

## FINANCIAL DERIVATIVE VALUATION - A DYNAMIC SEMIPARAMETRIC APPROACH

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*Abstract:* A dynamic semiparametric pricing method is proposed for financial derivatives, including European and American-type options and convertible bonds. The proposed method is an iterative procedure which uses nonparametric regression to approximate derivative values, and parametric asset models to derive the continuation values. Extension to higher-dimensional option pricing is also developed, in which the dependence structure of financial time series is modeled by copula functions. In the simulation study, we value one-dimensional American options, convertible bonds, multi-dimensional American geometric average options, and max options. The considered one-dimensional underlying asset models include the Black-Scholes, jump-diffusion, and NGARCH models and, for the multivariate case, we study copula models such as the Gaussian, Clayton, and Gumbel copulae. Convergence of the method is proved under continuity assumption on the transition densities of the underlying asset models, and the orders of the supnorm errors are derived. Both the theoretical findings and the simulation results show the proposed approach to be tractable for numerical implementation and that it provides a unified and accurate technique for financial derivative pricing.

*Key words and phrases:* American option, Black-Scholes model, convertible bond, copula, European option, jump-diffusion model, multi-dimensional option pricing, NGARCH model.

### 1. Introduction

Over the past 30 years, financial derivative pricing has become an important issue. Many models have been proposed to describe the processes underlying the derivatives, for example the binomial tree, Geometric Brownian motion, jump-diffusion process, stochastic volatility model (SVM), and the GARCH model (Bollerslev (1986)), etc. A variety of valuation techniques have been proposed to price European or American-type derivatives under various model assumptions. There are closed-form pricing formulae of European options for models such as the Geometric Brownian motion and the jump-diffusion process, see for example Black and Scholes (1973), Merton (1976), Kou (2002), Heston (1993), and Heston and Nandi (2000). options, there are in general no analytic pricing solutions; researchers or practitioners usually rely on numerical or Monte Carlo

methods. A common approach is to solve the stochastic differential or integral equations of the early exercise premium or early exercise boundary of the American option. Among which Barone-Adesi and Whaley (1987), Ju and Rui (1999), Ju (1998) and Lai and AitSahalia (2001) studied the Black-Scholes Model, and Bates (1991), Chesney and Jeanblanc (2004), and Chiarella and Ziogas (2005) developed methods for jump diffusion models. For more complicated varying volatility models such as the GARCH models, Ritchken and Trevor (1999) proposed a lattice method and Duan and Simonato (2001) used a Markov chain approximation. For simulation methods of American option pricing, see for example Broadie and Glasserman (2004), Glasserman (2004) and Deng and Lee (2004). In this paper, we propose a dynamic semiparametric derivative pricing method for parametric asset models. The proposed method is applicable to a variety of derivatives including European and American-type options and convertible bonds. One major difficulty in American derivative pricing is to derive the temporal continuation values. Duan and Simonato (2001) used step functions to approximate the option value and compute the continuation values by multiplying the step functions by a sparse transition matrix. We extend the step function approximation to quadratic regression functions and modify the transition matrix to a further sparse moment transition matrix. Generally speaking, the proposed approach comprises two parts, the first approximates the values of the derivatives by nonparametric regression functions, and the second computes the one-step-backward filtration by a parametric transition density. Here the semiparametric technique provides a flexible alternative to solving the complicated multiple integral involved in derivative pricing. For semiparametric models, see for example Härdle, Müller, Sperlich and Werwatz (2004).

Multi-dimensional derivative pricing is an important topic in financial markets (Franke, Härdle and Hafner (2004)), yet its progress is in general hampered by the curse of dimensionality. It is even more challenging to price American style derivatives in high-dimensional cases. Techniques including numerical methods and Monte Carlo simulation have been proposed to handle the problem. Deterministic numerical methods (see, eg. Barraquand and Martineau (1995) and Judd (1998)) can provide good approximations in specific cases, yet their convergence properties are unclear in general (Broadie and Glasserman (2004)). For the Monte Carlo approach, random samples are simulated to approximate the continuation values (see, eg. Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001)). As pointed out by Glasserman (2004), many of the simulation methods are related to the stochastic mesh method (Rust (1997)), and Broadie and Glasserman (2004) extend the method to multi-dimensional American option pricing. In the stochastic mesh method, random samples are generated by importance sampling of mesh density functions, and the continuation values are

approximated by a weighted sum of the sample. The mesh estimate is shown to converge under conditions imposed on moments of payoffs, weights and likelihood ratios (Broadie and Glasserman (2004)). In this study, we extend the aforementioned dynamic semiparametric approach to multi-dimensional option pricing. The multivariate joint distributions of the underlying assets are modeled by copula functions (Nelsen (2006)). Since copula models provide a new and flexible way to model multivariate dependence, it has recently become a very popular tool in financial studies (Cherubini, Luciano and Vecchiato (2004)). At each possible exercise date, we approximate the option values at pre-selected grid points by step functions of the asset values; the continuation values are evaluated by the Riemann-Stieltjes sums of recursively defined distributions, which are from weighted average densities of the pre-selected grid points. Since we use the Riemann-Stieltjes integral, the proposed multivariate extension is easily implemented under the framework of a copula model. Details of the proposed extension and the stochastic mesh method are given in Section 4. The computational effort of the proposed semiparametric scheme is linear in the number of exercise opportunities, and quadratic in the number of partition points. The convergence of the proposed approach is proved under a continuity assumption on the transition densities of the underlying models; the convergence orders of the supnorm errors are derived in terms of the the respective partition lengths of the asset values and maturity time. In the simulation study, one-dimensional derivative pricing and multi-dimensional American geometric average options and max options are obtained. Both the theoretical results and simulation studies show that the proposed approach is tractable for numerical implementation and that it provides an accurate and unified method for pricing financial derivatives.

This article is organized as follows. In Section 2, the proposed approach for valuing financial derivatives of an univariate underlying asset is introduced for the jump-diffusion and nonlinear asymmetric GARCH(1, 1), abbreviated by NGARCH(1, 1), models. In Section 3, the approximation orders of the proposed method and the one-step-backward filtration for the Black-Scholes, jump-diffusion, and the NGARCH(1,1) models are derived. The valuation of convertible bonds is also discussed. In Section 4, the extension of the proposed approach to multi-dimensional option valuation is illustrated. Simulation results are given in Section 5. Conclusions are in Section 6. All the proofs, tables, and figures can be found in the online supplement (<http://www.stat.sinica.edu.tw/statistica>).

## 2. Methodology

In this section, we describe in detail the proposed approach to valuing American options of an univariate underlying asset with dividend  $\delta$ , maturity time  $T$ ,

and strike price  $K$ . We start by dividing the maturity time period  $[0, T]$  into  $n$ -subintervals with equal length  $[t_{i-1}, t_i]$ ,  $i = 1, \dots, n$ , where  $0 = t_0 < t_1 < \dots < t_n = T$ , and denote the partition length by  $\Delta = t_i - t_{i-1}$ . Throughout, we consider an  $n$ -period model, in which the American option can only be exercised at time  $t_i$ ,  $i = 0, \dots, n$ .

Let  $V_i$  denote the time  $t_i$  value of the American put option and  $S_i$  be the corresponding underlying asset value. For a discrete time model, the no-arbitrage American put option value at time  $t_n$  is  $V_n = (K - S_n)^+$ , and at time  $t_i < t_n$  is

$$V_{i-1} = \max \left\{ (K - S_{i-1})^+, e^{-r\Delta} E_{i-1}(V_i) \right\}, \quad (2.1)$$

(see Shreve (2004, p.91)) where  $r > 0$  is the riskless interest rate and  $E_{i-1}$  is the conditional expectation under a risk-neutral probability measure given the information up to time  $t_{i-1}$ . In (2.1), the term  $(K - S_{i-1})^+$  is called the intrinsic value, and  $e^{-r\Delta} E_{i-1}(V_i)$  is the continuation value. The American put option will be exercised earlier at time  $t_{i-1}$  if  $(K - S_{i-1})^+ \geq e^{-r\Delta} E_{i-1}(V_i)$ , and will be held continuously if  $(K - S_{i-1})^+ < e^{-r\Delta} E_{i-1}(V_i)$ .

The main difficulty arises in evaluating the continuation value. In this study, we propose a new approach to tackle this problem. We illustrate the proposed schemes for jump-diffusion and NGARCH(1, 1) models in Sections 2.1 and 2.2, respectively.

### 2.1. The proposed scheme for the jump-diffusion model

Jump-diffusion processes have been widely used to model financial time series that reflect the discontinuity of asset returns. The parametric jump-diffusion process of the asset price  $S_t$ , under the risk neutral measure, can be written as

$$\frac{dS_t}{S_{t-}} = (r - \delta - \lambda\phi)dt + \sigma dW_t + (Y_t - 1)dN_t, \quad (2.2)$$

where  $r$  is the riskless interest rate,  $\delta$  is the continuously compounded dividend yield of  $S_t$ ,  $\sigma$  is the instantaneous volatility,  $W_t$  is a standard Wiener process, and  $N_t$  is a Poisson process with intensity rate  $\lambda$ . Herein, we consider a log-normal jump model, in which the  $\ln Y_t$  are *i.i.d.*  $N(\gamma - (1/2)\xi^2, \xi^2)$  random variables, where  $\gamma = \ln(1 + \phi)$  and  $\phi$  is the expected jump size. The processes  $W_t$ ,  $N_t$ , and  $Y_t$  are assumed to be independent. In particular, if the jump size is a constant  $\phi$  (i.e.,  $\xi = 0$ ), then it is called a constant jump-diffusion process,

$$\frac{dS_t}{S_{t-}} = (r - \delta - \lambda\phi)dt + \sigma dW_t + \phi dN_t. \quad (2.3)$$

In the case of no jump ( $\xi = 0$  and  $\phi = 0$ ), it reduces to the Black-Scholes model,

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t. \quad (2.4)$$

At time  $t_{n-1}$ , if the stock price is  $S_{n-1}$ , then the continuation value of an American put option is

$$\begin{aligned}
 e^{-r\Delta} E_{n-1}(V_n) &= e^{-r\Delta} E_{n-1}[(K - S_n)^+] \\
 &= \sum_{\nu=0}^{\infty} \frac{e^{-\lambda\Delta} (\lambda\Delta)^\nu}{\nu!} \left[ K e^{-r\Delta} \Phi\left(\frac{\ln(K/S_{n-1}) - \gamma\nu - \mu\Delta}{\sigma\sqrt{\Delta}}\right) \right. \\
 &\quad \left. - e^{\gamma\nu - (\xi + \lambda\phi)\Delta} S_{n-1} \Phi\left(\frac{\ln(K/S_{n-1}) - \gamma\nu - \mu\Delta - \sigma^2\Delta}{\sigma\sqrt{\Delta}}\right) \right]. \tag{2.5}
 \end{aligned}$$

By comparing the continuation and the intrinsic values, see (2.1), the American put option value function at time  $t_{n-1}$ ,  $V_{n-1}$ , can be obtained. Continuing to time  $t_{n-2}$ , to derive the continuation value one faces the difficulty of evaluating the intractable term  $E_{n-2}(V_{n-1})$ . To tackle this problem a new method is proposed, in which  $V_{n-1}$  is approximated by the multi-piece polynomials  $\hat{V}_{n-1}$  defined by (2.6) below. Throughout, we confine the domain of the stock price to the interval  $[0, 2K]$  since the put option values become valueless when the stock prices are greater than  $2K$ . Let  $0 = s_0 < s_1 < \dots < s_N = 2K$  denote the fixed equidistance stock values and  $0 = A^{(0)} < A^{(1)} < \dots < A^{(m)} = 2K$  be a partition of the stock price interval  $[0, 2K]$  such that each subinterval  $[A^{(j-1)}, A^{(j)}]$  contains at least three  $s_i$ 's. In the following, we use  $\tilde{V}_i$  to denote an approximate option value at time  $t_i$ ,  $i = 0, 1, \dots, n - 1$ , which is defined recursively in the proposed procedure. At time  $t_{n-1}$ , since the option value function  $V_{n-1}$  is derivable, we set  $\tilde{V}_{n-1} = V_{n-1}$ . The followings are the four main steps of the proposed method. The procedure starts from  $i = n - 1$ , that is time  $t_{n-1}$ .

**Algorithm 2.1.**

1. In each subinterval  $[A^{(j-1)}, A^{(j)}]$ , fit the approximate option value  $\tilde{V}_i(S_i)$  by a quadratic regression function to the stock points, denoted by

$$\hat{V}_i(S_i) = \sum_{j=1}^m Q_i^{(j)}(S_i), \tag{2.6}$$

where  $Q_i^{(j)}(S_i) = \sum_{k=0}^2 a_{i,k}^{(j)} S_i^k I^{(j)}$  with  $I^{(j)} = I_{\{A^{(j-1)} \leq S_i < A^{(j)}\}}$ ,  $j = 1, \dots, m$ .

2. Compute the one-step filtration of  $\hat{V}_i$  conditional on the stock price  $S_{i-1} = s_h$ , to obtain the continuation value at time  $t_{i-1}$ . That is,

$$E(\hat{V}_i | S_{i-1} = s_h) = \mathbf{P}_h \mathbf{a}_i, \quad h = 1, \dots, N, \tag{2.7}$$

where  $\mathbf{a}_i = (a_{i,0}^{(1)}, a_{i,1}^{(1)}, a_{i,2}^{(1)}, \dots, a_{i,2}^{(m)})'$  and  $\mathbf{P}_h$  is the  $h$ th row of the moment transition matrix  $\mathbf{P}$ ,

$$\mathbf{P}_h = (E_{s_h}(I^{(1)}), E_{s_h}(S_i I^{(1)}), E_{s_h}(S_i^2 I^{(1)}), \dots, E_{s_h}(I^{(m)}),$$

$$E_{s_h}(S_i I^{(m)}), E_{s_h}(S_i^2 I^{(m)})$$

with  $E_{s_h}(\cdot) = E(\cdot | S_{i-1} = s_h)$ . The closed-form expressions of the entries of  $\mathbf{P}$  are derived in Section 3 for several well-known models. Further note that the moment transition matrix  $\mathbf{P}$  remains the same for all times  $t_i$ ,  $i = 0, \dots, n-1$ .

3. The approximate option value function at time  $t_{i-1}$ , given the stock price  $S_{i-1} = s_h$ , is determined by  $\tilde{V}_{i-1}(s_h) = \max\{(K - s_h)^+, e^{-r\Delta} E(\hat{V}_i | S_{i-1} = s_h)\}$ .
4. If  $i - 1 = 0$ , then stop; otherwise set  $i = i - 1$  and return to Step-1.

The option price at the initial time  $t_0$  is obtained by following the above procedure iteratively. To apply Algorithm 2.1 to American call options with dividend  $\delta$ , one only needs to replace (2.1) by the American call option value at  $(t_i, S_i)$ ,

$$C_i = \begin{cases} (S_n - K)^+, & \text{for } i = n \\ \max\{(S_i - K)^+, e^{-(r-\delta)\Delta} E_i(C_{i+1})\}, & \text{for } i < n \end{cases}$$

**Remark 2.1.** Algorithm 2.1 can also be applied to estimate the early exercise boundary, for example, at time  $t_{i-1}$ , the early exercise boundary of an American put option is obtained by solving the value match equation  $(K - S_{i-1})^+ = e^{-r\Delta} E_{i-1}(\hat{V}_i)$  for  $S_{i-1}$ .

If the quadratic regression functions of (2.6) are changed to the following step functions,  $Q_i^{(j)}(S_i) = \tilde{V}_i(S_i)I^{(j)}$ ,  $j = 1, \dots, m$ , with  $m = N$  (that is  $A^{(0)} = s_0$ ,  $A^{(j)} = (s_j + s_{j+1})/2$  for  $j = 1, \dots, N-1$ , and  $A^{(N)} = s_N$ ), then Algorithm 2.1 is converted to the Markov chain approximation method of Duan and Simonato (2001). Using the quadratic regression functions in Algorithm 2.1 provides an alternative to the fitted curve of option values, which smoothes the fit and increases the sparsity of the matrix of the multiplication operation. The regression technique was introduced by Longstaff and Schwartz (2001) to value the American option by Monte Carlo simulation. But the regression steps in the two approaches are used differently. In Longstaff and Schwartz, the regression function is fitted to the discounted cash flows on the possibly early-exercised stock paths, and the continuation values of the sample paths are estimated by the fitted curve; in Algorithm 2.1, we use regression functions to approximate the option values and compute the continuation values by filtration of the regression functions.

The jump-diffusion model is a constant volatility model in which the option value  $V_t$  is a function of the stochastic stock price  $S_t$ . In the case of nonconstant

volatility,  $V_t$  becomes a function of both the stock price and the stochastic volatility. In the following section, we extend the proposed scheme to the nonconstant volatility NGARCH model.

**2.2. The proposed scheme for the NGARCH(1, 1) model**

Consider the heteroscedastic NGARCH(1, 1) model (Duan and Simonato (2001))

$$\begin{aligned} \ln \frac{S_i}{S_{i-1}} &= r\Delta - \frac{1}{2}\sigma_i^2 + \lambda\sigma_i + \sigma_i\varepsilon_i, \\ \sigma_i^2 &= \alpha_0 + \alpha_1\sigma_{i-1}^2(\varepsilon_{i-1} - \theta)^2 + \alpha_2\sigma_{i-1}^2, \end{aligned} \tag{2.8}$$

where the  $\varepsilon_i$ 's are i.i.d.  $N(0, 1)$  random variables under the dynamic measure,  $r$  is the riskless interest rate,  $\sigma_i$  is the conditional volatility at time  $t_i$ ,  $\lambda \geq 0$  is the risk-premium,  $\theta \geq 0$  determines the leverage effect, and  $\alpha_0, \alpha_1$  and  $\alpha_2$  are nonnegative constants. The asset return process under the risk-neutral measure, see Duan (1995), can be written as

$$\begin{aligned} \ln \frac{S_i}{S_{i-1}} &= r\Delta - \frac{1}{2}\sigma_i^2 + \sigma_i\varepsilon_i, \\ \sigma_i^2 &= \alpha_0 + \alpha_1\sigma_{i-1}^2(\varepsilon_{i-1} - \theta - \lambda)^2 + \alpha_2\sigma_{i-1}^2, \end{aligned} \tag{2.9}$$

where the  $\varepsilon_i$ 's are i.i.d.  $N(0, 1)$  random variables under the risk-neutral measure. The following features of Model (2.9) are to be noted.

- (F.1)  $\sigma_{i+1}$  is  $\mathcal{F}_i$ -measurable, where  $\mathcal{F}_i$  is the  $\sigma$ -field generated by  $\{(S_u, \sigma_{u+1}) : t_u \leq t_i\}$ .
- (F.2) If  $\sigma_i^2 \geq \alpha_0/(1 - \alpha_2)$ , then  $\sigma_{i+1}^2 \geq \alpha_0 + \alpha_2\sigma_i^2 \geq \alpha_0(1 - \alpha_2)$ . Consequently, if  $\sigma_1 \geq [\alpha_0/(1 - \alpha_2)]^{0.5}$ , then  $\sigma_i \geq [\alpha_0/(1 - \alpha_2)]^{0.5}, \forall i \geq 1$ . The stationary volatility under the dynamic measure of Model (2.8) is  $\sigma\sqrt{\Delta} = [\alpha_0/(1 - \alpha_1(1 + \theta^2) - \alpha_2)]^{0.5}$ , which is obviously larger than  $[\alpha_0/(1 - \alpha_2)]^{0.5}$ . In practice,  $\sigma_1$  is estimated by  $\sigma\sqrt{\Delta}$ , and thus we assume  $\sigma_1 \geq [\alpha_0/(1 - \alpha_2)]^{0.5}$  throughout.
- (F.3) Given the stock value and volatility at time  $t_{i-1}, (S_{i-1}, \sigma_i)$ , then by (2.9),  $\sigma_{i+1}$  can be expressed as a function of  $S_i, \sigma_{i+1}(S_i|S_{i-1}, \sigma_i) = \{\alpha_0 + \alpha_1[\ln(S_i/S_{i-1}) - r\Delta + 0.5\sigma_i^2 - \sigma_i(\theta + \lambda)]^2 + \alpha_2\sigma_i^2\}^{0.5}$ .

By (F.3), every future volatility is uniquely determined by a stock price if its preceding stock price and volatility level are given. Thus the bivariate system of  $(S_i, \sigma_{i+1})$  is reduced to the univariate case of  $S_i$  (see also Duan and Simonato (2001)). In Fig. 1(a), we show two examples of  $\sigma_{i+1}(S_i|S_{i-1}, \sigma_i)$  given  $(S_{i-1}, \sigma_i) = (49, 0.0105)$  (solid line) and  $(50, 0.0105)$  (dash line), respectively.

To carry out Algorithm 2.1 for the NGARCH model (called Algorithm 2.2), we partition the stock and volatility intervals as follows. Let  $\{s_i\}_{i=1}^N$  and  $\{A^{(j)}\}_{j=1}^m$  denote the fixed equidistance stock values and a partition of the stock price interval  $[0, 2K]$ , respectively, as in Section 2.1, and let  $[\alpha_0/(1 - \alpha_2)]^{0.5} = B^{(0)} < B^{(1)} < \dots < B^{(\ell)} < B^{(\ell+1)} = \infty$  be a partition of the volatility interval  $([\alpha_0/(1 - \alpha_2)]^{0.5}, \infty)$ , with the choice of  $B^{(\ell)}$  given in Remark B.3 in the online supplement. The partitions are kept fixed at all time points  $t_i$ ,  $i = 0, \dots, n - 1$ . Beginning with the time  $t_{n-1}$  and given  $(S_{n-1}, \sigma_n)$ , the NGARCH American put option value is  $V_{n-1}^G(S_{n-1}, \sigma_n) = \max\{(K - S_{n-1})^+, e^{-r\Delta} E_{n-1}[(K - S_n)^+]\}$ , in which the continuation value,  $e^{-r\Delta} E_{n-1}[(K - S_n)^+]$ , follows the Black-Scholes formula (since  $S_n$  has a conditional log-normal distribution given  $(S_{n-1}, \sigma_n)$ ), henceforth we assume  $V_{n-1}^G(S_{n-1}, \sigma_n)$  is known. We introduce an approximate NGARCH American option value at each time  $t_i$ , denoted by  $\tilde{V}_i^G(S_i, \sigma_{i+1})$ , in Algorithm 2.2 below. In particular, at time  $t_{n-1}$ , we define  $\tilde{V}_{n-1}^G(S_{n-1}, \sigma_n) = V_{n-1}^G(S_{n-1}, \sigma_n)$  for  $\sigma_n = B^{(h)}$ ,  $h = 1, \dots, \ell$ , that is  $\tilde{V}_{n-1}^G(S_{n-1}, \sigma_n)$  takes the same value as  $V_{n-1}$  on the volatility partition curves; for  $\sigma_n \neq B^{(h)}$ ,  $h = 1, \dots, \ell$ , we define  $\tilde{V}_{n-1}^G(S_{n-1}, \sigma_n)$  by the interpolation

$$\begin{cases} w_{\sigma_n} \tilde{V}_{n-1}^G(S_{n-1}, B^{(h)}) + (1 - w_{\sigma_n}) \tilde{V}_{n-1}^G(S_{n-1}, B^{(h-1)}), & \text{if } \sigma_n \in (B^{(h-1)}, B^{(h)}) \\ \min \left\{ K, w_{\sigma_n} \tilde{V}_{n-1}^G(S_{n-1}, B^{(\ell)}) + (1 - w_{\sigma_n}) \tilde{V}_{n-1}^G(S_{n-1}, B^{(\ell-1)}) \right\}, & \text{if } \sigma_n > B^{(\ell)} \end{cases}, \tag{2.10}$$

where  $w_{\sigma_n} = (\sigma_n - B^{(h-1)}) / (B^{(h)} - B^{(h-1)})$  for  $\sigma_n \in (B^{(h-1)}, B^{(h)})$ ,  $h = 0, \dots, \ell$ , and  $w_{\sigma_n} = (\sigma_n - B^{(\ell-1)}) / (B^{(\ell)} - B^{(\ell-1)})$  for  $\sigma_n > B^{(\ell)}$ . To accept  $\tilde{V}_{n-1}^G(S_{n-1}, \sigma_n)$ , abbreviated by  $\tilde{V}_{n-1}^G$ , to compute the option value one-step ahead, we have to cope with computing the continuation value  $e^{-r\Delta} E_{n-2}(\tilde{V}_{n-1}^G)$  at time  $t_{n-2}$ . Recall from (F.3), given  $(S_{n-2}, \sigma_{n-1})$  the volatility at time  $t_{n-1}$ ,  $\sigma_n$ , is a function of  $S_{n-1}$ ; as a result so is the function  $\tilde{V}_{n-1}^G$ . To make a distinction in the notation, we use  $\tilde{V}_i^G(S_i | \mathcal{F}_{i-1})$  to denote the corresponding curve of  $\tilde{V}_i^G$  given  $\mathcal{F}_{i-1} = (S_{i-1}, \sigma_i)$ . In Fig. 1(b), we give two examples of  $\tilde{V}_i^G(S_i | \mathcal{F}_{i-1})$ , given  $(S_{i-1}, \sigma_i) = (49, 0.0105)$  (by the symbol “\*”) and  $(50, 0.0105)$  (by the symbol “o”), respectively. Although the conditional expectation  $E_{n-2}(\tilde{V}_{n-1}^G)$  is now reduced to the one-dimensional integration  $E_{n-2}[\tilde{V}_{n-1}^G(S_{n-1} | \mathcal{F}_{n-2})]$ , the function  $\tilde{V}_{n-1}^G(S_{n-1} | \mathcal{F}_{n-2})$  is in general of a complicated nonlinear form. In Algorithm 2.2, we use a regression technique to tackle the computation of the continuation value. The following specifies Algorithm 2.2, with start-up point  $i = n - 1$ .

**Algorithm 2.2.**

- 1 Let  $(S_{i-1}, B^{(h)})$  be a point on the volatility partition curve  $\sigma_i = B^{(h)}$  at time  $t_{i-1}$ , and let  $\tilde{V}_i^G(S_i | \mathcal{F}_{i-1})$  denote the corresponding curve of  $\tilde{V}_i^G$  given  $\mathcal{F}_{i-1} = (S_{i-1}, B^{(h)})$ . In each stock partition interval  $[A^{(j-1)}, A^{(j)}]$  of  $S_i$ , fit a quadratic

regression function to the data  $\{(s_h, \tilde{V}_i^G(s_h|\mathcal{F}_{i-1})) : s_h \in [A^{(j-1)}, A^{(j)}], h = 1, \dots, N\}$ ,  $j = 1, \dots, m$ . The fitted regression function is denoted by

$$\hat{V}_i^G(S_i|\mathcal{F}_{i-1}) = \sum_{j=1}^m Q^{(j)}(S_i|\mathcal{F}_{i-1}), \tag{2.11}$$

where  $Q^{(j)}(S_i|\mathcal{F}_{i-1}) = \sum_{k=0}^2 a_{i,k}^{(j)} S_i^k I^{(j)}$ . The continuation value at  $(S_{i-1}, B^{(h)})$  is

$$E_{i-1}[\hat{V}_i^G(S_i|\mathcal{F}_{i-1})] = \sum_{j=1}^m E_{i-1}[Q^{(j)}(S_i|\mathcal{F}_{i-1})], \tag{2.12}$$

with the closed-form expression given in Proposition 3.4.

- 2 The approximate option value of  $(S_{i-1}, B^{(h)})$  on the volatility partition curve at time  $t_{i-1}$  is defined by  $\tilde{V}_{i-1}^G(S_{i-1}, B^{(h)}) = \max\left\{(K - S_{i-1})^+, e^{-r\Delta} E_{i-1}\left[\hat{V}_i^G(S_i|\mathcal{F}_{i-1})\right]\right\}$ ,  $h = 1, \dots, \ell$ ; for  $\sigma_i \neq B^{(h)}$ , define  $\tilde{V}_{i-1}^G(S_{i-1}, \sigma_i)$  by interpolation of (2.10), with  $n$  replaced by  $i$ .
3. If  $i - 1 = 0$ , then stop; otherwise set  $i = i - 1$  and return to Step-1.

**Remark 2.2.** Algorithm 2.2 can be applied to other GARCH models analogously. Similar to the case of the jump diffusion model, if fitting a piecewise constant regression model to the option value, then the algorithm is converted to the method of Duan and Simonato (2001). The early exercise boundaries of the GARCH models can be evaluated by solving the value match equation (see Remark 2.1) on each volatility partition point  $\sigma_i = B^{(h)}$ ,  $h = 0, \dots, \ell$ .

### 3. Continuation Values and the Orders of Approximation

In this section, we derive the closed-form formulae for the conditional expectations in (2.7) and (2.12) and the orders of approximation of Algorithm 2.1 and 2.2. In Section 3.1 and 3.2, we derive the continuation values and the orders of approximation of Algorithm 2.1 and 2.2 for the jump-diffusion and NGARCH(1,1) models, respectively. The proposed method can be extended to pricing convertible bonds as well (see Appendix A in the online supplement).

#### 3.1. Jump-diffusion models

In the following proposition, we give the closed-form expression of the continuation value defined by (2.7) for the jump-diffusion model.

**Proposition 3.1.** *Assume the asset price  $\{S_t\}$  follows the log-normal jump-*

diffusion model (2.2), and define  $\hat{V}_i$  by (2.6). Then we have

$$E(\hat{V}_i | S_{i-1} = s) = \sum_{j=1}^m \sum_{k=0}^2 a_{i,k}^{(j)} s^k \left( \Gamma_k^{(j)} - \Gamma_k^{(j-1)} \right),$$

where  $\Gamma_k^{(j)} = \sum_{\nu=0}^{\infty} [(\lambda\Delta)^\nu e^{-\lambda\Delta}] / \nu! \exp\{kR_{\nu,\Delta} + 1/2k^2\rho_{\nu,\Delta}^2\} \Phi(d_k^{(j)})$ ,  $d_k^{(j)} = (\ln A^{(j)} - \ln s - R_{\nu,\Delta}) / (\rho_{\nu,\Delta}) - k\rho_{\nu,\Delta}$ ,  $R_{\nu,\Delta}$ , and the  $\rho_{\nu,\Delta}$  are defined as in Section 2.1 for  $j = 1, 2, \dots, m$  and  $k = 0, 1, 2$ .

A similar approach can be applied to compute the American call options of the jump-diffusion model. For the special case,  $\xi = \phi = 0$ , we have the following result for the Black-Scholes Model.

**Corollary 3.2.** *Assume the asset price  $\{S_t\}$  follows the Black-Scholes Model (2.4). Then we have*

$$E(\hat{V}_i | S_{i-1} = s) = \sum_{j=1}^m \sum_{k=0}^2 a_{i,k}^{(j)} s^k \left( \tilde{\Gamma}_k^{(j)} - \tilde{\Gamma}_k^{(j-1)} \right),$$

where  $\tilde{\Gamma}_k^{(j)} = \exp\{k(r - \delta)\Delta + 1/2(k^2 - k)\sigma^2\Delta\} \Phi(\tilde{d}_k^{(j)})$  and  $\tilde{d}_k^{(j)} = [(\ln A^{(j)} - \ln s - (r - \delta - 1/2\sigma^2)\Delta) / (\sigma\sqrt{\Delta})] - k\sigma\sqrt{\Delta}$ , for  $j = 1, 2, \dots, m$  and  $k = 0, 1, 2$ .

In the following, we derive the order of approximation,  $\sup_{S_0} |V_0 - \tilde{V}_0|$ , of Algorithm 2.1 for the jump-diffusion model.

**Theorem 3.3.** *Assume the asset price  $\{S_t\}$  follows the jump diffusion model (2.2). Then  $\sup_{S_0} |V_0 - \tilde{V}_0| = O(\Delta_A^3/\Delta)$  as  $\Delta \rightarrow 0$ , and  $\Delta_A = \max_j (A^{(j)} - A^{(j-1)}) \rightarrow 0$ , where  $A^{(j)}$ 's are the stock partitions. Thus, if  $\Delta_A^3/\Delta \rightarrow 0$ , then  $\sup_{S_0} |V_0 - \tilde{V}_0| \rightarrow 0$ .*

The key property used in the proof of Theorem 3.3 is the continuity in  $y$  of the one-step transition density  $f_\Delta(x | y)$  of  $\ln S_{i+1} = x$  given  $\ln S_i = y$ . Generally speaking, the result of Theorem 3.3 can be extended to include other models satisfying this continuity condition. In the online supplement, Appendix C, we give the order of  $\sup_{S_0} |C_0 - \tilde{C}_0|$  for the American call option without dividends for Model (2.4).

### 3.2. NGARCH(1, 1) model

In this section, we give the closed-form expression of the continuation value at (2.12) for the NGARCH(1,1) model, and derive the order of approximation of Algorithm 2.2.

**Proposition 3.4.** *Assume the asset price  $\{S_t\}$  follows the NGARCH(1, 1) model (2.9), and define  $\hat{V}_i^G(S_i|\mathcal{F}_{i-1})$  by (2.11). Given  $\mathcal{F}_{i-1} = (S_{i-1} = s, \sigma_i = B^{(h)})$ , we have*

$$E_{i-1}[\hat{V}_i^G(S_i|\mathcal{F}_{i-1})] = \sum_{j=1}^m \sum_{k=0}^2 a_{i,k}^{(j,h)} s^k (\Gamma_k^{(j,h)} - \Gamma_k^{(j-1,h)}),$$

where  $\Gamma_k^{(j,h)} = \exp\left\{kr\Delta + 1/2(k^2 - k)(B^{(h)})^2\right\} \Phi(d_k^{(j,h)})$  and  $d_k^{(j,h)} = [(\ln A^{(j)} - \ln s - r\Delta + 1)/(2(B^{(h)})^2)/B^{(h)}] - kB^{(h)}$ , for  $h = 0, \dots, \ell$ ,  $j = 0, \dots, m$ , and  $k = 0, 1, 2$ .

Similar closed-form expressions can be obtained for NGARCH(1, 1) American call options. In the following, we derive the order of approximation for Algorithm 2.2 at the initial time  $t_0$ , that is,  $\sup_{\Theta_0} |V_0^G - \tilde{V}_0^G|$ , where  $\Theta_0 = \{(S_0, \sigma_1) : 0 \leq S_0 \leq 2K \text{ and } \sqrt{[\text{alpha}_0/(1 - \alpha_2)]} \leq \sigma_1 \leq \bar{B}_0\}$  and  $\bar{B}_0 = 3\sigma\sqrt{\Delta}$ , where  $\sigma\sqrt{\Delta}$  is the stationary volatility under the dynamic measure. Note that the NGARCH(1, 1) is a discrete-time model, consequentially the time partition length  $\Delta$  is determined by the sampling frequency of the data, for example,  $\Delta$  is 1/365 (or 1/52) for daily (or weekly) returns. Accordingly, based on the model one can only price the option for a fixed time length. Therefore, in the following theorem, we derive the order of approximation of Algorithm 2.2 for fixed  $\Delta$ .

**Theorem 3.5.** *Assume the asset price  $\{S_t\}$  follows the NGARCH(1, 1) model (2.9). Then  $\sup_{\Theta_0} |V_0^G - \tilde{V}_0^G| = (n - 1)O(\Delta_A^3) + nO(\Delta_B)$ , where  $\Delta_A = \max_{0 \leq j \leq m} (A^{(j)} - A^{(j-1)})$ ,  $\Delta_B = \max_{1 \leq h \leq \ell} (B^{(h)} - B^{(h-1)})$ , and  $n = T/\Delta$ .*

#### 4. Multi-Dimensional Options

Valuing multi-dimensional derivative securities recently became an important topic in financial studies. In this section, we extend Algorithm 2.1 to high-dimensional derivative pricing.

Consider a derivative of the  $d$ -dimensional underlying assets,  $\mathbf{S}_t = (S_{1,t}, \dots, S_{d,t})$ , with each asset satisfying the risk-neutral geometric Brownian process

$$\frac{dS_{\ell,t}}{S_{\ell,t}} = rdt + \sigma_\ell dW_{\ell,t}, \quad \ell = 1, \dots, d, \tag{4.1}$$

where  $r$  is the risk-free interest rate,  $\sigma_\ell$  is the instantaneous volatility of the  $\ell$ th asset and  $W_{\ell,t}$ 's are Brownian motions. Let  $G(\mathbf{S}_t)$  denote the payoff function of the derivative, for examples,  $G(\mathbf{S}_t) = (K - (\prod_{\ell=1}^d S_{\ell,t})^{1/d})^+$  for the put option on a geometric average with strike price  $K$ , and  $G(\mathbf{S}_t) = (K - \max\{S_{1,t}, \dots, S_{d,t}\})^+$  for the max-put option. If  $\mathbf{X}_t = (X_{1,t}, \dots, X_{d,t})' = \ln \mathbf{S}_t$  denotes the logarithm of the asset prices, then the (conditional) marginal distribution of  $X_{\ell,t}$  given

$X_{\ell,0}$  is  $N(X_{\ell,0} + (r - 1/2\sigma_\ell^2)t, \sigma_\ell^2 t)$  for  $\ell = 1, \dots, d$ . Owing to the flexibility of copula functions in modeling multivariate dependence (Sklar (1959), Cherubini et al. (2004) and Nelsen (2006)), it has recently become a significant new tool to model multivariate dependence in financial markets. In this work, we also adopt copula models to describe the conditional joint distribution of  $\mathbf{X}_t$  given  $\mathbf{X}_{t-1}$ . Let  $C(F_1(X_{1,t}|X_{1,t-1}), \dots, F_d(X_{d,t}|X_{d,t-1}))$  denote the copula function connecting the conditional univariate marginal distributions,  $F_\ell(\cdot|\cdot)$ , of  $X_{\ell,t}$  given  $X_{\ell,t-1}$  to their multivariate distribution. In the case of the Gaussian copula, since the marginal  $F_\ell$ 's of (4.1) are Gaussian, the joint distribution of  $\mathbf{X}_t$  is the multivariate normal distribution.

Huang and Guo (2008) extend Algorithm 2.1 directly to multidimensional Bermudan option pricing using grids of constant size throughout the time period. One major problem encountered is the exponentially increasing numbers of the grid points. For example, consider the multivariate geometric Brownian motion model (4.1); since the conditional standard deviation of  $X_{\ell,t}$  given  $X_{\ell,0}$  is proportional to  $\sqrt{t}$ , the volume of the  $d$ -dimensional region within 3 standard deviation of the mean is proportional to  $t^{d/2}$ . Thus if we expect to cover the  $\pm 3$  standard deviation regions of a given initial point  $\mathbf{X}_0$  by constant size grids, the number of partition grids will increase exponentially with time. Thus, if we choose  $N$  constant size grids at time  $t_1$ , then we need  $2^{d/2}N$  grids of the same size at time  $t_2$  and  $n^{d/2}N$  grids to cover the considered regions at time  $t_n$ . This exponentially increasing rate hinders the extension of Algorithm 2.1 to the high-dimensional case when the dimension  $d$ , or the time length  $t$ , is large. To overcome this difficulty, we apply the importance sampling idea to re-weight the grid probabilities, and consider only fixed number of grids through time. Details are explained below. Let  $I_i^{(j)}$  denote the  $j$ -th grid at time  $t_i$  and  $\mathbf{x}_i^{(j)}$  be an interior point in  $I_i^{(j)}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, N$ . Approximate the continuation value of the  $h$ th grid point,  $\mathbf{x}_{i-1}^{(h)}$ , at time  $t_{i-1}$  by

$$\begin{aligned} E(V_i|\mathbf{x}_{i-1}^{(h)}) &= \int V_i(\mathbf{u})f(\mathbf{u}|\mathbf{x}_{i-1}^{(h)})d\mathbf{u} = \int U_i(\mathbf{u}|\mathbf{x}_{i-1}^{(h)})g_i(\mathbf{u})d\mathbf{u} \\ &\approx \int \hat{U}_i(\mathbf{u}|\mathbf{x}_{i-1}^{(h)})g_i(\mathbf{u})d\mathbf{u} = \sum_{j=1}^N V_i(\mathbf{x}_i^{(j)}) \frac{f(\mathbf{x}_i^{(j)}|\mathbf{x}_{i-1}^{(h)})}{g_i(\mathbf{x}_i^{(j)})} P_{g_i}(I_i^{(j)}), \quad (4.2) \end{aligned}$$

where  $U_i(\mathbf{u}|\mathbf{x}_{i-1}^{(h)})=V_i(\mathbf{u})[(f(\mathbf{u}|\mathbf{x}_{i-1}^{(h)}))/(g_i(\mathbf{u}))]$  and  $\hat{U}_i(\mathbf{u}|\mathbf{x}_{i-1}^{(h)})=\sum_{j=1}^N U_i(\mathbf{x}_i^{(j)}|\mathbf{x}_{i-1}^{(h)})I(\mathbf{u} \in I_i^{(j)})$ ,  $V_i(\cdot)$  is the option value function at time  $t_i$ ,  $\mathbf{u}$  is a  $d$ -dimensional vector, and  $I(\cdot)$  denotes the indicator function. In (4.2), we approximate the integral by treating  $[(V_i(\mathbf{x}_i^{(j)})f(\mathbf{x}_i^{(j)}|\mathbf{x}_{i-1}^{(h)}))/(g_i(\mathbf{x}_i^{(j)}))]$  as piecewise constant on the partition grid  $I_i^{(j)}$ ,  $j = 1, \dots, N$ , and use  $g_i(\cdot)$  as a new density of  $\mathbf{X}_i$  at time

$t_i, i = 1, \dots, n$ . Here we use step function instead of the quadratic regression function (see Algorithm 2.1) to approximate the option value function. The main consideration is to improve computational efficiency of the algorithm. In (4.2),  $f(\mathbf{x}_i^{(j)}|\mathbf{x}_{i-1}^{(h)})$  denotes the conditional density function of  $\mathbf{X}_i = \mathbf{x}_i^{(j)}$  given  $\mathbf{X}_{i-1} = \mathbf{x}_{i-1}^{(h)}$ ,

$$f(\mathbf{x}_i^{(j)}|\mathbf{x}_{i-1}^{(h)}) = c\left(F_1(x_{1,i}^{(j)}|x_{1,i-1}^{(h)}), \dots, F_d(x_{d,i}^{(j)}|x_{d,i-1}^{(h)})\right) \prod_{\ell=1}^d f_\ell(x_{\ell,i}^{(j)}|x_{\ell,i-1}^{(h)}),$$

where  $c(\cdot)$  is the copula density of  $\mathbf{X}_i$  given  $\mathbf{X}_{i-1}$ , and  $f_\ell(\cdot|\cdot)$  is the univariate conditional density of  $X_{\ell,i}$  given  $X_{\ell,i-1}$ . The density functions  $g_i(\cdot)$  are defined recursively by setting

$$g_1(\mathbf{u}) = f(\mathbf{u}|\mathbf{x}_0) \text{ and } g_i(\mathbf{u}) = \sum_{h=1}^N f(\mathbf{u}|\mathbf{x}_{i-1}^{(h)})P_{g_{i-1}}(I_{i-1}^{(h)}) \text{ for } i > 1, \quad (4.3)$$

where

$$P_{g_1}(I_1^{(j)}) = \int_{I_1^{(j)}} f(\mathbf{u}|\mathbf{x}_0)d\mathbf{u} = \sum_{i_1=1}^2 \dots \sum_{i_d=1}^2 (-1)^{i_1+\dots+i_d} C(u_{1i_1}, \dots, u_{di_d}) \quad (4.4)$$

is the distribution function of  $g_i$ , with  $I_1^{(j)} = \prod_{\ell=1}^d [a_\ell, b_\ell]$ ,  $u_{\ell 1} = F_\ell(a_\ell|x_{\ell,0})$ , and  $u_{\ell 2} = F_\ell(b_\ell|x_{\ell,0})$  for  $\ell = 1, \dots, d$  (McNeil, Frey and Embrechts (2005, p.185)). For  $i > 1$ ,

$$P_{g_i}(I_i^{(j)}) = \sum_{h=1}^N P_f(I_i^{(j)}|\mathbf{x}_{i-1}^{(h)})P_{g_{i-1}}(I_{i-1}^{(h)}),$$

where  $P_f(I_i^{(j)}|\mathbf{x}_{i-1}^{(h)}) = \int_{I_i^{(j)}} f(\mathbf{u}|\mathbf{x}_{i-1}^{(h)})d\mathbf{u}$  is obtained by (4.4) analogously. Under these settings, the European option at the initial time  $t_0$  is

$$\begin{aligned} V_0 &= e^{-r\Delta} E(V_1|X_0) \approx e^{-r\Delta} \sum_{j_1=1}^N V_1(\mathbf{x}_1^{(j_1)})P_{g_1}(I_1^{(j_1)}) \\ &\approx e^{-2r\Delta} \sum_{j_1=1}^N P_{g_1}(I_1^{(j_1)}) \sum_{j_2=1}^N V_2(\mathbf{x}_2^{(j_2)}) \frac{f(\mathbf{x}_2^{(j_2)}|\mathbf{x}_1^{(j_1)})}{g_2(\mathbf{x}_2^{(j_2)})} P_{g_2}(I_2^{(j_2)}) \\ &\approx e^{-2r\Delta} \sum_{j_2=1}^N V_2(\mathbf{x}_2^{(j_2)})P_{g_2}(I_2^{(j_2)}) \approx \dots \approx e^{-nr\Delta} \sum_{j_n=1}^N V_n(\mathbf{x}_n^{(j_n)})P_{g_n}(I_n^{(j_n)}). \end{aligned} \quad (4.5)$$

In Algorithm 4.1, we give the procedure for pricing a  $d$ -dimensional American option with payoff function  $G(\cdot)$ . With the initial log stock price of  $\mathbf{x}_0 =$

$(x_{1,0}, \dots, x_{d,0})$ , the procedure starts backward from  $i = n$  and takes the approximate option value function at the time of expiration,  $t_n$ , to be  $\tilde{V}_n(\mathbf{u}) = V_n(\mathbf{u}) = G(\mathbf{u})$ .

**Algorithm 4.1.**

1. At time  $t_i$ ,  $i = 1, \dots, n$ , take the grid points,  $\mathbf{x}_i^{(j)} = (x_{1,i}^{(j)}, \dots, x_{d,i}^{(j)})'$ ,  $j = 1, \dots, N$ , to be  $x_{\ell,i}^{(1)} = x_{\ell,0} - 3\sqrt{t_i}\sigma_\ell$  and  $x_{\ell,i}^{(j_\ell)} = x_{\ell,i}^{(1)} + (j_\ell - 1)\Delta_{x_{\ell,i}}$ , where  $\Delta_{x_{\ell,i}} = [(6\sqrt{t_i}\sigma_\ell)/(N_\ell - 1)]$  for  $j_\ell = 2, \dots, N_\ell - 1$ ,  $\ell = 1, \dots, d$ , and  $N = \prod_{\ell=1}^d N_\ell$  is the total number of grid points. Let  $\{A_{\ell,i}^{(j)}\}_{j=0}^{N_\ell}$  be a partition of the  $\ell$ th stock price interval satisfying  $A_{\ell,i}^{(0)} = x_{\ell,0} - 5\sqrt{t_i}\sigma_\ell$ ,  $A_{\ell,i}^{(N_\ell)} = x_{\ell,0} + 5\sqrt{t_i}\sigma_\ell$ , and  $A_{\ell,i}^{(j_\ell)} = x_{\ell,i}^{(j_\ell)} + \kappa\Delta_{x_{\ell,i}}$  for  $j_\ell = 1, \dots, N_\ell - 1$  and  $0 \leq \kappa \leq 1$ , where  $\kappa$  is chosen to satisfy the condition imposed in Remark 4.6. Further denote the  $j$ th grid at time  $t_i$  by  $I_i^{(j)} = \prod_{\ell=1}^d [A_{\ell,i}^{(j_\ell-1)}, A_{\ell,i}^{(j_\ell)}]$  for  $j = 1, \dots, N$  and  $j_\ell = 1, \dots, N_\ell$ . Compute the continuation value at time  $t_{i-1}$  given  $\mathbf{X}_{i-1} = \mathbf{x}_{i-1}^{(h)}$ , denoted  $\hat{E}(\tilde{V}_i|\mathbf{x}_{i-1}^{(h)})$ , by (4.2) in which the step function  $\hat{U}_i$  is defined on the above grids  $\{I_i^{(j)}\}$  and  $V_i$  is replaced by  $\tilde{V}_i$ .
2. The approximate option value at time  $t_{i-1}$  given  $\mathbf{X}_{i-1} = \mathbf{x}_{i-1}^{(h)}$  is given by  $\tilde{V}_{i-1}(\mathbf{x}_{i-1}^{(h)}) = \max\{G(\mathbf{x}_{i-1}^{(h)}), e^{-r\Delta}\hat{E}(\tilde{V}_i|\mathbf{x}_{i-1}^{(h)})\}$ .
3. If  $i - 1 = 0$ , then stop; otherwise set  $i = i - 1$  and return to Step 1.

In Algorithm 4.1, if the  $I_i^{(j)}$ 's are chosen to satisfy  $P_{g_i}(I_i^{(j)}) = 1/N$ , then the densities in (4.3) are

$$g_1(\mathbf{u}) = f(\mathbf{u}|\mathbf{x}_0) \quad \text{and} \quad g_i(\mathbf{u}) = \frac{1}{N} \sum_{h=1}^N f(\mathbf{u}|\mathbf{x}_{i-1}^{(h)}), \quad \text{for } i > 1, \quad (4.6)$$

and (4.2) becomes

$$E(V_i|\mathbf{x}_{i-1}^{(h)}) \cong \sum_{j=1}^N V_i(\mathbf{x}_i^{(j)}) \frac{f(\mathbf{x}_i^{(j)}|\mathbf{x}_{i-1}^{(h)})}{Ng_i(\mathbf{x}_i^{(j)})}, \quad \text{for } i = 1, \dots, n. \quad (4.7)$$

In this case, the densities  $g_i(\cdot)$  and the conditional expectation  $E(V_i|\mathbf{x}_{i-1}^{(h)})$  have the same form as those of the stochastic mesh method of Broadie and Glasserman (2004) (Eq.(17), (18) and (6)). The stochastic mesh method was proposed by Rust (1997) and successfully extended to high-dimensional option pricing by Broadie and Glasserman (2004). The stochastic mesh method is based on random paths simulated from the mesh densities (4.6), and uses (4.7) to approximate the

conditional expectation at time  $t_i$  given  $\mathbf{x}_{i-1}^{(h)}$  as

$$E(V_i|\mathbf{x}_{i-1}^{(h)}) = \int V_i(\mathbf{u}) \frac{f(\mathbf{u}|\mathbf{x}_{i-1}^{(h)})}{g_i(\mathbf{u})} g_i(\mathbf{u}) d\mathbf{u}. \tag{4.8}$$

The mesh density,  $g_i(\cdot)$  in (4.6), is interpreted as an average density, which is useful in reducing the variance of the estimator. As aforementioned, if the grids are chosen to satisfy  $P_{g_i}(I_i^{(j)}) = 1/N$ , then the densities  $g_i$ 's at (4.3), and the approximate formula of the conditional expectation in our approach, (4.2), are the same as the ones, (4.6) and (4.7), in the stochastic mesh method. That is the weighted average density reduces to the equal-weighted average when particular partition grids are chosen. However, since choosing the grids satisfying  $P_{g_i}(I_i^{(j)}) = 1/N$  requires more computational effort on our part we consider the equidistance partition points as described in Step 1 of Algorithm 4.1 and use a weighted average to define the distributions recursively.

There are still several differences between the proposed approach and the stochastic mesh methods. One major difference is that the stochastic mesh method is based on simulated random paths and the proposed method adopts pre-selected grid points. The stochastic mesh method uses the means of the random samples to approximate the conditional expectation integrals of the continuation values; the proposed method uses closed-form formulae for the conditional expectation integrals of approximate option value functions (eg. piecewise quadratic regression functions in one-dimensional case and step functions for multidimensional case). For the high-dimensional case, we use the Riemann-Stieltjes sum of the recursively defined distributions  $P_{g_i}(\cdot)$  to compute the multiple integrals of the continuation values, which is easily implemented when the multivariate dependence of the underlying assets is modeled by copula functions. The stochastic mesh method is a consistent but biased-high estimator; Broadie and Glasserman (2004) use the average of the mesh estimator and another biased-low estimator (called a path estimator) to improve the estimation. In the proposed approach, we use function approximation (regression functions or step functions) and integral evaluation at pre-selected grid points to calculate the option prices. The convergence of our method for one dimensional case is derived under continuity assumptions on the transition densities of the underlying asset models, and the orders of the sup-norm errors are also derived as functions of partition lengths. Furthermore, since the approximate option value function  $\tilde{V}_i$  derived from Algorithm 4.1 is a continuous function of  $\mathbf{x}_i$ , the convergence proof for the multivariate case is similar to the one-dimensional case, see Theorem 4.7.

**Remark 4.6.** The European option can be valued using Algorithm 4.1 by setting  $\tilde{V}_{i-1}(\mathbf{x}_{i-1}^{(h)}) = e^{-r\Delta} \hat{E}(\tilde{V}_i|\mathbf{x}_{i-1}^{(h)})$  in Step 2. The constant  $\kappa$  in Step 1 is chosen

such that the European option values computed from Algorithm 4.1 meet the benchmark values, which can either be obtained analytically or by Monte Carlo simulation. For the European option on a geometric average for multivariate normal underlying assets, the benchmark values can be obtained via the Black-Scholes formula since it can be reduced to a one-dimensional problem.

In the following, we derive the order of the supnorm error,  $\sup_{\mathbf{X}_0} |V_0 - \tilde{V}_0|$ , of using Algorithm 4.1 to approximate the option values for Model (4.1).

**Theorem 4.7.** *Assume the  $d$ -dimensional asset prices  $\mathbf{S}_t = \{S_{1,t}, \dots, S_{d,t}\}$  follow Model (4.1). Let  $\tilde{V}_0$  denote the approximate  $d$ -dimensional American option value obtained from Algorithm 4.1, and let  $V_0$  denote the no-arbitrage price. Then we have  $\sup_{\mathbf{X}_0} |V_0 - \tilde{V}_0| = O(\|\mathbf{h}_n\|/\Delta)$  as  $\Delta \rightarrow 0$  and  $\|\mathbf{h}_n\| = (\sum_{\ell=1}^d \Delta_{x_{\ell,n}}^2)^{1/2}$ , where  $\Delta_{x_{\ell,n}}$  is defined in Step 1 of Algorithm 4.1 and  $\mathbf{X}_0 = \log \mathbf{S}_0$ . Thus, if  $(\|\mathbf{h}_n\|/\Delta) \rightarrow 0$ , then  $\sup_{\mathbf{X}_0} |V_0 - \tilde{V}_0| \rightarrow 0$ .*

## 5. Numerical Results

In this section, we report on the simulation of pricing European and American options by Algorithm 2.1, 2.2 and 4.1 for several well-known models. For the high-dimensional case, multi-dimensional American geometric average options and max options are included, in which the multivariate joint distributions are modeled by Gaussian, Clayton, and Gumbel copulae. All codes are implemented using MatLab 7.0 running on a computer with a Pentium M processor 750, and 1GB of RAM. All the simulation results are given in Table 1–Table 5 in the supplementary part (see Appendix D and E).

## 6. Conclusion

We propose a semiparametric approach to value financial derivatives when the one-step transition probability function of the underlying process is given. Valuation problems of various financial derivatives including European options, American options, and convertible bonds can be solved by the proposed method. The extension to multi-dimensional derivative pricing is also obtained by incorporating copula functions. Both the theoretical findings and the simulation results show that the proposed approach is very tractable for numerical implementation, and provides an accurate and unified method for pricing financial derivatives. Possible extensions of the proposed method include derivative pricing with path-dependent American options and credit risk derivatives.

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## References

- Barone-Adesi, G. and Whaley, R. (1987). Efficient analytic approximation of American option values. *J. Finance* **42**, 301-320.
- Barraquand, J. and Martineau, D. (1995). Numerical valuation of high dimensional multivariate American securities. *J. Finan. Quant. Anal.* **30**, 383-405.
- Bates, D. S. (1991). The Crash of 1987: What is expected. The evidence from options markets. *J. Finance* **46**, 1009-1044.
- Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *J. Polit. Economy* **81**, 637-654.
- Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *J. Econometrics* **31**, 307-327.
- Broadie, M. and Glasserman, P. (2004). A stochastic mesh method for pricing high-dimensional American options. *J. Comput. Finance* **7**, 35-72.
- Cherubini, U., Luciano, E. and Vecchiato, W. (2004). *Copula Methods in Finance*, Wiley, West Sussex.
- Chesney, M. and Jeanblanc, M. (2004). Pricing American currency options in an exponential Lévy model. *Applied Mathematical Finance* **11**, 207-225.
- Chiarella, C. and Ziogas, A. (2005). Pricing American options on jump-diffusion processes using Fourier Hermite series expansions. *Quantitative Finance Research Centre, University of Technology Sydney*, Research Paper, No. 145.
- Deng, S. and Lee, S. (2004). Pricing American-style options with adaptive simulation. *Preprint*, Georgia Tech.
- Duan, J. C. (1995). The GARCH option pricing model. *Math. Finance* **5**, 13-32.
- Duan, J. C. and Simonato, J. G. (2001). American option pricing under GARCH by a Markov chain approximation. *J. Econom. Dynam. Control* **25** 1689-1718.
- Franke, J., Härdle, W. and Hafner, C. M. (2004). *Statistics of Financial Markets*. Springer, Berlin.
- Glasserman, P. (2004). *Monte Carlo Methods in Financial Engineering*. Springer-Verlag, New York.
- Härdle, W., Müller, M., Sperlich, S. and Werwatz, A. (2004). *Nonparametric and Semiparametric Models*. Springer, Berlin.
- Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Finan. Stud.* **6**, 327-343.
- Heston, S. L. and Nandi, S. (2000). A closed-form GARCH option valuation model. *Rev. Finan. Stud.* **13**, 585-625.
- Huang, S. F. and Guo, M. H. (2008). Valuation of multidimensional bermudan options. In *Applied Quantitative Finance 2<sup>nd</sup> Edition* (Edited by W. Härdle), forthcoming, Springer, Berlin.
- Ju, N. (1998). Pricing an American option by approximating its early exercise boundary as multipiece exponential function. *Rev. Finan. Stud.* **11**, 627-646.
- Ju, N. and Rui, Z. (1999). An approximation formula for pricing American options. *J. Derivatives* **7**, 31-40.
- Judd, K. (1998). *Numerical Methods in Economics*. MIT Press, Cambridge, Mass.
- Kou, S. G. (2002). A jump-diffusion model for option pricing. *Management Science* **48**, 1086-1101.

- Lai, T. L. and AitSahalia, F. (2001). Exercise boundaries and efficient approximations to American option prices and hedge parameters. *J. Computational Finance* **4**, 85-103.
- Longstaff, F. A. and Schwartz, E. S. (2001). Valuing American options by simulation: a simple least-squares approach. *Rev. Finan. Stud.* **14**, 113-147.
- McNeil, A. J., Frey, R. and Embrechts, P. (2005). *Quantitative Risk Management*. Princeton University Press, New Jersey.
- Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *J. Finan. Econom.* **3**, 125-144.
- Nelsen, R. B. (2006). *An Introduction to Copulas*. Springer, Berlin.
- Ritchken, P. and Trevor, R. (1999). Pricing option under generalized GARCH and stochastic volatility processes. *J. Finance* **8**, 377-402.
- Rust, J. (1997). Using randomization to break the curse of dimensionality. *Econometrica* **65**, 487-516.
- Shreve, S. E. (2004). *Stochastic Calculus for Finance I*. Springer, New York.
- Sklar, A. (1959). Fonctions de repartition à  $n$  dimensions et leurs marges. *Pub. Inst. Statist. Univ. Paris* **8**, 229-231.
- Tsitsiklis, J. and Van Roy, B. (1999). Optimal stopping of Markov processes: Hilbert space theory, approximation algorithms, and an application to pricing high-dimensional financial derivatives. *IEEE Trans. Automat. Control* **44**, 1840-51.

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