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ESTIMATION OF TIME-VARYING PARAMETERS IN DETERMINISTIC DYNAMIC MODELS

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Supplementary Material

On-line Supplement Appendix: Proofs

In order to prove Theorem 2 in the paper, the lemma 1 (Fan and Zhang, 1999) is needed.

Lemma 1. *Let* $(t_1, Y_1), \ldots, (t_n, Y_n)$ *be i.i.d. random vectors, where the* $Y_i's$ *are scalar random variables. Assume further that* $E|Y|^s < \infty$ *and* $\sup_t \int |y|^s f(t,y)dy < \infty$ *where* f *denotes the joint density of* (t, Y). *Let* K *be a bounded positive function with a bounded support, satisfying Lipschitz condition. Then,*

$$\sup_{t\in D} \Big| n^{-1} \sum_{i=1}^{n} \Big\{ K_h(t_i - t_0)Y_i - E[K_h(t_i - t_0)Y_i] \Big\} \Big| = O_P \Big\{ [nh/\log(1/h)]^{-1/2} \Big\},$$

provided that $n^{2\varepsilon-1}h \to \infty$ *for* $\varepsilon < 1 - s^{-1}$.

Lemma 2. *Suppose that Conditions* (1)−(4) *in Section* 3.2. *Then for* $k = 1, \ldots, m$, *we have*

$$\begin{aligned}
E(U_{k;0,1}\mathbf{Y}_k|D) &= \mathbf{V}_k + \frac{1}{2}\mu_2 h_{k;0,1}^2 \mathbf{V}_k^{(2)} + o_P(h_{k;0,1}^2),\\
E(U_{k;1,1}\mathbf{Y}_k|D) &= \mathbf{V}_k^{(1)} + \frac{1}{3!}\frac{\mu_4}{\mu_2} h_{k;1,1}^2 \Big\{ \mathbf{V}_k^{(3)} + 3\mathbf{V}_k^{*(2)} \Big\} + o_P(h_{k;1,1}^2),\\
E(U_{k;0,2}\mathbf{Y}_k|D) &= \mathbf{V}_k + \frac{1}{4!}\frac{\mu_4^2 - \mu_2\mu_6}{\mu_4 - \mu_2^2} h_{k;0,2}^4 \Big\{ \mathbf{V}_k^{(4)} + 4\mathbf{V}_k^{*(3)} \Big\} + o_P(h_{0,2}^4),\\
E(U_{k;1,2}\mathbf{Y}_k|D) &= \mathbf{V}_k^{(1)} + \frac{1}{3!}\frac{\mu_4}{\mu_2} h_{k;1,2}^2 \mathbf{V}_k^{(3)} + o_P(h_{k;1,2}^2),
\end{aligned}$$

holds uniformly in a neighborhood of t_0, *where* $\mathbf{V}_k^{(l)} = (X_k^{(l)}(t_1), \ldots, X_k^{(l)}(t_n))^T$ *and* $\mathbf{V}_k^{*(l)} = (X_k^{(l)}(t_1)f'(t_1)/f(t_1), \ldots, X_k^{(l)}(t_n)f'(t_n)/f(t_n))^T$, $k = 1, \ldots, m$; $l = 0, \ldots, 4$.

Proof. Lemma 2 is easy to prove by using the above Lemma 1 and Theorem 3.1 in the book by Fan and Gijbels (1996), the details are omitted.

Proof of Theorem 2. (a) First of all, we prove the asymptotic conditional bias of the two-step local linear estimator $\hat{\theta}_{k,1}(t_0)$ for the kth component, $k = 1, \ldots, m$. By (3.8) and Lemma 2, the conditional mean of $\hat{\theta}_{k,1}(t_0)$ can be expressed as

$$\begin{aligned} E(\hat{\theta}_{k,1}(t_0)|D) &= e_{1,2}^T(\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1} \mathbf{Z}_1^T W_{b_k} \Big[E(U_{k;1,1}\mathbf{Y}_k|D) + \sum_{j=1}^m a_{k,j} E(U_{j;0,1}\mathbf{Y}_j|D) \Big] \\ &\equiv I_1^{(1)} + I_2^{(1)} + I_3^{(1)} + o_P(h_{0,1}^2 + h_{k;1,1}^2), \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} I_1^{(1)} &= e_{1,2}^T(\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1} \mathbf{Z}_1^T W_{b_k} \Big[\mathbf{V}_k^{(1)} + \sum_{j=1}^m a_{k,j} \mathbf{V}_j \Big], \\ I_2^{(2)} &= \frac{1}{2}\mu_2 \sum_{j=1}^m a_{k,j} h_{j;0,1}^2 e_{1,2}^T (\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1} \mathbf{Z}_1^T W_{b_k} \mathbf{V}_j^{(2)}, \\ I_3^{(3)} &= \frac{1}{3!}\frac{\mu_4}{\mu_2} h_{k;1,1}^2 e_{1,2}^T (\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1} \mathbf{Z}_1^T W_{b_k} \Big[\mathbf{V}_k^{(3)} + 3\mathbf{V}_k^{*(2)} \Big]. \end{aligned}$$

By using $\mathbf{V}_k^{(1)} + \sum_{j=1}^m a_{k,j}\mathbf{V}_j = [\theta_k(t_1), \ldots, \theta_k(t_n)]^T$ and Taylors expansion, we have

$$\begin{aligned} I_1^{(1)} &= e_{1,2}^T(\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1} \mathbf{Z}_1^T W_{b_k} \begin{pmatrix} \theta_k(t_0) + \theta_k'(t_0)(t_1 - t_0) + \frac{1}{2}\theta_k''(\eta_1)(\eta_1 - t_0)^2 \\ \vdots \\ \theta_k(t_0) + \theta_k'(t_0)(t_n - t_0) + \frac{1}{2}\theta_k''(\eta_n)(\eta_n - t_0)^2 \end{pmatrix} \\ &= \theta_k(t_0) + e_{1,2}^T(\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1} \mathbf{Z}_1^T W_{b_k} \begin{pmatrix} \frac{1}{2}\theta_k''(\eta_1)(\eta_1 - t_0)^2 \\ \vdots \\ \frac{1}{2}\theta_k''(\eta_n)(\eta_n - t_0)^2 \end{pmatrix}, \end{aligned} \tag{A.2}$$

where η_i is between t_i and t_0 for $i = 1, \ldots, n$. By calculating the mean and variance, we can easily get

$$\begin{aligned} &\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1 = n f(t_0) \begin{pmatrix} 1 & 0 \\ 0 & \mu_2 b_k^2 \end{pmatrix} (1 + o_P(1)), \\ &\frac{1}{2}\mathbf{Z}_1^T W_{b_k} \begin{pmatrix} \theta_k''(\eta_1)(\eta_1 - t_0)^2 \\ \vdots \\ \theta_k''(\eta_n)(\eta_n - t_0)^2 \end{pmatrix} = \frac{1}{2} n b_k^2 f(t_0)\theta_k''(t_0)[\mu_2, 0]^T (1 + o_P(1)), \\ &\mathbf{Z}_1^T W_{b_k} \begin{pmatrix} X_k^{(l)}(t_1) \\ \vdots \\ X_k^{(l)}(t_n) \end{pmatrix} = n f(t_0) X_k^{(l)}(t_0)[\mu_0, 0]^T (1 + o_P(1)), \quad l = 0, 1, \ldots, 3. \end{aligned} \tag{A.3}$$

By using the results (A.1), (A.2) and (A.3), we find that

$$
\begin{aligned}
I_1^{(1)} &= \theta(t_0) + \frac{1}{2} b_k^2 \mu_2 \theta_k''(t_0)(1 + o_P(1)),\\
I_2^{(1)} &= \frac{1}{2}\mu_2 \sum_{j=1}^{m} a_{k,j} h_{j;0,1}^2 X_j''(t_0)(1 + o_P(1)),\\
I_3^{(1)} &= \frac{1}{3!}\frac{\mu_4}{\mu_2} h_{k;1,1}^2 \Big[X_k^{(3)}(t_0) + 3X_k^{(2)}(t_0)\frac{f(t_0)}{f(t_0)}\Big](1 + o_P(1)). \qquad \text{(A.4)}
\end{aligned}
$$

Combining (A.1) and (A.4), the asymptotic conditional bias in Part (a) is obtained. Now we prove the asymptotic conditional variance of $\hat{\theta}_{k,1}(t_0)$. It follows from (3.8) that

$$
\begin{aligned}
\mathrm{Var}\,(\hat{\theta}_{k,1}(t_0)|D) = e_{1,2}^T(\mathbf{Z}_1^T W_{b_k}\mathbf{Z}_1)^{-1}\mathbf{Z}_1^T W_{b_k}\,\mathrm{Cov}\Bigg(&U_{k;1,1}\mathbf{Y}_k + \sum_{j=1}^{m} a_{k,j}U_{j;0,1}\mathbf{Y}_j,\\
&U_{k;1,1}\mathbf{Y}_k + \sum_{j=1}^{m} a_{k,j}U_{j;0,1}\mathbf{Y}_j | D\Bigg) W_{b_k}\mathbf{Z}_1(\mathbf{Z}_1^T W_{b_k}\mathbf{Z}_1)^{-1}e_{1,2}. \quad \text{(A.5)}
\end{aligned}
$$

By Lemma 1 and the simple calculation, for $k = 1\ldots,m$; $v,l = 0,1$; $i,j = 1,\ldots,n$, we can show that

$$
C_{k;v,1}^{(1)}(i) \equiv e_{v+1,2}^T(\mathbf{Z}_{1(i)}^T W_{h_{k;v,1}(i)}\mathbf{Z}_{1(i)})^{-1} = \frac{1}{nf(t_i)\mu_{2v}h_{k;v,1}^{2v}} e_{v+1,2}^T(1+o_P(1)), \quad \text{(A.6)}
$$

$$
\begin{aligned}
C_{k;v,1}^{(2)}(j,i) &\equiv e_{v+1,2}^T(\mathbf{Z}_{1(i)}^T W_{h_{k;v,1}(i)}\mathbf{Z}_{1(i)})^{-1}K_{h_{k;v,1}}(t_j - t_i)[1,(t_j - t_i)]^T\\
&= \frac{\nu_{2v}}{\mu_{2v}f(t_i)nh_{k;v,1}^{2v}}(t_j - t_i)^v K_{h_{k;v,1}}(t_j - t_i)(1 + o_P(1)), \qquad \text{(A.7)}
\end{aligned}
$$

and

$$
\begin{aligned}
C_{k;v,l}^{(3)}(i) &\equiv e_{v+1,2}^T(\mathbf{Z}_{1(i)}^T W_{h_{k;v,1}(i)}\mathbf{Z}_{1(i)})^{-1}\mathbf{Z}_{1(i)}^T W_{h_{k;v,1}(i)}\mathrm{Var}\,(Y) W_{h_{k;l,1}(i)}\mathbf{Z}_{1(i)}\\
&\quad \times(\mathbf{Z}_{1(i)}^T W_{h_{k;l,1}(i)}\mathbf{Z}_{1(i)})^{-1}e_{l+1,2}\\
&= \begin{cases} \frac{\nu_0\sigma_k^2(t_i)}{f(t_i)nh_{k;0,1}}(1 + o_P(1)), & v = l = 0,\\ \frac{\nu_2\sigma_k^2(t_i)}{\mu_2^2 f(t_i)nh_{k;1,1}^3}(1 + o_P(1)), & v = l = 1,\\ o_P(1), & v \neq l. \end{cases} \qquad \text{(A.8)}
\end{aligned}
$$

Note that the term $o_P(1)$ holds uniformly in i such that t_i falls in the neighborhood of t_0. Further, based on the results (3.8), (A.6), (A.7) and (A.8), we have

$$
\mathbf{Z}_1^T W_{b_k}\,\mathrm{Cov}\,\Bigg(U_{k;1,1}\mathbf{Y}_k + \sum_{j=1}^{m} a_{k,j}U_{j;0,1}\mathbf{Y}_j, U_{k;1,1}\mathbf{Y}_k + \sum_{j=1}^{m} a_{k,j}U_{j;0,1}\mathbf{Y}_j|D\Bigg)W_{b_k}\mathbf{Z}_1
$$

$$= \Big(A_{r,s}(k,1) + B_{r,s}(k,1)\Big)_{2\times 2}, \qquad 0 \le r, \quad s \le 1, \tag{A.9}$$

where

$$\begin{aligned} A_{r,s}(k,1) &= \sum_{i=1}^{n} K_{b_k}^2(t_i - t_0)(t_i - t_0)^{r+s}\Big(C_{k;1,1}^{(3)}(i) + 2C_{k;0,1}^{(3)}(i) + \sum_{j=1}^{m} a_{k,j}^2 C_{j;0,0}^{(3)}(i)\Big) \\ &\quad \times (1 + o_P(1)) \\ &= n\sigma_k^2(t_0)\nu_{r+s} b_k^{r+s-1}\Big(\frac{\nu_2}{\mu_2^2 n h_{k;1,1}^3} + \sum_{j=1}^{m} \frac{a_{k,j}^2 \nu_0}{n h_{j;0,1}}\Big)(1 + o_P(1)), \end{aligned}$$

and

$$\begin{aligned} B_{r,s}(k,1) &= \sum_{i=1}^{n}\Big[\sum_{j=1}^{n} K_{b_k}(t_j - t_0)(t_j - t_0)^r\Big(C_{k;1,1}^{(2)}(j,i) + \sum_{j=1}^{m} a_{k,j} C_{j;0,1}^{(2)}(j,i)\Big)\Big] \\ &\quad \times\Big[\sum_{l=1,l\neq j}^{n} K_b(t_l - t_0)(t_l - t_0)^s\Big(C_{k;1,1}^{(2)}(l,i) + \sum_{j=1}^{m} a_{k,j} C_{j;0,1}^{(2)}(l,i)\Big)\Big] \\ &= \sum_{i=1}^{n} K_{b_k}^2(t_i - t_0)(t_i - t_0)^{r+s}\sigma_k^2(t_i)\sum_{j=1}^{m}\frac{a_{k,j}^2(n-1)}{n}(1 + o_P(1)) \\ &= \sum_{j=1}^{m} a_{k,j}^2 \sigma_k^2(t_0)(n-1) f(t_0)\nu_{r+s} b_k^{r+s-1}(1 + o_P(1)). \end{aligned}$$

By using the results (A.3), (A.5) and (A.9), the asymptotic conditional variance of the two-step local linear estimator $\hat{\theta}_{k,1}(t_0)$, $k = 1, \ldots, m$, is given by

$$\begin{aligned} &\mathrm{Var}\,(\hat{\theta}_{k,1}(t_0)|D) \\ &= \frac{1}{(nf(t_0))^2}(1,0)\begin{pmatrix} A_{0,0}(k,1)+B_{0,0}(k,1) & 0 \\ 0 & A_{1,1}(k,1)+B_{1,1}(k,1)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}(1+o_P(1)) \\ &= \frac{\nu_0\sigma_k^2(t_0)}{f(t_0)nb_k}\Big[\sum_{j=1}^{m}\Big(\frac{n-1}{n}a_{k,j}^2 + \frac{a_{k,j}^2\nu_0}{nh_{j;0,1}f(t_0)}\Big) + \frac{\nu_2}{\mu_2^2 f(t_0) n h_{k;1,1}^3}\Big](1 + o_P(1)). \end{aligned}$$

Therefore, the asymptotic conditional variance in Part (a) is obtained by using (A.5), (A.6) and the above result.

(b) The proof of Part (b) is quite similar to that given in Part (a). We outline the key idea of the proof. The asymptotic conditional expectation of the two-step local quadratic estimator $\hat{\theta}_{k,2}(t_0)$, $k = 1, \ldots, m$, is given by

$$\begin{aligned} &E(\hat{\theta}_{k,2}(t_0)|D) \\ &\quad = e_{1,2}^T(\mathbf{Z}_1^T W_{b_k}\mathbf{Z}_1)^{-1}\mathbf{Z}_1^T W_{b_k}\Big[\mathbf{V}_k^{(1)} + \sum_{j=1}^{m} a_{k,j}\mathbf{V}_j\Big] + \frac{1}{3!}\frac{\mu_4}{\mu_2}h_{k;1,2}^2 e_{1,2}^T(\mathbf{Z}_1^T W_{b_k}\mathbf{Z}_1)^{-1} \end{aligned}$$

$$\times \mathbf{Z}_1^T W_{b_k} \mathbf{V}_k^{(3)} + \frac{1}{4!} \sum_{j=1}^{m} a_{k,j} \frac{\mu_4^2 - \mu_2\mu_6}{\mu_4 - \mu_2^2} h_{j;0,2}^4 e_{1,2}^T (\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1}$$
$$\times \mathbf{Z}_1^T W_{b_k} \Big[\mathbf{V}_j^{(4)} + 4\mathbf{V}_j^{*(3)} \Big] + o_P(h_{k;1,2}^2 + h_{0,2}^4)$$
$$\equiv I_1^{(2)} + I_2^{(2)} + I_3^{(2)} + o_P(h_{k;1,2}^2 + h_{0,2}^4). \tag{A.10}$$

Note that the first term in the above is the same as that in (A.1), and then $I_1^{(2)} = I_1^{(1)} = \theta_k(t_0) + \frac{1}{2} b_k^2 \mu_2 \theta_k''(t_0)(1 + o_P(1))$. Further, by (A.3), the second and third terms in (A.10) can be expressed as

$$I_2^{(2)} = e_{1,2}^T (\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1} \mathbf{Z}_1^T W_{b_k} \frac{1}{3!} \frac{\mu_4}{\mu_2} h_{k;1,2}^2 \mathbf{V}_k^{(3)}$$
$$= \frac{1}{3!} \frac{\mu_4}{\mu_2} h_{k;1,2}^2 X_k^{(3)}(t_0)(1 + o_P(1)). \tag{A.11}$$

and

$$I_2^{(3)} = \frac{1}{4!} \sum_{j=1}^{m} a_{k,j} \frac{\mu_4^2 - \mu_2\mu_6}{\mu_4 - \mu_2^2} h_{j;0,2}^4 e_{1,2}^T (\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1} \mathbf{Z}_1^T W_{b_k} \Big[\mathbf{V}_j^{(4)} + 4\mathbf{V}_j^{*(3)} \Big]$$
$$= \frac{1}{4!} \sum_{j=1}^{m} a_{k,j} \frac{\mu_4^2 - \mu_2\mu_6}{\mu_4 - \mu_2^2} h_{j;0,2}^4 \Big[X_j^{(4)}(t_0) + 4\frac{f'(t_0)}{f(t_0)} X_j^{(3)}(t_0) \Big]. \tag{A.12}$$

Therefore, we obtain the asymptotic conditional bias of the two-step local quadratic estimator $\hat{\theta}_{k,2}(t_0)$, $k = 1, \ldots, m$, in Theorem 2 by using (A.10), (A.11) and (A.12). For the asymptotic conditional variance of $\hat{\theta}_{k,2}(t_0)$, we have

$$\mathrm{Var}\,(\hat{\theta}_{k,2}(t_0)|D) = e_{1,2}^T (\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1} \mathbf{Z}_1^T W_{b_k} \mathrm{Var}\,\Big(U_{k;1,2} \mathbf{Y}_k + \sum_{j=1}^{m} a_{k,j} U_{j;0,2} \mathbf{Y}_j | D \Big)$$
$$\times W_{b_k} \mathbf{Z}_1 (\mathbf{Z}_1^T W_{b_k} \mathbf{Z}_1)^{-1} e_{1,2}. \tag{A.13}$$

Similar to the proof of Theorem 1, we have

$$C_{k;v,2}^{(1)}(i) \equiv e_{v+1,3}^T (\mathbf{Z}_{1(i)}^T W_{h_{k;v,2}(i)} \mathbf{Z}_{1(i)})^{-1}$$
$$= \frac{1}{n f(t_i) \mu_{2v} h_{k;v,2}^{2v}} e_{v+1,3}^T (1 + o_P(1)), \tag{A.14}$$
$$C_{k;v,2}^{(2)}(j,i) \equiv e_{v+1,3}^T (\mathbf{Z}_{1(i)}^T W_{h_{k;v,2}(i)} \mathbf{Z}_{1(i)})^{-1} K_{h_{k;v,2}}(t_k - t_i)[1, (t_j - t_i), (t_j - t_i)^2]^T$$
$$= \frac{\nu_{2v}}{\mu_{2v} f(t_i) n h_{k;v,2}^{2v}} (t_j - t_i)^v K_{h_{k;v,2}}(t_j - t_i)(1 + o_P(1)), \tag{A.15}$$

and

$$C_{k;v,l}^{(3)}(i) \equiv e_{v+1,3}^T (\mathbf{Z}_{1(i)}^T W_{h_{k;v,2}(i)} \mathbf{Z}_{1(i)})^{-1} \mathbf{Z}_{1(i)}^T W_{h_{k;v,2}(i)} \mathrm{Var}\,(Y) W_{h_{k;l,2}(i)} \mathbf{Z}_{1(i)}$$
$$\times (\mathbf{Z}_{1(i)}^T W_{h_{k;l,2}(i)} \mathbf{Z}_{1(i)})^{-1} e_{l+1,3}$$

$$
= \begin{cases} \frac{(\nu_0\mu_4^2-2\nu_2\mu_2\mu_4+\mu_2^2\nu_4)\sigma_k^2(t_i)}{(\mu_4-\mu_2^2)^2 f(t_i)nh_{k;0,2}}(1+o_P(1)), & v=l=0, \\ \frac{\nu_2\sigma_k^2(t_i)}{\mu_2^2 f(t_i)nh_{k;1,2}^3}(1+o_P(1)), & v=l=1, \\ o_P(1), & v\neq l. \end{cases} \tag{A.16}
$$

Note that the term $o_P(1)$ holds uniformly in i such that t_i falls in the neighborhood of t_0. Based on the results in (3.9) and (A.14)−(A.16), for $k=1,\ldots,m$, we find that

$$
\begin{aligned}
&\mathbf{Z}_1^T W_b \operatorname{Cov}\left(U_{k;1,2}\mathbf{Y}_k + \sum_{j=1}^m a_{k,j}U_{j;0,2}\mathbf{Y}_j, U_{k;1,2}\mathbf{Y}_k + \sum_{j=1}^m a_{k,j}U_{j;0,2}\mathbf{Y}_j | D\right) W_{b_k}\mathbf{Z}_1 \\
&= \Big(A_{r,s}(k,2) + B_{r,s}(k,2)\Big)_{2\times 2}, \quad 0\le r, \quad s\le 2,
\end{aligned} \tag{A.17}
$$

where

$$
\begin{aligned}
A_{r,s}(k,2) &= \sum_{i=1}^n K_{b_k}^2(t_i-t_0)(t_i-t_0)^{r+s}\left(C_{k;1,1}^{(3)}(i) + 2C_{k;0,1}^{(3)}(i) + \sum_{j=1}^m a_{k,j}^2 C_{j;0,0}^{(3)}(i)\right) \\
&= n\sigma_k^2(t_0)\nu_{r+s}b_k^{r+s-1}\left(\frac{\nu_2}{\mu_2^2 nh_{k;1,2}^3} + \sum_{j=1}^m \frac{a_{k,j}^2(\nu_0\mu_4^2 - 2\nu_2\mu_2\mu_4 + \mu_2^2\nu_4)}{(\mu_4-\mu_2^2)^2 nh_{j;0,2}}\right) \\
&\quad \times (1+o_P(1)),
\end{aligned}
$$

and

$$
\begin{aligned}
B_{r,s}(k,2) &= \sum_{i=1}^n \left[\sum_{j=1}^n K_{b_k}(t_j-t_0)(t_j-t_0)^r\left(C_{k;1,2}^{(2)}(j,i) + \sum_{j_1=1}^m a_{k,j_1}C_{j_1;0,2}^{(2)}(j,i)\right)\right] \\
&\quad \times \left[\sum_{l=1,l\neq j}^n K_{b_k}(t_l-t_0)(t_l-t_0)^s\left(C_{k;1,2}^{(2)}(l,i) + \sum_{j_1=1}^m a_{k,j_1}C_{j_1;0,2}^{(2)}(l,i)\right)\right] \\
&= \sum_{i=1}^n K_{b_k}^2(t_i-t_0)(t_i-t_0)^{r+s}\sigma_k^2(t_i)\sum_{j_1=1}^m \frac{a_{k,j_1}^2(n-1)}{n}(1+o_P(1)) \\
&= \sum_{j=1}^m a_{k,j}^2\sigma_k^2(t_0)(n-1)f(t_0)\nu_{r+s}b_k^{r+s-1}(1+o_P(1)).
\end{aligned}
$$

Combining the results (A.3), (A.13) and (A.17), the asymptotic conditional variance of the two-step local quadratic estimator $\hat{\theta}_{k,2}(t_0)$, $k=1,\ldots,m$, can be obtained. Therefore the proof of the Part (b) in Theorem 2 is completed.

(c) Based on similar arguments in the above procedure, the proof of Theorem 2(c) can be completed similarly. Here we omit the details.

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