Asymptotically Efficient Product-limit Estimators with Censoring Indicators Missing at Random

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Appendix A: Proof for Strongly Uniform Consistency

(A.m): $m(x)$ is a continuous function.

(A.H): $H(\cdot)$ has uniformly continuous probability density function $h(\cdot)$.

(A.W): $W(\cdot)$ is a kernel function with bounded variation and bounded support satisfying $\int W(u)\,du = 1$ and $\int |W(u)|\,du < \infty$.

(A.K): $K(\cdot)$ is a bounded kernel function with bounded support satisfying $\int K(u)\,du = 1$.

(A.b_n): $b_n \to 0, (nb_n)^{-1}\log n \to 0$.

(A.h_n): $h_n \to 0, (nh_n)^{-1}\log n \to 0$.

Let

$$
\begin{aligned}
\widehat{\Lambda}_{n,W}(t) &= \sum_{i:X_i\le t} \frac{\widehat{m}_n(X_i)}{n-R_i+1},\\
\widehat{\Lambda}_{n,I}(t) &= \sum_{i:X_i\le t} \frac{\xi_i\delta_i + (1-\xi_i)\widehat{m}_n(X_i)}{n-R_i+1},\\
\widetilde{\Lambda}_{n,I}(t) &= \sum_{i:X_i\le t} \frac{\xi_i\delta_i + (1-\xi_i)\widetilde{m}_n(X_i)}{n-R_i+1},\\
\widetilde{\Lambda}_{n,W}(t) &= \sum_{i:X_i\le t} \frac{\xi_i\delta_i/\pi_n(X_i) + (1-\xi_i/\pi_n(X_i))\widetilde{m}_n(X_i)}{n-R_i+1}.
\end{aligned}
$$

Let $\widehat{\Lambda}_n(t)$ denote the one of $\widehat{\Lambda}_{n,W}(t), \widehat{\Lambda}_{n,I}(t), \widetilde{\Lambda}_{n,I}(t)$ and $\widetilde{\Lambda}_{n,W}(t)$. To prove Theorem 3.1, we first prove the following lemma.

Lemma A.1. Under assumptions of Theorem 3.1, we have

$$\sup_{0\le t\le\tau_0} |\widehat{\Lambda}_n(t) - \Lambda(t)| \xrightarrow{a.s.} 0,$$

where $0 < \tau_0 < \tau_H$ and τ_H is as defined in Section 3.

Proof. We only prove that Lemma A.1 is true for $\widetilde{\Lambda}_{n,W}(t)$. The other three cases can be proved similarly.

For $\widetilde{\Lambda}_{n,W}(t)$, we have

$$\begin{aligned}
\widetilde{\Lambda}_{n,W}(t) - \Lambda(t) &= \left(\sum_{i:X_i \le t} \frac{\xi_i \delta_i/\pi_n(X_i) + (1-\xi_i/\pi_n(X_i))\, \widetilde{m}_n(X_i)}{n - R_i + 1} \right. \\
&\quad \left. - \sum_{i:X_i \le t} \frac{\xi_i \delta_i/\pi_n(X_i) + (1-\xi_i/\pi_n(X_i))\, m(X_i)}{n - R_i + 1} \right) \\
&\quad + \left(\sum_{i:X_i \le t} \frac{\xi_i \delta_i/\pi_n(X_i) + (1-\xi_i/\pi_n(X_i))\, m(X_i)}{n - R_i + 1} \right. \\
&\quad \left. - \sum_{i:X_i \le t} \frac{\xi_i \delta_i/\pi(X_i) + (1-\xi_i/\pi(X_i))\, m(X_i)}{n - R_i + 1} \right) \\
&\quad + \left(\sum_{i:X_i \le t} \frac{\xi_i \delta_i/\pi(X_i) + (1-\xi_i/\pi(X_i))\, m(X_i)}{n - R_i + 1} - \Lambda(t) \right) \\
&:= \zeta_{n1}(t) + \zeta_{n2}(t) + \zeta_{n3}(t). \qquad \text{(A.1)}
\end{aligned}$$

Clearly

$$\begin{aligned}
\zeta_{n1}(t) &= \sum_{i:X_i \le t} \frac{\widetilde{m}_n(X_i) - m(X_i)}{n - R_i + 1} - \sum_{i:X_i \le t} \frac{(\xi_i/\pi_n(X_i))(\widetilde{m}_n(X_i) - m(X_i))}{n - R_i + 1} \\
&:= \zeta_{n1,1}(t) + \zeta_{n1,2}(t). \qquad \text{(A.2)}
\end{aligned}$$

Let $L_1(t) = P(X \le t, \xi = 1)$ and $L_{n1}(t) = n^{-1} \sum_{i=1}^n I[X_i \le t, \xi_i = 1]$. Then

$$\zeta_{n1,2}(t) = -n^{-1} \sum_{i=1}^n \int_0^t \frac{\widetilde{m}_n(s) - m(s)}{\pi_n(s)(1 - H_n(s-))}\, dL_{n1}(s).$$

Hence, by the following facts

$$\sup_t |H_n(t-) - H(t-)| \xrightarrow{a.s.} 0, \quad \sup_{0 \le t \le \tau_0} |\widetilde{m}_n(t) - m(t)| \xrightarrow{a.s.} 0 \qquad \text{(A.3)}$$

and

$$\sup_t |L_{n1}(t) - L_1(t)| \xrightarrow{a.s.} 0, \quad \sup_{0 \le t \le \tau_0} |\pi_n(t) - \pi(t)| \xrightarrow{a.s.} 0 \qquad \text{(A.4)}$$

it follows $\sup_{0 \le t \le \tau_0} |\zeta_{n1,2}(t)| \xrightarrow{a.s.} 0$. Similarly, it can be proved $\sup_{0 \le t \le \tau_0} |\zeta_{n1,1}(t)| \xrightarrow{a.s.} 0$. This together with (A.2) yields

$$\sup_{0 \le t \le \tau_0} |\zeta_{n1}(t)| \xrightarrow{a.s.} 0. \qquad \text{(A.5)}$$

Similar arguments can be used to prove

$$\sup_{0 \le t \le \tau_0} |\zeta_{n2}(t)| \xrightarrow{a.s.} 0. \qquad \text{(A.6)}$$

Wright

$$\zeta_{n3}(t) = \sum_{i:X_i\le t}\frac{(\eta_i/\pi(X_i))(\delta_i - m(X_i))}{n-R_i+1} + \left(\sum_{i:X_i\le t}\frac{m(X_i)}{n-R_i+1} - \Lambda(t)\right)$$
$$:= \zeta_{n3,1}(t) + \zeta_{n3,2}(t). \tag{A.7}$$

Let $L_{11}(t) = P(X \le t, \delta = 1, \xi = 1)$ and $L_{n11}(t) = n^{-1}\sum_{i=1}^n I[X_i \le t, \delta_i = 1, \xi_i = 1]$. Note that $L_{11}(t) = \int_0^t m(s)\,dL_1(t)$ under MAR. We have

$$\zeta_{n3,1}(t) = \int_0^t \frac{d(L_{n11}(s) - L_{11}(s))}{\pi(s)(1-H_n(s-))} - \int_0^t \frac{m(s)\,d(L_{n1}(s) - L_1(s))}{\pi(s)(1-H_n(s-))}. \tag{A.8}$$

From the first formulas of (A.3) and (A.4) and the fact

$$\sup_t |L_{n11}(t) - L_{11}(t)| \xrightarrow{a.s.} 0, \tag{A.9}$$

integration by part can be applied to (A.8) to prove

$$\sup_{0\le t\le\tau_0} |\zeta_{n3,1}(t)| \xrightarrow{a.s.} 0. \tag{A.10}$$

For $\zeta_{n3,2}(t)$, we have

$$\zeta_{n3,2}(t) = \int_0^t \frac{m(s)(H_n(s-) - H(s-))}{(1-H_n(s-))(1-H(s-))}\,dH_n(s)$$
$$+ \int_0^t \frac{m(s)}{1-H(s)}\,d(H_n(s) - H(s)). \tag{A.11}$$

This together with the first formula in (A.3) implies that

$$\sup_{0\le t\le\tau_0} \zeta_{n32}(t) \xrightarrow{a.s.} 0. \tag{A.12}$$

Hence, by (A.7), (A.10) and (A.12) it follows that $\sup_{0\le t\le\tau_0}|\zeta_{n3}(t)| \overset{a.s.}{\to} 0$. This together with (A.1),(A.5) and (A.6) proves Lemma A.1 is true for $\widetilde{\Lambda}_{n,W}(t)$.

Proof of Theorem 3.1. By Taylor expansion, it is easy to obtain

$$\widehat{S}_n(t) - S(t) = (-\widehat{\Lambda}_n(t) + \Lambda(t))\exp\{-\Lambda(t)\} + R_n(t), \tag{A.13}$$

where

$$R_n(t) = (\log\widehat{S}_n(t) + \widehat{\Lambda}_n(t))\exp\{-\Lambda(t)\} + \frac{\exp\{c_n(t)\}}{2}(\log\widehat{S}_n(t) + \Lambda(t))^2$$

with

$$\min\{\log\widehat{S}_n(t), -\Lambda(t)\} \le c_n(t) \le \max\{\log\widehat{S}_n(t), -\Lambda(t)\}.$$

Next, we prove

$$\sup_{0\le t\le\tau_0} |\log \widehat{S}_n(t) + \widehat{\Lambda}_n(t)| \xrightarrow{a.s.} 0 \tag{A.14}$$

for any τ_0 such that $0 < \tau_0 < \tau_H$. We only prove the result is true when $\widehat{S}_n(t)$ and $\widehat{\Lambda}_n(t)$ are taken to be $\widetilde{S}_{n,W}(t)$ and $\widetilde{\Lambda}_{n,W}(t)$, respectively. Other cases can be proved similarly.

Similar to Major and Rejtö (1988)), using the following inequality

$$-\log\left(1-\frac{1}{x+1}\right) - \frac{1}{x+1} = \frac{1}{2(x+1)^2} + \frac{1}{3(x+1)^3} + \cdots \le \frac{1}{x(x+1)},$$

it follows

$$\begin{aligned} 0 \le -\log \widetilde{S}_{n,W}(t) - \widetilde{\Lambda}_{n,W}(t) &< \sum_{i:X_i\le t} \frac{|(\xi_i/\pi_n(X_i))\delta_i + (1-\xi_i/\pi_n(X_i))\widetilde{m}_n(X_i)|}{(n-R_i)(n-R_i+1)} \\ &\le \frac{1}{n}\int_0^t \frac{\widetilde{m}_n(s)}{(1-H_n(s))(1-H_n(s-))}\,ds \\ &\quad +\frac{1}{n}\int_0^t \frac{dL_{n11}(s)}{\pi_n(s)(1-H_n(s))(1-H_n(s-))} \\ &\quad +\frac{1}{n}\int_0^t \frac{dL_{n1}(s)}{\pi_n(s)(1-H_n(s))(1-H_n(s-))} \end{aligned} \tag{A.15}$$

Applying (A.3), (A.4) and (A.9) to (A.15), it is easy to obtain (A.14). Lemma A.1 together with (A.13) and (A.14) proves the strongly uniform consistency of $\widehat{S}_n(t)$ on $[0, \tau_0]$.

Appendix B: Proofs for Asymptotic Representation, weak convergence and Asymptotic efficiency

(C.m). $m(\cdot)$ has continuous derivatives up to order $k > 1$.

(C.K). $K(\cdot)$ is a kernel function of order k with bounded support.

(C.W)i. $W(\cdot)$ is a probability density kernel function with bounded support.

ii. $\int W^2(s)\,ds < \infty$.

(C.H). $H(\cdot)$ has probability density $h(\cdot)$ and $h(\cdot)$ has derivatives up to order of $k > 1$.

(C.π)i. $\pi(\cdot)$ has derivatives up to order of $k > 1$.

ii. $\inf_t \pi(t) > 0$.

(C.h_n). $nh_n \to \infty$ and $nh_n^{2k} \to 0$ for k in (C.H).

(C.b_n). $b_n \to 0$ and $nb_n \to \infty$.

LEMMA A.2. Under assumptions of Theorem 3.2, we have

$$\widehat{\Lambda}_n(t) - \Lambda(t) = -\frac{1}{n}\sum_{i=1}^{n}\frac{\mathrm{IC}(X_i, \delta_i, \xi_i)}{S(t)} + o_p(n^{-\frac{1}{2}}),$$

where $\mathrm{IC}(X, \delta, \xi)$ is as defined in Theorem 3.2.

Proof.(a) We first prove Lemma A.2 is true for $\widehat{\Lambda}_{n,W}(t)$. For $\widehat{\Lambda}_{n,W}(t)$, we have

$$\begin{aligned}\widehat{\Lambda}_{n,W}(t) - \Lambda(t) &= \left(\sum_{i:X_i\le t}\frac{\widehat{m}_n(X_i)}{n-R_i+1} - \sum_{i:X_i\le t}\frac{m(X_i)}{n-R_i+1}\right) + \left(\sum_{i:X_i\le t}\frac{m(X_i)}{n-R_i+1} - \Lambda(t)\right)\\ &:= I_{n1}(t) + I_{n2}(t). \qquad \text{(A.16)}\end{aligned}$$

Write

$$I_{n1}(t) = \int_0^t \frac{\widehat{m}_n(s) - m(s)}{1 - H_n(s-)}\, dH_n(s) \qquad \text{(A.17)}$$

By (A.17) and the fact $\sup_s |H_n(s-) - H(s-)| = O_p(n^{-\frac{1}{2}})$, we have

$$\begin{aligned}&I_{n1}(t)\\ =\ &\int_0^t \frac{\widehat{m}_n(s) - m(s)}{1 - H(s-)}\, dH(s) + o_p(n^{-\frac{1}{2}})\\ =\ &\int_0^t \frac{(nh_n)^{-1}\sum_{i=1}^n(\xi_i/\pi(X_i))(\delta_i - m(s))K((s-X_i)/h_n)}{1-H(s)}\, ds\\ &+\int_0^t \frac{(nh_n)^{-1}\sum_{i=1}^n(\xi_i/\pi_n(X_i)\pi(X_i))(\pi(X_i)-\pi_n(X_i))(\delta_i - m(s))K((s-X_i)/h_n)}{1-H(s)}\, ds\\ &+o_p(n^{-\frac{1}{2}})\\ :=\ &I_{n1,1}(t) + I_{n1,2}(t) + o_p(n^{-\frac{1}{2}}). \qquad \text{(A.18)}\end{aligned}$$

Observe that

$$\begin{aligned}I_{n1,1}(t) &:= \int_0^t \frac{(nh_n)^{-1}\sum_{i=1}^n(\xi_i/\pi(X_i))(\delta_i - m(X_i))K((s-X_i)/h_n)}{1-H(s)}\, ds\\ &\quad+\int_0^t \frac{(nh_n)^{-1}\sum_{i=1}^n(\xi_i/\pi(X_i))(m(X_i) - m(s))K((s-X_i)/h_n)}{1-H(s)}\, ds.\\ &:= I_{n1,11}(t) + I_{n1,12}(t) \qquad \text{(A.19)}\end{aligned}$$

By (C.K) and (C.H), it follows that

$$\begin{aligned}I_{n1,11}(t) &= (nh_n)^{-1}\sum_{i=1}^{n}\frac{\xi_i(\delta_i - m(X_i))}{\pi(X_i)}\int_0^t \frac{K((s-X_i)/h_n)}{1-H(s)}\, ds\\ &= n^{-1}\sum_{i=1}^{n}\frac{\xi_i(\delta_i - m(X_i))}{\pi(X_i)(1-H(X_i))} I[X_i \le t] + o_p(h_n^k). \qquad \text{(A.20)}\end{aligned}$$

By (C.m), (C.H) and (C.K), under MAR we have

$$
\begin{aligned}
I_{n1,12}(t) &= E\int_0^t \frac{(nh_n)^{-1}\sum_{i=1}^n(\xi_i/\pi(X_i))(m(X_i)-m(s))K((s-X_i)/h_n)}{1-H(s)}\,ds + o_p(n^{-\frac{1}{2}}) \\
&= (-1)^k h_n^k \sum_{l=0}^k \left(\int_0^t \frac{m^{(l)}(s)h^{(k-l)}(s)}{l!(k-l)!(1-H(s))}\,ds\right)\int u^k K(u)\,du + o(h_n^k) + o_p(n^{-\frac{1}{2}}).
\end{aligned}
$$

This proves

$$
I_{n1,12}(t) = o_p(n^{-\frac{1}{2}}) \tag{A.21}
$$

by (C.h_n). (A.19), (A.20) and (A.21) together prove

$$
I_{n1,1}(t) = n^{-1}\sum_{i=1}^n \frac{\xi_i(\delta_i - m(X_i))}{\pi(X_i)(1-H(X_i))} I[X_i \le t] + o_p(n^{-\frac{1}{2}})). \tag{A.22}
$$

By (C.m), (C.π), (C.K) and (A.4), we have

$$
\begin{aligned}
I_{n1,2}(t) &= n^{-1}\sum_{i=1}^n \frac{\xi_i(\delta_i - m(X_i))I[X_i\le t]}{\pi^2(X_i)h(X_i)(1-H(X_i))}(nb_n)^{-1}\sum_{j\ne i} W\left(\frac{X_i-X_j}{b_n}\right)(\pi(X_j)-\xi_j) \\
&\quad + n^{-1}\sum_{i=1}^n \frac{\xi_i(\delta_i - m(X_i))I[X_i\le t]}{\pi^2(X_i)h(X_i)(1-H(X_i))}(nb_n)^{-1}\sum_{j\ne i} W\left(\frac{X_i-X_j}{b_n}\right)(\pi(X_i)-\pi(X_j)) \\
&\quad + o_p(n^{-\frac{1}{2}}) := I_{n1,21}(t) + I_{n1,22}(t).
\end{aligned} \tag{A.23}
$$

From (C.π)ii and (C.W)ii, under MAR we have $EI_{n1,21}^2(t) = O(n^{-2}b_n^{-1})$. This implies $I_{n1,21}(t) = o_p(n^{-\frac{1}{2}})$ if $nb_n \to \infty$. Similarly, it can be proved $I_{n1,22}(t) = o_p(n^{-\frac{1}{2}})$. Hence, we have

$$
I_{n1,2}(t) = o_p(n^{-\frac{1}{2}}). \tag{A.24}
$$

This together with (A.18) and (A.22) proves

$$
I_{n1}(t) = n^{-1}\sum_{i=1}^n \frac{\xi_i(\delta_i - m(X_i))}{\pi(X_i)(1-H(X_i))} I[X_i \le t] + o_p(n^{-\frac{1}{2}})). \tag{A.25}
$$

By Dikta (1998), for $I_{n2}(t)$ in (A.16) we have

$$
\begin{aligned}
I_{n2}(t) &= \int_0^t \frac{m(s)(H_n(s)-H(s))}{(1-H(s))^2}\,dH(s) + \int_0^t \frac{m(s)}{1-H(s)}\,d(H_n(s)-H(s)) + o_p(n^{-\frac{1}{2}}) \\
&= \frac{1}{n}\sum_{i=1}^n \int_0^{t\wedge X_i} \frac{d\widetilde{H}_1(s)}{(1-H(s))^2} + \frac{1}{n}\sum_{i=1}^n \frac{m(X_i)I[X_i\le t]}{1-H(X_i)} + o_p(n^{-\frac{1}{2}}) \\
&= \frac{1}{n}\sum_{i=1}^n \int_0^{t\wedge X_i} \frac{d\widetilde{H}_1(s)}{(1-H(s))^2} + \frac{1}{n}\sum_{i=1}^n \frac{I[X_i\le t, \delta_i = 1]}{1-H(X_i)} + \frac{1}{n}\sum_{i=1}^n \frac{m(X_i)-\delta_i}{1-H(X_i)} I[X_i \le t] \\
&\quad + o_p(n^{-\frac{1}{2}}).
\end{aligned} \tag{A.26}
$$

Lemma A.2 is then proved to be true for $\widehat{\Lambda}_{n,W}(t)$ from (A.16), (A.25) and (A.26).

(b). Secondly, we prove Lemma A.2 is true for $\widehat{\Lambda}_{n,I}(t)$. For $\widehat{\Lambda}_{n,I}(t)$, we have

$$\begin{aligned}
\widehat{\Lambda}_{n,I}(t)-\Lambda(t) &= \left(\sum_{i:X_i\le t}\frac{\xi_i\delta_i+(1-\xi_i)m(X_i)}{n-R_i+1}-\Lambda(t)\right)\\
&\quad+\left(\sum_{i:X_i\le t}\frac{\xi_i\delta_i+(1-\xi_i)\widehat{m}_n(X_i)}{n-R_i+1}-\sum_{i:X_i\le t}\frac{\xi_i\delta_i+(1-\xi_i)m(X_i)}{n-R_i+1}\right)\\
&:= T_{n1}(t)+T_{n2}(t). \qquad\qquad (A.27)
\end{aligned}$$

Clearly

$$\begin{aligned}
T_{n1}(t) &= \left(\sum_{i:X_i\le t}\frac{\xi_i\delta_i+(1-\xi_i)m(X_i)}{n-R_i+1}-\sum_{i:X_i\le t}\frac{\delta_i}{n-R_i+1}\right)\\
&\quad+\left(\sum_{i:X_i\le t}\frac{\delta_i}{n-R_i+1}-\Lambda(t)\right)\\
&:= T_{n1,1}(t)+T_{n1,2}(t) \qquad\qquad (A.28)
\end{aligned}$$

Again, by the fact $\sup_s|H_n(s-)-H(s-)|=O_p(n^{-\frac{1}{2}})$, we have

$$\begin{aligned}
T_{n1,1}(t) &= \frac{1}{n}\sum_{i=1}^n\frac{(1-\xi_i)(m(X_i)-\delta_i)}{1-H(X_i)}I[X_i\le t]\\
&\quad+\frac{1}{n}\sum_{i=1}^n\frac{(1-\xi_i)(m(X_i)-\delta_i)(H_n(X_i-)-H(X_i-))}{(1-H(X_i))^2}I[X_i\le t]+o_p(n^{-\frac{1}{2}})\\
&:= T_{n1,11}(t)+T_{n1,12}(t)+o_p(n^{-\frac{1}{2}}). \qquad\qquad (A.29)
\end{aligned}$$

Note that $E[(m(X_i)-\delta_i)(m(X_j)-\delta_j)|X_i,X_j]=0$ for $i\neq j$. Then, we have

$$\begin{aligned}
ET^2_{n1,12}(t) &= \frac{1}{n^2}\sum_{i=1}^n E\left[\frac{(1-\xi_i)^2(m(X_i)-\delta_i)^2}{(1-H(X_i))^2}(H_n(X_i-)-H(X_i))^2I[X_i\le t]\right]\\
&\le \frac{\alpha}{n}E(H_n(X_i-)-H(X_i-))^2=o(\frac{1}{n}), \qquad\qquad (A.30)
\end{aligned}$$

where α is a constant. (6.12) implies $T_{n1,12}(t)=o_p(n^{-\frac{1}{2}})$. By (A.28), (A.29), (A.30) and the following fact

$$T_{n1,2}(t)=\frac{1}{n}\sum_{i=1}^n\left(\int_0^{t\wedge X_i}\frac{d\widetilde{H}_1(s)}{(1-H(s))^2}+\frac{I[X_i\le t,\delta_i=1]}{1-H(X_i)}\right)+o_p(n^{-\frac{1}{2}}), \qquad (A.31)$$

(see Lo and Singh (1986)), it follows

$$\begin{aligned}
T_{n1}(t) &= \frac{1}{n}\sum_{i=1}^n\frac{(1-\xi_i)(m(X_i)-\delta_i)}{1-H(X_i)}I[X_i\le t]\\
&\quad+\frac{1}{n}\sum_{i=1}^n\left(\int_0^{t\wedge X_i}\frac{d\widetilde{H}_1(s)}{(1-H(s))^2}+\frac{I[X_i\le t,\delta_i=1]}{1-H(X_i)}\right)+o_p(n^{-\frac{1}{2}}) \quad (A.32)
\end{aligned}$$

For $T_{n2}(t)$ in (A.27), we have

$$
\begin{aligned}
T_{n2}(t) &= \sum_{i:X_i\le t}\frac{(1-\xi_i)(\widehat{m}_n(X_i)-m(X_i))}{n-R_i+1}\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{(1-\xi_i)I[X_i\le t]\left[(nh_n)^{-1}\sum_{j=1}^{n}\frac{\xi_j}{\pi(X_j)}(\delta_j-m(X_j))K\left(\frac{X_i-X_j}{h_n}\right)\right]}{(1-H(X_i))h(X_i)}\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\frac{(1-\xi_i)I[X_i\le t]\left[(nh_n)^{-1}\sum_{j=1}^{n}\frac{\xi_j}{\pi(X_j)}(m(X_j)-m(X_i))K\left(\frac{X_i-X_j}{h_n}\right)\right]}{(1-H(X_i))h(X_i)}+o_p(n^{-\frac{1}{2}})\\
&:= T_{n2,1}(t)+T_{n2,2}(t)+o_p(n^{-\frac{1}{2}}). \qquad (A.33)
\end{aligned}
$$

By (C.K), (C.π) and (C,H), we have

$$
\begin{aligned}
T_{n2,1}(t) &= \frac{1}{n}\sum_{j=1}^{n}\frac{\xi_j}{\pi(X_j)}(\delta_j-m(X_j))\frac{1}{h_n}\int_0^t\frac{(1-\pi(s))K\left(\frac{s-X_j}{h_n}\right)}{1-H(s)}ds+o_p(n^{-\frac{1}{2}})\\
&= \frac{1}{n}\sum_{j=1}^{n}\frac{(1-\pi(X_j))\xi_j(\delta_j-m(X_j))}{\pi(X_j)(1-H(X_j))}I[X_j\le t]+o_p(n^{-\frac{1}{2}}). \qquad (A.34)
\end{aligned}
$$

Similar to (A.21), it can be shown

$$T_{n2,2}(t)=o_p(n^{-\frac{1}{2}}). \qquad (A.35)$$

(A.33), (A.34) and (A.35) together prove

$$T_{n2}(t)=\frac{1}{n}\sum_{j=1}^{n}\frac{(1-\pi(X_j))\xi_j(\delta_j-m(X_j))}{\pi(X_j)(1-H(X_j))}I[X_j\le t]+o_p(n^{-\frac{1}{2}}). \qquad (A.36)$$

From (A.27), (A.32) and (A.36), Lemma A.2 is then proved for $\widehat{\Lambda}_{n,I}(t)$.

c) Similar to b), it can be proved Lemma A.2 is also true for $\widetilde{\Lambda}_{n,I}(t)$.

d) Finally, we prove that Lemma A.2 is true for $\widetilde{\Lambda}_{n,W}(t)$.

For $\zeta_{n1}(t)$ in (A.1), we have

$$
\begin{aligned}
\zeta_{n1}(t) &= \frac{1}{n}\sum_{i=1}^{n}\frac{(1-\xi_i/\pi_n(X_i))\left(\widetilde{m}_n(X_i)-m(X_i)\right)}{1-H_n(X_i-)}I[X_i\le t]\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{(1-\xi_i/\pi(X_i))\left(\widetilde{m}_n(X_i)-m(X_i)\right)}{1-H(X_i)}I[X_i\le t]+o_p(n^{-\frac{1}{2}})\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{(1-\xi_i/\pi(X_i))I[X_i\le t]\left[(nh_n)^{-1}\sum_{j=1}^{n}\xi_j(\delta_j-m(X_j))K\left(\frac{X_i-X_j}{h_n}\right)\right]}{(1-H(X_i))h(X_i)\pi(X_i)}\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\frac{(1-\xi_i/\pi(X_i))I[X_i\le t]\left[(nh_n)^{-1}\sum_{j=1}^{n}\xi_j(m(X_j)-m(X_i))K\left(\frac{X_i-X_j}{h_n}\right)\right]}{(1-H(X_i))h(X_i)\pi(X_i)}\\
&\quad+o_p(n^{-\frac{1}{2}}):=\zeta_{n1,1}(t)+\zeta_{n1,2}(t)+o_p(n^{-\frac{1}{2}}). \qquad (A.37)
\end{aligned}
$$

It is observed that

$$\zeta_{n1,1}(t) = \frac{1}{n}\sum_{j=1}^{n}\xi_j(\delta_j - m(X_j))\frac{1}{nh_n}\sum_{i=1}^{n}\frac{\left(1-\frac{\xi_i}{\pi(X_i)}\right)I[X_i \le t]K\left(\frac{X_i-X_j}{h_n}\right)}{(1-H(X_i))h(X_i)\pi(X_i)}.$$

Let

$$\psi(X_i,\xi_i;X_j,\delta_j,\xi_j) = \xi_j(\delta_j - m(X_j))\left[\frac{\left(1-\frac{\xi_i}{\triangle(X_i)}\right)I[X_i \le t]K\left(\frac{X_i-X_j}{h_n}\right)}{(1-H(X_i))h(X_i)\pi(X_i)}\right].$$

Under MAR, we have

$$E[\psi(X_i,\xi_i;X_j,\delta_j,\xi_j)\psi(X_k,\xi_k;X_l,\delta_l,\xi_l)] = 0$$

for $i \ne j,k$ and l, or $j \ne i,k$ and l, or $k \ne i,j$ and l or $l \ne i,j$ and k. Then, we have

$$\begin{aligned}
& E(\zeta_{n1,1}^2(t)) \\
\le\ & \frac{1}{n^2}\sum_{j=1}^{n}E\left\{[(\xi_j(\delta_j - m(X_j))]^2\left[\frac{1}{nh_n}\sum_{i=1,i\ne j}^{n}\frac{(1-\xi_i/\pi(X_i))I[X_i \le t]K\left((X_i-X_j)/h_n\right)}{(1-H(X_i))h(X_i)\pi(X_i)}\right]^2\right\} \\
& +O(\frac{1}{n^2h_n}) \\
\le\ & \frac{1}{n^4h_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left[\frac{(1-\xi_i/\pi(X_i))^2I[X_i \le t]K((X_i-X_j)/h_n)}{(1-H(X_i))h(X_i)\pi(X_i)}\right]^2 + O(\frac{1}{n^2h_n}) = O(\frac{1}{n^2h_n}).
\end{aligned}$$

This proves $\zeta_{n1,1}(t) = o_p(n^{-\frac{1}{2}})$ as $nh_n \to \infty$. Similarly, it can be proved $E\zeta_{n1,2}^2(t) = O(h_n/n)$, which implies $\zeta_{n1,2}(t) = o_p(n^{-\frac{1}{2}})$ by the condition that $m(\cdot)$ has bounded derivative of order 1 and $K(\cdot)$ is a kernel function with bounded support. By (A.37), we then have

$$\zeta_{n1}(t) = o_p(n^{-\frac{1}{2}}). \tag{A.38}$$

Note that

$$\zeta_{n2}(t) = -\frac{1}{n}\sum_{i=1}^{n}\frac{\xi_i(\delta_i - m(X_i))(\pi_n(X_i)-\pi(X_i))}{(1-H_n(X_i-))\pi_n(X_i)\pi(X_i)}.$$

for $\zeta_{n2}(t)$ defined in (A.1). Similar to (A.38), we have

$$\zeta_{n2}(t) = o_p(n^{-\frac{1}{2}}). \tag{A.39}$$

For $\zeta_{n3}(t)$ defined in (A.1), we have

$$\begin{aligned}
\zeta_{n3}(t) &= \sum_{i:X_i\le t} \frac{\frac{\xi_i\delta_i}{\pi(X_i)} + (1-\frac{\xi_i}{\pi(X_i)})m(X_i)}{n-R_i+1} - \Lambda(t) \\
&= \left(\sum_{i:X_i\le t} \frac{\delta_i}{n-R_i+1} - \Lambda(t)\right) + \sum_{i:X_i\le t} \frac{\left(\frac{\xi_i}{\pi(X_i)}-1\right)(\delta_i - m(X_i))}{n-R_i+1}. \\
&:= \zeta_{n3,1} + \zeta_{n3,2} \qquad \text{(A.40)}
\end{aligned}$$

For $\zeta_{n3,2}(t)$, we have

$$\begin{aligned}
\zeta_{n3,2}(t) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\left(\frac{\xi_i}{\pi(X_i)}-1\right)(\delta_i - m(X_i))}{1-H_n(X_i-)} I[X_i\le t] \\
&= \frac{1}{n}\sum_{i=1}^{n} \frac{(\xi_i - \pi(X_i))(\delta_i - m(X_i))}{\pi(X_i)(1-H(X_i))} I[X_i\le t] + o_p(n^{-\frac{1}{2}}). \qquad \text{(A.41)}
\end{aligned}$$

Note that $\xi_{n3,1}(t) = T_{n1,2}(t)$, where $T_{n1,2}(t)$ is defined in (A.28). (A.1), (A.31), (A.39) and (A.41) together prove Lemma A.2 is true for $\widetilde{\Lambda}_{n,W}(t)$.

Proof of Theorem 3.2. Similar to (A.15), it can be proved

$$\log \widehat{S}_n(t) + \widehat{\Lambda}_n(t) = o_p(n^{-\frac{1}{2}}). \qquad \text{(A.42)}$$

This can be seen to be true from (A.15) for $\widehat{\Lambda}_{n.W}(t)$. Similarly, we have (A.42) for the remaining three estimators $\widehat{\Lambda}_{n,I}(t), \widetilde{\Lambda}_{n,I}(t)$ and $\widetilde{\Lambda}_{n,W}(t)$. By Lemma A.2, it follows that

$$\widehat{\Lambda}_n(t) - \Lambda(t) = O_p(n^{-\frac{1}{2}}). \qquad \text{(A.43)}$$

This together with (A.13), (A.42) and Lemma A.2 proves Theorem 3.2.

Proof of Theorem 3.3. Van der Laan and McKeague (1998) used the iid representation of their estimator to prove the asymptotic efficiency and weak convergence. To prove Theorem 3.3, we need only to prove that our estimators have the same asymptotic representation as that of Van der Laan and McKeague (1998) by proving that the influence curves $I(X,\delta,\xi;t)$ for our estimators are equal to the efficient influence curve $IC_t^*(X,\delta,\xi)$.

By the fact $k(x) = m(x)$, which is easily seen by $H_1(t) = \int_0^t m(x)\,dH(x)$, we have

$$\begin{aligned}
IC_t^*(X,\delta,\xi) &= S(t)\Bigg[-\frac{I[X\le t](\pi(X)-\xi)m(X)}{\pi(X)S(X)} \\
&\quad -\frac{I[X\le t](\pi(X)-\xi)\delta}{\pi(X)S(X)} + \int_0^{t\wedge} \frac{dH_1(x)}{(1-H(x))^2}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\frac{(\xi - \pi(X))(\delta - m(X))}{\pi(X)(1 - H(X))} I[X \le t] \\
& - \int_0^{t \wedge X} \frac{d\widetilde{H}_1(s)}{(1 - H(s))^2} - \frac{I[X \le t, \delta = 1]}{1 - H(X)} = IC(X, \delta, \xi; t).
\end{aligned}
$$