2\ln(1/L'_{np}))^{(5p-6)/2} L'_{np}.$$

Finally note that by (20) we obtain

$$U_{np} \le \frac{\sup_j |b_{nj}|^{p-2}}{(\sum b_{nj}^2)^{(p-2)/2}} \to 0,$$

and consequently $L'_{np} \to 0$. Therefore, $\Lambda_n(x^4, x^5, \epsilon) \to 0$. Note also that the quantity L_{np} in Theorem 5.2 satisfies $L_{np} \le L'_{np} \to 0$. Therefore if $x^2 - 2\ln(L'_{np})^{-1} - (p-1)\ln\ln(L'_{np})^{-1} \to -\infty$ we have that $x^2 - 2\ln(L_{np}^{-1}) - (p-1)\ln\ln(L_{np}^{-1}) \to -\infty$ and the result holds for such a positive x.

It remains to show that $x^2 \le 2\ln(U_{np}^{-1})$ implies $x^2 - 2\ln(L'_{np})^{-1} - (p-1)\ln\ln(L'_{np})^{-1} \to -\infty$, which holds provided that

$$2\ln((U_{np}^{-1}) L'_{np}[\ln(L'_{np})^{-1}]^{(1-p)/2}) \to -\infty.$$

This last divergence is equivalent to

$$(U_{np}^{-1}) L'_{np}[\ln(L'_{np})^{-1}]^{(1-p)/2} \to 0.$$

Clearly, because $L'_{np} = (2C_p + 1)U_{np}\mathbb{E}|\xi_0|^p$ and the fact that we have shown that $L'_{np} \to 0$ the result follows.

Proof of part (i). The proof is similar to the proof of Theorem 2.1 and Corollary 2.1. We have only to show that Proposition 5.1 is still valid in this context if we let $k_n = \infty$. The proof is similar to the proof of (iii) but more involved, since the sequence of truncated variables is not centered. Denote

$$X'_{ni} = b_{ni}\xi_i I(b_{ni}\xi_i \le \varepsilon x \sigma_n) = b_{ni}\xi'_i.$$

For k_n a sequence of integers, denote $R'_n = \sum_{|i|>k_n} b_{ni}\xi'_i$ and note that R'_n is also well defined in $\mathbb{L}_p$. By Rosenthal inequality, after centering we obtain

$$\mathbb{E}|R'_n|^p \le C'_p[(\sum_{|i|>k_n} b_{ni}^2)^{p/2} + \mathbb{E}|\xi_0|^p \sum_{|i|>k_n} |b_{ni}|^p + |\mathbb{E}(R'_n)|^p].$$

Because $x \ge c > 0$ and the fact that $\mathbb{E}(X'_{ni}) = -\mathbb{E}(b_{ni}\xi_i I(b_{ni}\xi_i > \varepsilon x\sigma_n))$ we obtain

$$|\mathbb{E}(R'_n)| \le \frac{1}{\varepsilon x \sigma_n} \sum_{|i|>k_n} b_{ni}^2 \le \frac{1}{\varepsilon c \sigma_n} \sum_{|i|>k_n} b_{ni}^2.$$

We select now k_n, depending on c, ε and the distribution of ξ_0 and the coefficients (a_k), large enough such that

$$(\sum_{|i|>k_n} b_{ni}^2)^{p/2} + (\frac{1}{\varepsilon c \sigma_n} \sum_{|i|>k_n} b_{ni}^2)^p \le \mathbb{E}|\xi_0|^p \sum_i |b_{ni}|^p,$$

and so

$$\mathbb{E}|R'_n|^p \le 2C'_p \mathbb{E}|\xi_0|^p \sum_i |b_{ni}|^p.$$

Write now $S'_n = \sum_{|i|\le k_n} b_{ni}\xi'_i + R'_n$ and view S'_n as the sum of k_n+1 independent random variables and then apply Proposition 5.1. Similar computations as in the proof of the point (iii) show that L_{np} in Proposition 5.1 is bounded by

$$L_{np} \le \frac{2C'_p + 1}{\sigma_n^p} \mathbb{E}|\xi_0|^p \sum_i |b_{ni}|^p = (2C'_p + 1) U_{np} \mathbb{E}|\xi_0|^p.$$

Then, by Proposition 5.1 if $x^2 \le c\ln((2C'_p + 1)U_{np}\mathbb{E}|\xi_0|^p)^{-1}$ for $c < 1/\varepsilon$, we have $x^2 \le c\ln(L_{np}^{-1})$ for $c < 1/\varepsilon$ and

$$\mathbb{P}\left(\sum\nolimits_i X'_{nj} \ge x\sigma_n\right) = (1 - \Phi(x))(1 + o(1)).$$

It remains to notice that because $U_{np} \to 0$, we also have the result for $x^2 \le c\ln(U_{np})^{-1}$ for any $c < 1/\varepsilon$, for all n sufficiently large. $\Diamond$

4.7 Proof of Corollary 2.4

This Corollary follows from Corollary 2.3 via Lemma 5.1 in the Appendix. It remains to give an explicit form of the intervals moderate deviation and large deviation boundaries. Without loss of generality, we assume that $\mathbb{E}\xi_0^2 = 1$. For proving the large deviation part of this corollary we have to analyze the condition on x from part (i) of Corollary 2.3, namely $x > a(\ln U_{nt}^{-1})^{1/2}$ with $a = \sqrt{2}$. By Lemma 5.1

$$B_{n2} = \sum_i b_{ni}^2 \sim c_r n^{3-2r} l^2(n)$$

and

$$C_1 l^t(n) n^{(1-r)t+1} \leq \sum_{j=1}^{\infty} b_{nj}^t \leq C_2 l^t(n) n^{(1-r)t+1}.$$

Then, for certain constants K_1 and K_2 and because $U_{nt}^{-1} = B_{n2}^{t/2}/B_{nt}$, we have for n sufficiently large

$$K_1 + \ln n^{(t-2)/2} \leq \ln U_{nt}^{-1} \leq K_2 + \ln n^{(t-2)/2}.$$

So, the asymptotic result (12) holds for $x \geq c_1(\ln n)^{1/2}$ where $c_1 > (t-2)^{1/2}$. Furthermore, (13) holds for $0 < x \leq c_2(\ln n)^{1/2}$ where $c_2 < (t-2)^{1/2}$. $\Diamond$

4.8 Proof of Theorem 2.4

Without restricting the generality we assume $\kappa > 0$, since similar computations can be done when $\kappa < 0$. Let $A_n = \sum_{i=n}^{\infty} a_i^2$. Using the argument of Theorem 5 in Wu (2006), under Condition B, we have

$$\|\mathcal{P}_0(K(X_n) - \kappa X_n)\|_q = O(\theta_n), \text{ where } \theta_n = |a_n|^{p/q} + |a_n| A_n^{1/2}.$$

Let $\theta_i = 0$ if $i \leq 0$ and $\Theta_n = \sum_{i=1}^{n} \theta_i$. Then by Theorem 1 in Wu (2007), there exists a constant $B_q \geq 1$ such that

$$\frac{\|S_{n,1}\|_q^2}{B_q^2} \leq \sum_{i \in \mathbb{Z}} (\Theta_{n+i} - \Theta_i)^2 \leq 2n\Theta_{2n}^2 + \sum_{i=n+1}^{\infty} (\Theta_{n+i} - \Theta_i)^2. \tag{36}$$

By Karamata's theorem, $A_n \sim (2r-1)^{-1} n^{1-2r} l(n)^2$, and if $i > n$, $\Theta_{n+i} - \Theta_i = O(n\theta_i)$ and $\sum_{i=n+1}^{\infty} \theta_i^2 = O(n\theta_n^2)$. Let $\ell(\cdot)$ be a slowly varying function and $\beta \in \mathbb{R}$. Again by Karamata's theorem, there exists another slowly varying function $\ell_0(\cdot)$ such that $\sum_{i=1}^{n} i^{-\beta}\ell(i) = O(1 + n^{1-\beta})\ell_0(n)$. Hence by (36), there exists a slowly varying function $\ell_1(\cdot)$ such that

$$\|S_{n,1}\|_q = O(\sqrt{n})(1 + n^{1-rp/q} + n^{1-r+(1-2r)/2})\ell_1(n). \tag{37}$$

For $n \geq 3$ let $g_n = (\ln n)^{-1}$. Then

$$\mathbb{P}(S_n \geq (x + g_n)\sigma_n) - \mathbb{P}(H_n \geq \kappa x \sigma_n) \leq \mathbb{P}(|S_{n,1}| \geq \kappa g_n \sigma_n). \tag{38}$$

Since $x^2 \leq c \ln n$ and $g_n = (\ln n)^{-1}$, we have that $1 - \Phi(x \pm g_n) \sim 1 - \Phi(x)$. Hence by Corollary 2.5, (23) follows from (38) in view of

$$\begin{aligned}\mathbb{P}(|S_{n,1}| \geq \kappa g_n \sigma_n) &\leq \frac{\|S_{n,1}\|_q^q}{|\kappa|^q g_n^q \sigma_n^q} = \frac{O(\sqrt{n}^q)(1 + n^{q-rp} + n^{(3/2-2r)q})\ell_1^q(n)}{g_n^q (n^{3/2-r} l(n))^q} \qquad (39)\\ &= n^{-p\rho(r)} \frac{\ell_1^q(n)}{g_n^q l^q(n)} = \frac{o(n^{-c/2})}{\ln n} = o(xe^{-x/2}) = o[1 - \Phi(x)],\end{aligned}$$

since $c/2 < p\rho(r)$. Here we note that $\ell_1(n)/(g_n l(n))$ is also slowly varying in n and $x \le c \ln n$. By (37) and (39), it is easily seen that the normalizing constant $\kappa\sigma_n$ can be replaced by $\sqrt{var(H_n)}$. The proof of the upper bound is similar and it is left to the reader. $\Diamond$

5 Appendix

The following Theorem is a slight reformulation of Fuk–Nagaev inequality (see Corollary 1.7, S. Nagaev, 1979):

Theorem 5.1 *Let $X_1, \cdots, X_{k_n}$ be independent random variables. Assume $m \ge 2$. Suppose $\mathbb{E}X_i = 0$, $i = 1, \cdots, k_n$, $\beta = m/(m+2)$, and $\alpha = 1-\beta = 2/(m+2)$. For $y > 0$, define $X_i^{(y)} = X_i I(X_i \le y)$, $A_n(m; 0, y) := \sum_{i=1}^{k_n} \mathbb{E}[X_i^m I(0 < X_i < y)]$ and $B_n^2(-\infty, y) := \sum_{i=1}^{k_n} \mathbb{E}[X_i^2 I(X_i < y)]$. Then for any $x > 0$ and $y > 0$*

$$\mathbb{P}(\sum_{i=1}^{k_n} X_i^{(y)} \ge x) \le \exp(-\frac{\alpha^2 x^2}{2e^m B_n^2(-\infty, y)}) + (\frac{A_n(m; 0, y)}{\beta x y^{m-1}})^{\beta x/y}. \tag{40}$$

Remark 5.1 *Let $X_1, X_2, \cdots,$ be independent random variables. Assume that the sum $S = \sum_{i=1}^{\infty} X_i$ exists almost surely. By the same argument as in Lemma 4.2, $\sum_{i=1}^{\infty} X_i^{(y)}$ converges almost surely for all $y > 0$. By passing to the limit in (40) we note that this version of Fuk-Nagaev inequality is still valid for $\mathbb{P}(\sum_{i=1}^{\infty} X_i^{(y)} \ge x)$.*

We shall also use the following result which is an immediate consequence of Theorem 1.1 in Frolov (2005).

Theorem 5.2 *Let $(X_{nj})_{1\le j\le k_n}$ be an array of row-wise independent centered random variables. Let $p > 2$ and denote $S_n = \sum_{j=1}^{k_n} X_{nj}$, $\sigma_n^2 = \sum_{j=1}^{k_n} \mathbb{E}X_{nj}^2$, $M_{np} = \sum_{j=1}^{k_n} \mathbb{E}X_{nj}^p I(X_{nj} \ge 0) < \infty$, $L_{np} = \sigma_n^{-p} M_{np}$ and denote*

$$\Lambda_n(u, s, \epsilon) = \frac{u}{\sigma_n^2} \sum_{j=1}^{k_n} \mathbb{E}X_{nj}^2 I(X_{nj} \le -\epsilon\sigma_n/s).$$

Furthermore, assume $L_{np} \to 0$ and $\Lambda_n(x^4, x^5, \epsilon) \to 0$ for any $\epsilon > 0$. Then if $x \ge 0$ and $x^2 - 2\ln(L_{np}^{-1}) - (p-1)\ln\ln(L_{np}^{-1}) \to -\infty$, we have

$$\mathbb{P}\left(S_n \ge x\sigma_n\right) = (1 - \Phi(x))(1 + o(1)).$$

For truncated random variables by following the proof of Theorem 1.1 in Frolov (2005) we can present his relation (3.17) as a proposition.

Proposition 5.1 *Assume the conditions in Theorem 5.2 are satisfied. Fix $\varepsilon > 0$. Define*

$$X_{nj}^{(\varepsilon x\sigma_n)} = X_{nj} I(X_{nj} \le \varepsilon x \sigma_n) \text{ and } S_n^{(\varepsilon x\sigma_n)} = \sum_{j=1}^{k_n} X_{nj}^{(\varepsilon x\sigma_n)}.$$

Then if $x^2 \le c\ln(L_{np}^{-1})$ with $c < 1/\varepsilon$, for all n sufficiently large we have

$$\mathbb{P}\left(S_n^{(\varepsilon x\sigma_n)} \ge x\sigma_n\right) = (1-\Phi(x))(1+o(1)).$$

The following facts about the series are going to be used to analyze a class of linear processes:

Lemma 5.1 *Assume $a_i = l(i)i^{-r}$ with $1/2 < r < 1$. Let $b_j := b_{nj} := \sum_{i=1}^{j} a_i$ if $1 \le j \le n$ and $b_{nj} := \sum_{i=j-n+1}^{j} a_i$ if $j > n$. Then, for two positive constants C_1 and C_2, we have*

$$C_1(l^t(n)n^{(1-r)t+1}) \le \sum_{j=1}^{\infty} b_{nj}^t \le C_2(l^t(n)n^{(1-r)t+1}),$$

for any $t \ge 2$. In the case $t = 2$, $\sum_{j=1}^{\infty} b_{nj}^2 = c_r n^{3-2r} l^2(n)$ with

$$c_r = \{\int_0^{\infty} [x^{1-r} - \max(x-1,0)^{1-r}]^2 dx\}/(1-r)^2.$$

Proof. It is easy to see that $b_{nj} \ll j^{1-r}l(j)$ for $j \le 2n$ and $b_{nj} \ll n(j-n)^{-r}l(j)$ for $j > 2n$ from the Karamata theorem (see part 1 of Lemma 5.4 in Peligrad and Sang (2012)). Therefore,

$$\begin{aligned}\sum_{j=1}^{\infty} b_{nj}^t &= \sum_{j=1}^{2n} b_{nj}^t + \sum_{j=2n+1}^{\infty} b_{nj}^t \\ &\ll \sum_{j=1}^{2n} j^{(1-r)t} l^t(j) + \sum_{j=2n+1}^{\infty} n^t (j-n)^{-rt} l^t(j) = O(l^t(n)n^{(1-r)t+1}).\end{aligned}$$

The proof in the other direction is similar. The result of case $t = 2$ is well known. See for instance Theorem 2 in Wu and Min (2005). $\Diamond$