Statistica Sinica: Supplement

ON LEAST FAVORABLE CONFIGURATIONS FOR STEP-UP-DOWN TESTS

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Supplementary Material

This is the supplement of the paper Blanchard, Dickhaus, Roquain and Villers (2013).

S1 Additional results of interest

S1.1 FDR(SD) can exceed FDR(SU) in an extreme configuration

Lemma S1.1. *Consider the* $\mathrm{FM}(m, m_0, F)$ *model with* $F(x) = \mathbf{1}\{x \geq 1\}$ *(i.e., all the p-values under the alternative are constantly equal to* 1*). Consider the threshold collection* $\boldsymbol{t}$ *defined by* $t_k = t_0$, $1 \leq k \leq m-1$ *and* $t_m = 1$, *for some* $t_0 \in (0,1)$. *Then we have for any* $\lambda \in \{1, \ldots, m-1\}$,

$$FDR(SUD_\lambda(\boldsymbol{t})) = 1 - (1-t_0)^{m_0};$$
$$FDR(SU(\boldsymbol{t})) = m_0/m.$$

In particular, $FDR(SD(\boldsymbol{t})) > FDR(SU(\boldsymbol{t}))$ *for* $t_0 > 1 - (1 - m_0/m)^{1/m_0}$.

The proof is straightforward and is left to the reader. As an illustration, for $m = 10$ and $m_0 = 7$, $1 - (1 - m_0/m)^{1/m_0} \simeq 0.158$.

S1.2 DU is an LFC for the k-FWER

We state here for the sake of completeness a straightforward generalization of Lemma 1 of Finner and Gontscharuk (2009) (see also Lemma 2.2 of Gontscharuk (2010)) concerning the LFCs of multiple testing procedures under a class of type I criteria containing in particular the k-FWER (but not the FDR, as pointed out in the introduction of the main text). This result should be considered as already known by experts in the field, although we failed to locate a precise reference for it. The setting considered assumes independence of p-values corresponding to true nulls, but is more general than the fixed mixture model, since the p-values corresponding to true null hypotheses are only assumed to be stochastically lower bounded by a uniform variable on [0,1]; also, the p-values corresponding to alternatives are not assumed to be identically distributed nor independent.

Lemma S1.2. *Let* $m \geq 1$ *and* $m_0 \in \{0, \ldots, m\}$ *be fixed. Let* $\mathbf{p} = (p_1, \ldots, p_m)$ *be a family of p-values with distribution* P *such that* $(p_i)_{1 \leq i \leq m_0}$ *form an independent family of variables, each stochastically lower bounded by a uniform variable. Assume that* δ *is a multiple testing procedure rejecting all hypotheses having p-value less than a data-dependent threshold* $t^*(\mathbf{p})$ *and denote* $\delta(\mathbf{p})$ *the set of indices of rejected hypotheses. Let* R *be a type I error criterion taking the form*

$$R(P, \delta) = \mathbb{E}_{\mathbf{p} \sim P}[\phi(V_m(\delta(\mathbf{p})))],$$

where $V_m(\delta(\mathbf{p})) = |\delta(\mathbf{p}) \cap \{i \in \mathbb{N} : 1 \leq i \leq m_0\}|$ *, and* ϕ *is a function from* $\mathbb{N}$ *to* $\mathbb{R}$*.*

Assume the two following conditions are satisfied:

(i) t^* *is a nonincreasing function of each* p*-value;*
(ii) ϕ *is nondecreasing.*

Then it holds that

$$R(P, \delta) \leq R(\mathrm{DU}(m, m_0), \delta),$$

that is, $\mathrm{DU}(m, m_0)$ *is an LFC for* δ *among the set of distributions satisfying the properties described above.*

Proof. Using (i) and (ii) together entails that $\mathbf{p} \mapsto \phi(V_m(\delta(\mathbf{p})))$ is a nonincreasing function of each p-value. Denote $\mathbf{p}_0 = (p_1, \ldots, p_{m_0}, 0, \ldots, 0)$ the p-value family obtained by replacing p_i by 0 for $i > m_0$, and P_0 the distribution of $\mathbf{p}_0$ when $\mathbf{p}$ has distribution P. Obviously we have

$$\mathbb{E}_{\mathbf{p}\sim P}[\phi(V_m(\delta(\mathbf{p})))] \leq \mathbb{E}_{\mathbf{p}\sim P}[\phi(V_m(\delta(\mathbf{p}_0)))] = \mathbb{E}_{\mathbf{p}\sim P_0}[\phi(V_m(\delta(\mathbf{p})))].$$

Now, applying Lemma A.11 as cited by Gontscharuk (2010), we obtain

$$\mathbb{E}_{\mathbf{p}\sim P_0}[\phi(V_m(\delta(\mathbf{p})))] \leq \mathbb{E}_{\mathbf{p}\sim \mathrm{DU}(m,m_0)}[\phi(V_m(\delta(\mathbf{p})))]\,,$$

and thus the conclusion. □

A straightforward (though less immediately interpretable) extension of this result to procedures that are not necessarily threshold-based is to replace assumption (i) by (i'): $\mathbf{p} \mapsto V_m(\delta(\mathbf{p}))$ is a non-increasing function of each p-value.

S2 Displaying the exact distributions of the FDP

We use Theorem 5.2 to display the exact full distribution of the FDP in the case of a simple Gaussian location model with parameter μ. Figure 1 was obtained from these exact formulas for $m = 100$ and varying values of π_0 and μ. Note that the unrealistically large choice of $\alpha = 1/2$ has only been used for reasons of a good resolution of the figures; similar plots are also obtained when choosing α smaller (the variance of the FDP actually increases with smaller α, because this entails a smaller number of rejections). These graphs are discussed in Section 6.2.

Bibliography

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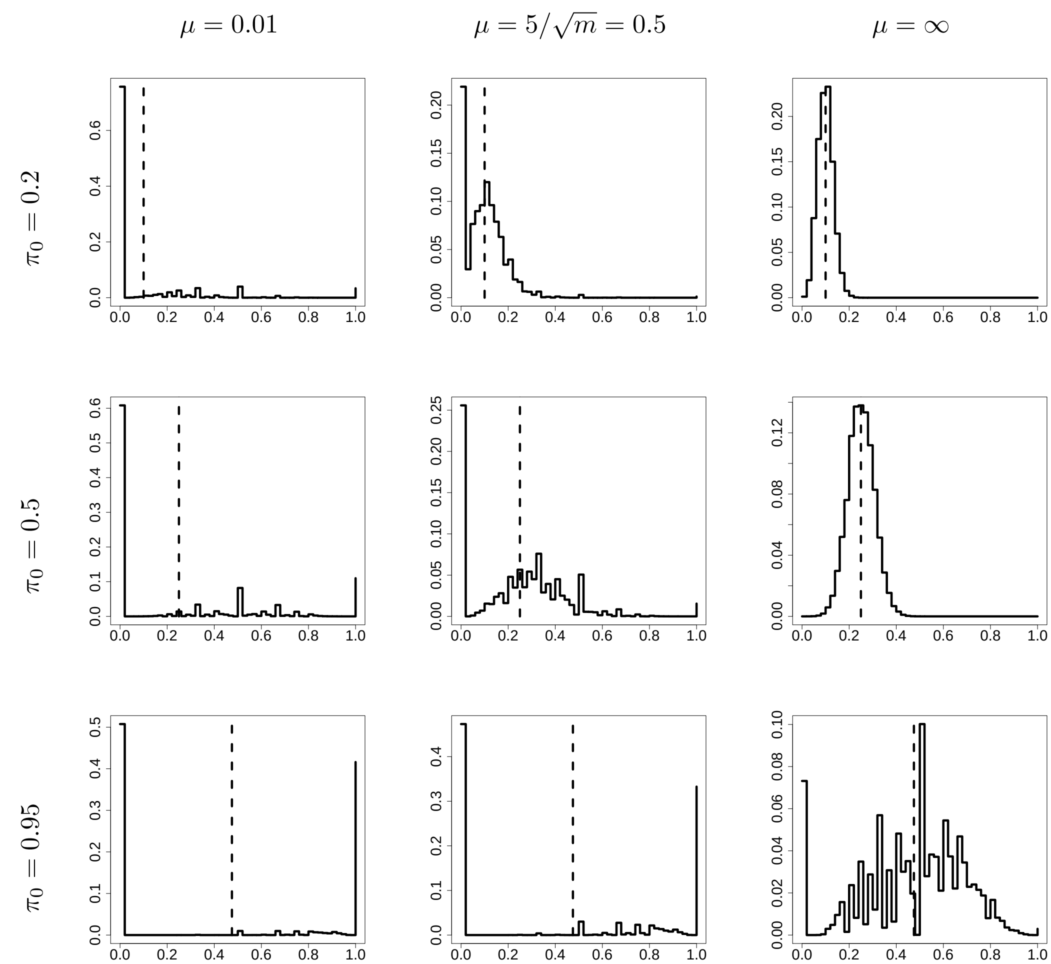

Figure 1: Exact probability $\mathbb{P}(\mathrm{FDP}(\mathrm{LSU}) \in [i/50, (i+1)/50))$ for $0 \leq i \leq 50$ under the random mixture model. The value of $\mathrm{FDR}(\mathrm{LSU}) = \pi_0\alpha$ is displayed by the vertical dashed line. The target FDR level was set to $\alpha = 0.5$ and $m = 100$ hypotheses were considered. Test statistics under alternatives are normally distributed with mean μ and variance 1.