NONLINEAR FILTER CONTROL CHART FOR DETECTING DYNAMIC CHANGES

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Supplementary Material

S1 Proof of Theorem in the Paper "A Nonlinear Filter Control Chart for Detecting Dynamic Changes

Let $Y_1, Y_2, \ldots$ be *i.i.d.*, $F(x) = P(Y_i \le x)$ and $E(.)$ denotes the expectation. Suppose the distribution, $F(x)$, satisfies the following two conditions:

(I) The moment-generating function is $h(\theta) = E(e^{\theta Y_i}) < \infty$ for some $\theta > 0$.

(II) For $x > E(Y_i)$ there is a $\theta(x) \in (0, \theta_1)$ such that $x = h'(\theta(x))/h(\theta(x))$, where $\theta_1 = \sup\{\theta : h(\theta) < \infty\}$.

Let $E(Y_i) < 0$. Since $h'(0) = E(Y_i) < 0$, $h'(\theta)/h(\theta)$ is strictly increasing (see Durrett(1991), p.60) and $\log h(\theta) \to +\infty$ as $\theta \to \theta_1$, it follows that there exits at most one $\theta^* \in (\theta(0), \theta_1)$ such that $h(\theta^*) = 1$ or $\log h(\theta^*) = 0$, where $\theta(0) > 0$ satisfies $0 = h'(\theta(0))/h(\theta(0))$. That is, $h(\theta)$ attains its minimum value at $\theta(0) > 0$. We can call θ^* an **exponential rate** of $F(x)$ of random variable Y_i. The meaning of θ^* is given in Theorem 1.

Let $u = h'(\theta^*)$ and $\theta(u) = \theta^*$. It is clear that $u > 0$, and $\log h(\theta(x)) < 0$ for $x < u$ and $\log h(\theta(x)) > 0$ for $x > u$. Thus,

$$\theta(\frac{1}{x}) - x \log h(\theta(\frac{1}{x})) \ge \theta^* \tag{S1.1}$$

for $x > 0$. In fact, if

$$H(x) = \theta(\frac{1}{x}) - x \log h(\theta(\frac{1}{x})) - \theta^*,$$

we have $H(1/u) = 0$ and

$$\begin{aligned} H'(x) &= -\theta'(\frac{1}{x})\frac{1}{x^2} - \log h(\theta(\frac{1}{x})) + x\frac{h'(\theta(\frac{1}{x}))}{h(\theta(\frac{1}{x}))}\theta'(\frac{1}{x})\frac{1}{x^2} \\ &= -\log h(\theta(\frac{1}{x})). \end{aligned}$$

It follows that $H'(x) > 0$ for $x > 1/u$ and $H'(x) < 0$ for $0 < x < 1/u$. Thus, (S1.1) is true. Since $H'(x) > 0$ for $x > 1/u$, we can take

$$b = \inf\{x > 1/u : \theta(\frac{1}{x}) - x\log h(\theta(\frac{1}{x})) \geq 2\theta^*\} \tag{S1.2}$$

such that

$$\theta(\frac{1}{x}) - x\log h(\theta(\frac{1}{x})) \geq 2\theta^* \tag{S1.3}$$

for $x \geq b$.

Now we define the stopping time of a control chart, T,

$$T = \inf\{n : \max_{1\leq k\leq n}[\sum_{i=n-k+1}^{n} Y_i] \geq c\}, \tag{S1.4}$$

where $c > 0$ is the control limit. For this chart, we have the following theorem.

Theorem 1. *Suppose the conditions (I) and (II) hold. If $E(Y_i) < 0$, then*

$$E(T) \sim D(c)e^{c\theta^*} \tag{S1.5}$$

for large c, where $\theta^* > 0$ is the exponential rate satisfying $h(\theta^*) = 1$, $1/bc \leq D(c) \leq c/u$, $u = h'(\theta^*) > 0$ and b is the positive constant defined in (S1.2). If $E(Y_i) > 0$, then

$$E(T) \sim \frac{c}{E(Y_i)} \tag{S1.6}$$

for large c, where $x \sim y$ means that $x/y \to 1$ as $x, y \to \infty$.

Proof of Theorem 1. In order to prove (S1.5) we need only to prove

$$e^{c\theta^*}/bc \leq E(T) \leq ce^{c\theta^*}/u \tag{S1.7}$$

for large c. Some results of large deviations theory will be used in the proof. We first prove the upward inequality of (S1.7). Choose $\lambda \in (\theta^*, \theta_1)$ and $v > h'(\lambda)/h(\lambda)$ and let $m = tu^{-1}c\exp\{c(v\lambda/u - \log h(\lambda)/u)\}$ for $t > 0$ and $m_k = ku^{-1}c$ for $k \geq 0$, we have

$$P(T > m) = P(\sum_{i=n-k+1}^{n} Y_i < c, \quad 1 \leq k \leq n, 1 \leq n \leq m) \leq [P(\sum_{i=1}^{m_1} Y_i < c)]^{m/m_1}$$

for large c, where the last equality holds since the events

$$\{\sum_{i=m_{j-1}+1}^{m_j} Y_i\},$$

$1 \leq j \leq k$, are mutually independent and have an identity distribution. Let $n = c/u$. It follows from Theorem 9.5 of Chapter 1 in Durrett (1991) that

$$\begin{aligned} P(\sum_{i=1}^{m_1} Y_i \geq c) &= P(\sum_{i=1}^{n} Y_i \geq nu) \\ &\geq \exp\{-n(v\lambda - \log h(\lambda) + o(1/n))\} \\ &= \exp\{-c(v\lambda/u - \log h(\lambda)/u + o(1/c))\} \end{aligned}$$

for large c, and therefore

$$\begin{aligned} & [P(\sum_{i=1}^{m_1} Y_i < c)]^{m/m_1} \\ \leq\ & [1 - \exp\{-c(v\lambda/u - \log h(\lambda)/u + o(1/c))\}]^{m/m_1} \\ =\ & (1 - \frac{tm_1}{me^{o(1)}})^{m/m_1} \to e^{-t} \end{aligned}$$

as $c \to \infty$. That is, $P(T > m) \leq e^{-t}$ for large c. Thus, by the properties of exponential distribution, we have

$$E(T) \leq \frac{c}{u} \exp\{c(v\lambda/u - \log h(\lambda)/u)\}$$

for large c. Since $\lambda > \theta^*$ and $v > h'(\lambda)/h(\lambda)$ are arbitrary, the upward inequality of (S1.7) is true. To prove the downward inequality of (S1.7), let

$$U_m = \{\sum_{i=n-k+1}^{n} Y_i < c, \ \ 1 \leq k \leq \min\{n, bc-1\}, \ 1 \leq n \leq m\}$$

and

$$V_m = \{\sum_{i=n-k+1}^{n} Y_i < c, \ \ bc \leq k \leq n, \ bc \leq n \leq m\}$$

for large c, where b is defined in (S1.2). Obviously, $\{T > m\} = U_m V_m$. For $k \geq 1$, take $x > 0$ such that $xc = k$ and $1/x = h'(\theta(1/x))/h(\theta(1/x))$. By Chebyshev's inequality, we have

$$\begin{aligned} & e^{\theta(1/x)k/x} P(\sum_{i=n-k+1}^{n} Y_i \geq c) \\ =\ & e^{\theta(1/x)k/x} P(\sum_{i=1}^{k} Y_i \geq k/x) \leq E(\exp\{\theta(1/x)\sum_{i=1}^{k} Y_i\}) = h(\theta(1/x))^k \end{aligned}$$

or

$$\begin{aligned} P(\sum_{i=n-k+1}^{n} Y_i < c) &\geq 1 - \exp\{-k[\theta(1/x)/x - \log h(\theta(1/x))]\} \\ &= 1 - \exp\{-c[\theta(1/x) - x \log h(\theta(1/x))]\} \geq 1 - e^{-c\theta^*}, \end{aligned}$$

where the last inequality follows from (S1.1). Thus, take $m = te^{c\theta^*}/bc$ for $t > 0$. We have

$$\begin{aligned} P(U_m) &\geq \prod_{n=1}^{m} \prod_{k=1}^{\min\{n,bc\}} P(\sum_{i=n-k+1}^{n} Y_i < c) \\ &\geq [1 - e^{-c\theta^*}]^{bcm} \to e^{-t}, \end{aligned}$$

as $c \to +\infty$. The first inequality follows from Theorem 5.1 in Esary, Proschan and Walkup (1967). Similarly, taking $xc = k$ and using (S1.3) we have (Note that $x \geq b$ if $k \geq bc$)

$$\begin{aligned} P(V_m) &\geq \prod_{n=bc}^{m} \prod_{k=bc}^{n} P(\sum_{i=n-k+1}^{n} Y_i < c) \\ &= \prod_{n=bc}^{m} \prod_{k=bc}^{n} P(\sum_{i=n-k+1}^{n} Y_i < k/x) \\ &\geq \prod_{n=bc}^{m} \prod_{k=bc}^{n} [1 - \exp\{-c[\theta(1/x) - x \log h(\theta(1/x))]\}] \\ &\geq [1 - e^{-2c\theta^*}]^{(m-bc)^2} \to 1, \end{aligned}$$

as $c \to +\infty$. Hence, $P(T > m) \geq P(U_m)P(V_m) \to e^{-t}$ as $c \to +\infty$. This proves the downward inequality of (S1.7).

To prove (S1.6), we first mention some known results (see Chapters V and VIII in Petrov's book (1975)). Let $\Phi(\cdot)$ be a standard normal distribution and $F_n(x)$ the distribution function of the sum $S_n = (nD_1^2)^{-1/2} \sum_{k=1}^{n} (Y_k - E_1))$, where $D_1^2 = Var(Y_i))$ and $E_1 = E(Y_i))$. Then

$$F_n(x) - \Phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}|x|}[O(\frac{|x|^3}{\sqrt{n}}) + o(\frac{1}{|x|})], \tag{S1.8}$$

as $|x| \to +\infty$ and $|x|^3/\sqrt{n} \to 0$, and

$$|F_n(x) - \Phi(x)| \leq \frac{Aa^3}{a_0\sqrt{n}(1 + |x|)^3} \tag{S1.9}$$

for every x and $n \geq 1$, where A is a constant, $a^3 = E(Y_i - E_1))^3$ and $a_0 = (D_1)^{2/3}$. The following elementary facts also will be used:

$$1 - \Phi(x) < \frac{e^{-x^2/2}}{\sqrt{2\pi}x}$$

for $x > 0$, and

$$1 - \Phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}x}(1 + O(\frac{1}{x^2}))$$

for large x. Let

$$A_m = \{ \sum_{i=n-k+1}^{n} Y_i < c; \quad 1 \le k \le n, 1 \le n \le m \}. \tag{S1.10}$$

Obviously, $\{T > m\} = A_m$ and

$$\begin{aligned} A_m &= \{ \frac{\sum_{i=n-k+1}^{n}(Y_i - E_1)}{\sqrt{kD_1^2}} < \frac{c - kE_1}{\sqrt{kD_1^2}}, 1 \le k \le n, 1 < n \le m \} \\ &\subset \{ \frac{\sum_{i=1}^{m}(Y_i - E_1)}{\sqrt{mD_1^2}} < \frac{c - mE_1}{\sqrt{mD_1^2}} \}. \end{aligned}$$

Let $N = c/E_1 + d\sqrt{2c \ln c}$ and $n = N + k$, where $d = D_1/(E_1)^{3/2}$. It follows that

$$\begin{aligned} \frac{c - E_1 n}{D_1\sqrt{n}} &= -\frac{E_1}{D_1}\sqrt{N+k}\{1 - \frac{1}{1 + D_1/\sqrt{E_1}\sqrt{2\ln c/c} + E_1 k/c}\} \\ &\le -\frac{E_1}{D_1} A_N \sqrt{N+k} \sim -\sqrt{2\ln c} \to -\infty, \end{aligned}$$

as $c \to \infty$ since $((E_1/D_1)A_N)^2(N) \to 2\ln c$ as $c \to \infty$, where $A_N = [1 - (1 + D_1\sqrt{E_1}\sqrt{2\ln c/c})^{-1}]$. Thus, by (S1.8), we have

$$\begin{aligned} \sum_{n=N+1}^{\infty} P(T > n) &\le \sum_{n=N+1}^{\infty} P(\frac{\sum_{i=1}^{n}(Y_i - nE_1)}{D_1\sqrt{n}} < \frac{c - E_1 n}{D_1\sqrt{n}}) \\ &\le (1 + o(1)) \sum_{k=1}^{\infty} \frac{e^{-((E_1/D_1)A_N)^2(N+k)/2}}{\sqrt{2\pi}E_1/D_1 A_N \sqrt{N}} \\ &\le (1 + o(1)) \frac{\exp\{-\frac{1}{2}(E_1/D_1 A_N)^2(N))\}}{\sqrt{2\pi}E_1/D_1 A_N \sqrt{N}(1 - e^{-\frac{1}{2}(E_1/D_1 A_N)^2})} \\ &\le (1 + o(1)) \frac{1}{4\sqrt{\pi}E_1(\ln c)^{3/2}} \end{aligned}$$

for large c, and therefore,

$$\begin{aligned} E(T) &\le \sum_{n=1}^{N} P(T > n) + \frac{1}{4\sqrt{\pi}E_0(\ln c)^{3/2}} \\ &\le N + \frac{1}{4\sqrt{\pi}E_1(\ln c)^{3/2}} \\ &\le c/E_1 + d\sqrt{2c\ln c} + o(\frac{1}{\ln c}) \end{aligned} \tag{S1.11}$$

for large c. This proves the upward inequality of (S1.6).

On the other hand, let $M = c/E_1 - d\sqrt{6c\ln c}$, where $d = D_1/(E_1)^{3/2}$. Since

$$\frac{c - E_1 k}{D_1\sqrt{k}} \geq \frac{c - E_1 M}{D_1\sqrt{M}} \sim \sqrt{6\ln c} \tag{S1.12}$$

for $k \leq M$ and $c \to \infty$, it follows that $\Phi(\frac{c-E_1 M}{D_1\sqrt{M}}) \sim 1 - (2\sqrt{3\pi\ln c}c^3)^{-1}$ as $c \to \infty$. Let $l_c = (\ln c)^4$,

$$A_{m,l_c} = \{\sum_{i=n-k+1}^{n} Y_{n,k,i} < c, \quad 1 \leq k \leq l_c, l_c \leq n \leq m\}$$

and

$$B_{m,l_c} = \{\sum_{i=n-k+1}^{n} Y_{n,k,i} < c; \quad l_c < k \leq n, l_c < n \leq m\}$$

for $m > l_c$. Obviously, $A_m = A_{m,l_c} B_{m,l_c}$ for $m > l_c$. Then,

$$\begin{aligned}
&\sum_{m=l_c+1}^{M} P(B_{m,l_c}) \\
\geq\ &\sum_{m=l_c+1}^{M} P\{\frac{\sum_{i=n-k+1}^{n}(Y_i - E_1)}{D_1\sqrt{k}} < \frac{c - E_1 k}{D_1\sqrt{k}}, \\
&\qquad l_c < k \leq n, l_c < n \leq m\} \\
\geq\ &\sum_{m=l_c+1}^{M} \prod_{n=l_c+1}^{m} \prod_{k=l_c+1}^{n} P\{\frac{\sum_{i=n-k+1}^{n}(Y_i - E_1)}{D_1\sqrt{k}} < \frac{c - E_1 k}{D_1\sqrt{k}}\} \\
\geq\ &\sum_{m=l_c+1}^{M} \prod_{n=l_c+1}^{m} \prod_{k=l_c+1}^{n} \Phi(\frac{c - E_1 k}{D_1\sqrt{k}}) \geq (M - l_c)[\Phi(\frac{c - E_1 M}{D_1\sqrt{M}})]^{(M-l_c)(M-l_c+1)/2} \\
\sim\ &(M - l_c)(1 - O(\frac{1}{c\sqrt{\ln c}})) \sim M - l_c - O(\frac{1}{\sqrt{\ln c}})
\end{aligned}$$

for large c. Here, the second inequality follows from Theorem 5.1 in Esary, Proschan and Walkup (1967) and the third inequality from (S1.8). Similarly, by using (S1.9) we have $1 - O(1/(l_c c^3)) \geq P(A_m) \geq (1 - O(1/c^3))^{l_c^2} \sim 1 - O(l_c^2/c^3)$ for $m \leq l_c$ and

$$P(A_{m,l_c}) \geq (1 - O(\frac{1}{c^3}))^{(M-l_c)l_c} \sim 1 - O(\frac{l_c}{c^2})$$

for $l_c < m \leq M$ as $c \to \infty$. Thus

$$E(T) \geq \sum_{m=1}^{l_c} P(A_m) + \sum_{m=l_c+1}^{M} P(A_{m,l_c} B_{m,l_c})$$

$$
\begin{aligned}
&\geq \sum_{m=1}^{l_c} P(A_m) + \sum_{m=l_c+1}^{M} P(A_{m,l_c})P(B_{m,l_c}) \\
&\geq l_c(1 - O(\frac{l_c^2}{c^3})) + (1 - O(\frac{l_c}{c^2}))(M - l_c - O(\frac{1}{\sqrt{\ln c}})) \\
&\sim c/E_1 - 2d'\sqrt{c \ln c} - O(\frac{1}{\sqrt{\ln c}}) \qquad \text{(S1.13)}
\end{aligned}
$$

for large c. From (S1.10) and (S1.11) we see that (S1.6) is true. This completes the proof of Theorem 1.

References

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