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OPTIMAL BLOCKING AND FOLDOVER PLANS FOR REGULAR TWO-LEVEL DESIGNS

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Abstract: This article considers the problem of choosing optimal designs when both blocking and foldover techniques are employed. Based on a general decomposition structure, the treatment and block split wordlength patterns of the combined blocked design under a general foldover plan are defined. They are proved to be independent of the choice of the block foldover plans. It is shown that, for an initial unblocked design, a pair of blocking and foldover plans has minimum aberration for the combined blocked design if and only if the foldover plan has minimum aberration without consideration of blocking plans and the blocking plan has minimum aberration without consideration of foldover plans. The clear effects in the combined blocked designs are also characterized. Based on these theoretical results, a catalogue of optimal blocking and foldover plans in terms of the aberration and clear effect criteria is tabulated.

Key words and phrases: Clear effect, minimum aberration, split wordlength pattern.

1. Introduction

Two-level fractional factorial (FF) designs are commonly used in experimental investigations. An FF design is called regular if any two effects are either orthogonal or aliased. This simple aliasing structure makes regular FF designs popular in practice. In order to break the aliasing in two-level designs, a standard strategy is to augment a foldover design of the same size by exchanging the signs of one or more columns of the initial design. A foldover plan refers to the collection of columns whose signs are exchanged in the foldover design. For detailed discussion on this technique, refer to Wu and Hamada (2000), and Box, Hunter and Hunter (2005).

Typically, a classic foldover plan is to exchange the signs of all columns. This plan is usually called a full foldover plan, by which we can increase the resolution of the resulting combined design, consisting of the initial design and the foldover design, from III to at least IV. Based on the minimum aberration (MA) criterion (Fries and Hunter (1980)), Li and Mee (2002) and Li and Lin (2003) presented all optimal foldover plans for regular two-level designs with small runs. Recently, Ye and Li (2003) provided some theoretical insight into the relationships between an initial design and the resulting combined design under a general foldover plan.

Blocking, a fundamental technique in design of experiments, can effectively improve the efficiency of an experiment by eliminating systematic variations due to inhomogeneities of experimental units. How to block a design in an optimal way is a problem of practical importance. For a given blocked regular two-level design, Li and Jacroux (2007) searched by an algorithm the optimal treatment foldover plans under two proposed optimality criteria, but provided no theoretical insight. Our research focuses on the optimal plans for regular two-level designs when both blocking and foldover techniques are employed. Some general theoretical properties are obtained and described.

The paper is organized as follows. Section 2 reviews the blocking schemes of regular two-level designs and the related optimality criteria. Section 3 introduces a general decomposition structure of a blocked regular two-level foldover design. In Section 4 the treatment and block split wordlength patterns of the combined blocked design are defined under a general foldover plan. The relationships between the treatment and block wordlength patterns of an initial design and its combined blocked design are obtained in Section 5. The clear effects in the combined blocked designs are also characterized. Based on these theoretical results, a catalogue of optimal blocking and foldover plans in terms of the aberration and clear effect criteria is tabulated and compared for 8, 16, 32 and 64-run initial designs in Section 6. Section 7 concludes with some remarks.

2. Blocked Regular Two-level Designs and Related Optimality Criteria

A regular FF 2^{n-p} design D is determined by p aliasing relations, which generate p treatment defining words. The group formed by these p defining words is called the treatment contrast subgroup, denoted by G_t. Every element in G_t is called a word. The number of letters in a word is called its length. Specifically, the identity I is a word of length 0. For $i = 1, \ldots, n$, let $A_i(D)$ be the number of words of length i in G_t. Then the vector

$$W_t(D) = (A_1(D), \ldots, A_n(D)) \tag{2.1}$$

is called the treatment wordlength pattern (TWP) of D. Since $A_1(D) = A_2(D) = 0$, for simplicity, only $A_i(D)$'s ($i \geq 3$) of the TWP are displayed in this paper. For any two 2^{n-p} designs D_1 and D_2, D_1 is said to have less aberration than D_2, denoted by $W_t(D_1) < W_t(D_2)$, if $A_r(D_1) < A_r(D_2)$, where r is the smallest integer such that $A_r(D_1) \neq A_r(D_2)$. If there exists no other design with less aberration than D_1, then D_1 is said to have MA. Hereafter, all other MA criteria are defined analogously based on the relevant wordlength pattern vectors.

Arranging the 2^{n-p} design D into 2^k blocks is equivalent to selecting k independent interactions $v_1, \ldots, v_k$ of the $(n-p)$ independent factors as the k blocking factors $b_1, \ldots, b_k$, referred to as a blocking plan. We call $v_1 b_1, \ldots, v_k b_k$ the k block defining words. This blocked design is formally denoted by $(2^{n-p} : 2^k)$ design D^b. The group formed by the p treatment defining words together with the k block defining words is denoted by G_{t+b}. Let $G_{b\otimes t} = G_{t+b} \setminus G_t$. Based on the usual assumptions that the block-by-treatment interactions are negligible and that the interactions between block factors are as important as the main effects of block factors, every element in $G_{b\otimes t}$ represents a treatment effect confounded with a block effect. For $i = 1, \ldots, n$, let $A_i^b(D^b)$ be the number of words containing i treatment letters in $G_{b\otimes t}$. Then the vector

$$W_b(D^b) = (A_1^b(D^b), \ldots, A_n^b(D^b)) \tag{2.2}$$

is called the block wordlength pattern (BWP) of the blocked design D^b. For convenience, we rewrite $A_i^t(D^b) = A_i(D)$ and $W_t(D^b) = W_t(D)$. For a given design D, an MA blocking plan sequentially minimizes the components in $W_b(D^b)$. A catalogue of the MA blocking plans for some two-level designs with small runs can be found in Sun, Wu and Chen (1997).

Since there are two wordlength patterns for blocked FF designs, one should combine the components of the two wordlength patterns into one combined wordlength pattern according to an ordering scheme. Then an MA blocked design sequentially minimizes the combined wordlength pattern. Along these lines, several different combined wordlength patterns have been proposed by Sitter, Chen and Feder (1997), Chen and Cheng (1999), Zhang and Park (2000) and Cheng and Wu (2002). For detailed illustrations and comparisons on these criteria, refer to Ai and Zhang (2004). Although the orderings are different, it is essential that the number A_i^b is always put behind the number A_i^t for $i = 1, \ldots, n$, based on the popular effect hierarchy principle (Wu and Hamada (2000, Sec 3.5)).

As argued in Chen, Sun and Wu (1993), when there is no design with resolution V or higher, the MA criterion does not always lead to the best designs. To address this problem, Wu and Chen (1992) introduced the concept of clear effect. A main effect or two-factor interaction (2fi) is called clear if it is not aliased with any other main effect or 2fi. For blocked designs, a main effect or a 2fi is clear if, in addition, it does not confound with any block effect. For detailed discussion, refer to Mukerjee and Wu (2006).

3. General Foldover Structure of A Blocked Regular Two-level Design

As in Mukerjee and Wu (2006), an FF 2^{n-p} design D and its $(2^{n-p} : 2^k)$ blocked design D^b, whose levels are denoted by the elements 0 and 1 in $GF(2) =$

$\{0,1\}$, can be expressed as $D = R(C)$ and $D^b = R(C, B)$, where C and B are, respectively, $(n-p) \times n$ and $(n-p) \times k$ matrices indicating the treatment factors and the blocking factors, and $R(C)$ denotes the row space of a matrix C over $GF(2)$. Moreover, $\text{rank}(C) = n - p$, $\text{rank}(B) = k$, and no column of C belongs to the column space of B. Note that Chen and Hedayat (1996) called the matrix C the factor representation of design D.

Now a foldover plan can be denoted by a $(n+k)$-dimensional row vector $\xi = (\xi^t, \xi^b)$ with elements 0 or 1, where ξ^t and ξ^b represent the foldover plans of the treatment factors and blocking factors, respectively. For convenience, the factors corresponding to 1 are called foldover factors with respect to the foldover plan ξ, while those corresponding to 0 are referred to as unfoldover factors. Without loss of generality, we assume that, for a blocked regular $(2^{n-p} : 2^k)$ design D^b, the first $(n-p)$ columns are independent columns and the remaining $p + k$ columns are additional columns, i.e., linear combinations of the first $(n-p)$ columns. Then the factor representation of design D^b can be expressed as $(C, B) = (\mathbf{I}, C_1, B)$, where $\mathbf{I}$ is the identity matrix of order $(n-p)$ and C_1 is an $(n-p) \times p$ matrix. This split structure is called the standard factor representation of D^b. For a foldover plan $\xi = (\xi^t, \xi^b)$, the combined blocked design, denoted by D^{b*}, consisting of the initial design and its foldover design, can be described as

$$D^{b*} = R \begin{pmatrix} \mathbf{I} & C_1 & B \\ \xi_1^t & \xi_2^t & \xi^b \end{pmatrix} = R \begin{pmatrix} \mathbf{I} & C_1 & B \\ \mathbf{0}_{n-p}^T & \xi_2^{t*} & \xi^{b*} \end{pmatrix}, \tag{3.1}$$

where $\xi^t = (\xi_1^t, \xi_2^t)$, $\xi_2^{t*} = \xi_2^t - \xi_1^t C_1$ and $\xi^{b*} = \xi^b - \xi_1^t B$. Therefore we only need to consider the foldover plans of the treatment factors of the form $\xi^t = (\xi_1^t, \xi_2^t)$ with $\xi_1^t = \mathbf{0}$. These foldover plans are called the *core foldover plans* by Li and Lin (2003). Any other foldover plan must generate the same combined design as one of the core foldover plans does. In particular, when ξ is a null vector, called the null foldover plan, the combined blocked design reduces to two replicates of the blocked design D^b. For simplicity, only ξ_2^t is displayed hereafter to represent the foldover plans of the additional treatment factors.

Thus, for a $(2^{n-p} : 2^k)$ blocked design with the standard factor representation $D^b = R(\mathbf{I}, C_1, B)$, a vector $z = (z_1^T, z_2^T, z_3^T)$ with a similar dimensional partition is a word of the combined blocked design under a treatment core foldover plan ξ_2^t and a block foldover plan ξ^b if and only if z is a word of D^b and $\xi_2^t z_2 + \xi^b z_3 = 0$ (mod 2). We will show in Section 4 that the treatment and block wordlength patterns of the combined blocked design are both independent of the choice of ξ^b when the implicit blocking factor is included.

Example 1. Consider a $(2^{6-2}:2^2)$ blocked design D^b in which the six treatment factors consist of the four independent columns 1, 2, 4, 8 and the two additional columns 3 and 12, and the two blocking factors are columns 5 and 10. Note that the related FF design is denoted by 6-2.3 in Chen et al. (1993). This blocked design can be expressed as $D^b = R(\mathbf{I}, C_1, B)$, where

$$C_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^T \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}^T .$$

The overall defining contrast subgroup G_{t+b} of design D^b is expressed as $R(S)$, where

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

in which a row vector represents a word. Its treatment and block wordlength patterns are, respectively, $W_t(D^b) = (2, 0, 0, 1)$ and $W_b(D^b) = (0, 3, 6, 3)$.

Under the treatment core foldover plan $\xi_2^t = (1, 1)$ and the block foldover plan $\xi^b = (1, 0)$, the combined blocked design is a blocked $(2^{6-1}:2^2)$ design D^{b*}, whose G_{t+b} consists of the row vectors in $R(W)$ whose sum of the 5-, 6- and 7-components is 0 modulus 2. It is easily checked that G_{t+b} is of the form $R(S^*)$ with

$$S^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consequently, the treatment and block wordlength patterns of D^{b*} are, respectively, $W_t(D^{b*}) = (0, 0, 0, 1)$ and $W_b(D^{b*}) = (0, 1, 4, 1)$.

4. Treatment and Block Split Wordlength Patterns

For a pair of treatment and block foldover plans (ξ_2^t, ξ^b), let $r_t = wt(\xi_2^t)$ and $r_b = wt(\xi^b)$, where $wt(u)$ is the number of nonzero elements of the vector u. Then after permutations of treatment factors and blocking factors, a combined blocked design can be decomposed as follows:

$$R\begin{pmatrix} \mathbf{I} & C_{11} & C_{12} & B_1 & B_2 \\ \mathbf{0}_{n-p}^T & \mathbf{0}_{p-r_t}^T & \mathbf{1}_{r_t}^T & \mathbf{0}_{k-r_b}^T & \mathbf{1}_{r_b}^T \end{pmatrix},$$

where $\mathbf{0}_r$ and $\mathbf{1}_r$ are the r-dimensional column vectors of zeros and ones, respectively. For the blocked design $D^b = R(C_1, C_2, B_1, B_2)$, both the treatment factors and blocking factors are split into two parts according to whether it is folded or not. Let $A_{i_1,i_2;j_1,j_2}(D^b)$ be the number of words in the overall defining contrast subgroup G_{t+b} of D^b that consist of i_1 factors in $\mathbf{I} \cup C_{11}$, i_2 factors in C_{12}, j_1 factors in B_1 and j_2 factors in B_2. A collection of these numbers is called the *split wordlength pattern* of D^b. Let $A^t_{i_1,i_2}(D^b) = A_{i_1,i_2;0,0}(D^b)$ and $A^b_{i_1,i_2}(D^b) = \sum_{j_1+j_2\geq 1} A_{i_1,i_2;j_1,j_2}(D^b)$. The matrices $[A^t_{i_1,i_2}(D^b)]$ and $[A^b_{i_1,i_2}(D^b)]$ are called, respectively, the *treatment and block split wordlength patterns* of D^b under the foldover plan (ξ^t_2, ξ^b).

Because of the sequential nature of a foldover plan, the combined design has been explicitly classified into the initial design and the foldover design, i.e., there exists an additional blocking factor which takes the value 0 for the first half and 1 for the other half. This blocking factor is called the implicit blocking factor. In this way, the combined design is blocked into 2^{k+1} blocks. Its design matrix has the form

$$D^{b*} = R\begin{pmatrix} \mathbf{I} & C_{11} & C_{12} & B_1 & B_2 & \mathbf{0}_{n-p} \\ \mathbf{0}^T_{n-p} & \mathbf{0}^T_{p-r_t} & \mathbf{1}^T_{r_t} & \mathbf{0}^T_{k-r_b} & \mathbf{1}^T_{r_b} & 1 \end{pmatrix}. \tag{4.1}$$

Hereafter, we always consider the combined blocked designs with the implicit blocking factor in the last column.

As before, let $A_{i_1,i_2;j_1,j_2,j_3}(D^{b*})$ be the number of words in the overall defining contrast subgroup of D^{b*} that consist of i_1 factors in $\mathbf{I} \cup C_{11}$, i_2 factors in C_{12}, j_1 factors in B_1, j_2 factors in B_2, and j_3 factor in the last blocking factor. It can be easily checked that

$$A_{i_1,i_2;j_1,j_2,j_3}(D^{b*}) = \frac{1}{2}\Big[1 + (-1)^{i_2+j_2+j_3}\Big] A_{i_1,i_2;j_1,j_2}(D^b). \tag{4.2}$$

Thus, the treatment and block split wordlength patterns of D^{b*} can be expressed in terms of those of D^b as follows:

$$A^t_{i_1,i_2}(D^{b*}) = A_{i_1,i_2;0,0,0}(D^{b*}) = \frac{1}{2}\Big[1 + (-1)^{i_2}\Big] A^t_{i_1,i_2}(D^b), \tag{4.3}$$

$$\begin{aligned} A^b_{i_1,i_2}(D^{b*}) &= \sum_{j_1+j_2+j_3\geq 1} A_{i_1,i_2;j_1,j_2,j_3}(D^{b*}) \\ &= \sum_{j_1+j_2\geq 1} \sum_{j_3=0}^{1} A_{i_1,i_2;j_1,j_2,j_3}(D^{b*}) + A_{i_1,i_2;0,0,1}(D^{b*}) \\ &= \sum_{j_1+j_2\geq 1} A_{i_1,i_2;j_1,j_2}(D^b) + \frac{1}{2}\Big[1 - (-1)^{i_2}\Big] A_{i_1,i_2;0,0}(D^b) \\ &= A^b_{i_1,i_2}(D^b) + \frac{1}{2}\Big[1 - (-1)^{i_2}\Big] A^t_{i_1,i_2}(D^b). \end{aligned} \tag{4.4}$$

Let $\mathcal{E}$ be the set of all nonnegative even numbers. Then the following two relationships follow from formulas (4.3) and (4.4), respectively.

Theorem 1. *The treatment and block split wordlength patterns of the blocked design D^b and its combined blocked design D^{b*} under a foldover plan have the relationships*

$$A^t_{i_1,i_2}(D^{b*}) = 1_{i_2\in\mathcal{E}} A^t_{i_1,i_2}(D^b), \tag{4.5}$$

$$A^b_{i_1,i_2}(D^{b*}) = A^b_{i_1,i_2}(D^b) + A^t_{i_1,i_2}(D^b) - 1_{i_2\in\mathcal{E}} A^t_{i_1,i_2}(D^b), \tag{4.6}$$

where $1_{[\cdot]}$ takes on the value 1 or 0 depending on whether the condition $[\cdot]$ holds or not.

Note that the above treatment and block split wordlength patterns of D^{b*} are both independent of the numbers j_1 and j_2. This implies that they are *independent of the split structure of the blocking factor columns* and so are *independent of the choice of the block foldover plan ξ^b*. In view of this, we hereafter only need to consider the treatment core foldover plans and simply call them foldover plans without consideration of the block foldover plans.

5. Characterization of Optimal Blocking and Foldover Plans

From Theorem 1, we can get the treatment and block wordlength patterns of the combined design D^{b*}:

$$A^t_i(D^{b*}) = \sum_{i_1+i_2=i} A^t_{i_1,i_2}(D^{b*}) = \sum_{i_2\in\mathcal{E}, i_2\le i} A^t_{i-i_2,i_2}(D^b), \tag{5.1}$$

$$A^b_i(D^{b*}) = \sum_{i_1+i_2=i} A^b_{i_1,i_2}(D^{b*}) = A^b_i(D^b) + A^t_i(D^b) - A^t_i(D^{b*}). \tag{5.2}$$

In order to compare and select optimal blocked designs, one usually arranges the treatment and block wordlength patterns into a combined sequence based on some ordering scheme and then sequentially minimizes this sequence among all blocked designs with the same parameters. For a fixed blocked design D^b, if there exists a foldover plan for which the corresponding combined blocked design has MA, then it is called a *minimum aberration foldover plan* for the combined blocked design D^{b*}. We have the following conclusion.

Theorem 2. *For a given blocked design D^b, a foldover plan has minimum aberration for the combined blocked design D^{b*} if and only if it sequentially minimizes $A^t_i(D^{b*})$ for $i = 3, \ldots, k$.*

Although the orderings of the treatment and block wordlength patterns vary with the optimality criteria, the validity of our results only requires that $A^t_i(D^{b*})$

is always put ahead of $A_i^b(D^{b*})$. Since all the optimality criteria reviewed in Section 2 satisfy this requirement, all the results in this section hold true for these criteria.

Note that for $p = 1$, there are only two distinct ways to generate a combined design: the null foldover plan and the foldover plan. Because the combined design under the foldover plan is the 2^n full factorial design, it is an optimal foldover plan. Theorem 2 implies that for a fixed initial blocked design, its MA foldover plan is the MA foldover plan for the unblocked case. Therefore, we need only consider the MA foldover plans for the unblocked cases, some of which can be found in Li and Lin (2003).

For a given 2^{n-p} FF design D, a blocking plan is said to have MA if it sequentially minimizes the block wordlength pattern $W_b(D^b)$ of the blocked design D^b. For the FF design D and a fixed foldover plan ξ, a blocking plan has MA if it sequentially minimizes the block wordlength pattern $W_b(D^{b*})$ of the combined blocked design D^{b*}. By noting that the last two terms on the right side of (5.2) are fixed in this case, we have the following conclusion.

Theorem 3. *For a 2^{n-p} FF design D with a fixed foldover plan, a blocking plan has MA if and only if it has MA for D without consideration of the foldover plans.*

As for the optimal blocking and foldover plans, which together ensure that the combined blocked design of an FF design has MA, by combining Theorems 2 and 3 we have the following result.

Theorem 4. *For a 2^{n-p} FF design D, a pair of blocking and foldover plans has MA for the combined blocked design D^{b*} if and only if the foldover plan has MA for the design D without consideration of the blocking plans, and the blocking plan has MA for D without consideration of the foldover plans.*

Next we consider the problem of efficient enumeration of the number of clear effects in the combined blocked designs. First, we consider a 2^{n-p} FF design D. Let $C = (\mathbf{I}, C_1)$ be its factor representation and $\xi^t = (\mathbf{0}_{n-p}^T, \xi_2^t)$ be a core foldover plan. Because a vector z is a word of a 2^{n-p} design $D = R(C)$ if and only if $Cz = 0$, z is a word in the treatment defining contrast subgroup of the combined design if and only if z is a word in the defining contrast subgroup of design D and $\xi z = 0$. Thus, we can get the treatment defining contrast subgroup of the combined design from the defining contrast subgroup of design D and then search for the clear effects without consideration of the blocking factors.

Furthermore, a treatment effect is confounded with a block effect in the initial blocked $(2^{n-p} : 2^k)$ design $D^b = R(C, B)$ if and only if a word $z = (z_1, z_2)$ with

n and k dimensional partition appears in $G_{b\otimes t}$ of D^b. Based on the relationship between the words of the blocked design D^b and the combined blocked design D^{b*}, the following conclusion can be obtained.

Theorem 5. *A treatment effect is confounded with a block effect in the combined blocked design D^{b*} if and only if it is confounded with a block effect in the initial blocked design D^b.*

Let M be the $2^{n-p} \times (k+1)$ submatrix of design D^b consisting of the k blocking columns and a treatment effect column under consideration, which is a main effect or a two-factor interaction. Ai and He (2006) showed that the treatment effect is not confounded with any block effect if and only if the number of null row vectors of M is exactly $2^{n-p-k-1}$. This can be used to efficiently identify whether a treatment effect is confounded with block effects and to screen out the clear treatment effects in the combined blocked design D^{b*}.

6. Optimal Blocking and Foldover Plans for 8, 16, 32 and 64-run Designs

In this section, we present some optimal blocking and foldover plans for given 2^{n-p} FF designs.

As in Chen et al. (1993) and Mukerjee and Wu (2006), we put the column set of the factor representation of saturated designs in Table 1 in Yates order. To save space, we represent a 2^{n-p} design as a collection of columns from the factor representation matrices in Table 1. For example, the factor representation matrix consists of the first 4 rows and 15 columns for the 16-run saturated design. Let $1, 2, 3$, and 4 be the four independent factors. Then all interactions are denoted as their products. A column $z = (z_1, z_2, z_3, z_4)'$ with z_i's being 1 or 0 corresponds to an effect of the form $1^{z_1}2^{z_2}3^{z_3}4^{z_4}$. Search starts with the initial 2^{n-p} designs given in the catalogue of Chen et al. (1993). In the tables, the notation $n - p.j$ under the column "Design" is the same as in Chen et al.. Their factor representations are given by a subset of n columns, consisting of $(n-p)$ independent columns and p additional columns. Only the latter are presented in the second columns of each table. Under the "Additional Col." of each design, the optimal core foldover plan is displayed. The optimal blocking columns are indicated in the third column of each table.

For ease of comparison and selection of optimal FF designs, the treatment wordlength pattern (TWP) and the block wordlength pattern (BWP) of the optimal combined blocked design are listed in the fourth and fifth columns, respectively. Note that only some initial components of TWP and BWP are given for the sake of brevity. The numbers of clear main effects and clear 2fi's, denoted by cc_1 and cc_2, are given in the sixth column.

In order to compare combined blocked designs in terms of their aberration, we adopt the following combined wordlength pattern for illustration:

$$W_c = (A_1^b, A_3^t, A_2^b, A_4^t, A_5^t, A_3^b, A_6^t, \ldots, A_{2j-1}^t, A_j^b, A_{2j}^t, \ldots). \tag{6.1}$$

This wordlength pattern was suggested by Zhang and Park (2000) and Cheng and Wu (2002). These designs are rank-ordered according to (6.1). The rank of each design is given in the last column. Certainly, other criteria based on different combined wordlength patterns can be adopted to rank-order designs.

In selecting and recommending good combined blocked designs, we use the following five criteria: TWP, BWP, cc_1, cc_2, and rank, which are given in the last four columns of the tables. Suppose q criteria are under consideration. An FF design D is called *inadmissible* if there exists another design D_1 such that D_1 is at least as "good" as D for all q criteria and better than D for at least one of the criteria. Otherwise, D is admissible. Here we use TWP, BWP, cc_1, cc_2, and rank as the criteria to consider admissibility. Only admissible initial designs are provided in the tables. The complete tables are available upon request.

Example 2. Revisit the 2^{6-2} FF design D in Example 1. It is known from Li and Lin (2003) that the MA treatment core foldover plan is $\xi_0 = (1, 1)$. On the other hand, the MA blocking plan for D in 2^2 blocks will select columns 5 and 10 as the two blocking factors, denoted by $\mathbf{b}_0 = (5, 10)$. Thus, the pair of blocking and foldover plans $\mathbf{b}_0$ and ξ_0 is simply the optimal plan for the combined blocked designs with consideration of the implicit blocking factor. Here the optimal combined blocked design is a $(2^{6-1} : 2^3)$ blocked design D^{b*}. Furthermore, it is independent of the choice of the block foldover plans. Thus, D^{b*} has the form

$$D^{b*} = R \begin{pmatrix} \mathbf{I}_4 & C_1 & B & \mathbf{0}_4 \\ \mathbf{0}_4^T & \mathbf{1}_2^T & \mathbf{0}_2^T & 1 \end{pmatrix}.$$

Based on the overall defining contrast subgroup $R(S)$ of the blocked design D^b in Example 1, we can easily obtain its treatment and block split wordlength patterns as

$$\left[A_{i_1,i_2}^t(D^b)\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \left[A_{i_1,i_2}^b(D^b)\right] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 6 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now, by applying formulas (5.1) and (5.2), it can be shown that the TWP and BWP of design D^{b*} are $(0,0,0,1)$ and $(0,3,8,3)$, respectively.

To judge whether a main effect or 2fi is clear, we need to check if the treatment effect is aliased with any other main effect or 2fi, and then to check if the treatment effect is confounded with a block effect. Based on $R(S)$, we can easily obtain the treatment defining contrast subgroup of the combined blocked design D^{b*} as $G_t(D^{b*}) = R(\mathbf{1}_6^T)$. Apparently, none of the main effects or 2fi's are aliased with other main effects or 2fi's.

Next, by Theorem 5 we need only check if the number of null row vectors of M, a submatrix of design D^b consisting of the two blocking columns and the treatment effect column under consideration, is exactly $2^{4-2-1} = 2$. By following this process for every main effect and 2fi, we obtain $cc_1 = 6$ and $cc_2 = 12$.

7. Concluding Remarks and Further Work

In this paper, based on a general decomposition structure of both blocking and foldover plans, we show that, for a 2^{n-p} FF design D, a pair of blocking and foldover plans has MA for the combined blocked design D^{b*} if and only if the foldover plan has MA for design D without consideration of the blocking plans and the blocking plan has MA for D without consideration of the foldover plans. We also show that a treatment effect is confounded with a block effect in the combined blocked design D^{b*} if and only if it is confounded with a block effect in the blocked design D^b.

Note that all of these minimum aberration results hold true for any aberration criterion for blocked designs which orders A_i^b behind A_i^t. This greatly simplifies the search from the joint pairs of blocking and foldover plans to two separate searches for optimal blocking and foldover plans respectively.

Finally, it should be mentioned that all the previous results are based on the assumption that the implicit blocking factor that takes 0 for the initial design and 1 for the foldover part is included and its effect is significant. Otherwise, some results may not hold. The most important is that, in the latter case, the block split wordlength pattern in Theorem 1 may depend on the choice of the block foldover plans, although it can be similarly expressed in terms of the initial blocked design. Some further work in this direction is in progress.

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Appendix

Table 1. Factor representation matrices for 8, 16, 32 and 64-run two-level designs.

1	**2**	3	**4**	5	6	7	**8**	9	10	11	12	13	14	15	**16**	17	18	19	20	21
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	23	24	25	26	27	28	29	30	31	**32**	33	34	35	36	37	38	39	40	41	42
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0
0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Note: The factor representation matrix consists of the first 3 rows and 7 columns for the 8-run saturated design, consists of the first 4 rows and 15 columns for the 16-run saturated design, consists of the first 5 rows and 31 columns for the 32-run saturated design, and is the whole matrix for the 64-run saturated design. Independent columns are numbered 1, 2, 4, 8, 16 and 32 in bold face.

Table 2. Optimal blocking and foldover plans for 8-run ($2^{n-p}:2$) designs.

Design	Additional Col.	Blocking Col.	TWP	BWP		
	Foldover plan		$(A_3^t,\ldots,A_6^t)$	$(A_1^b,\ldots,A_4^b)$	cc_1, cc_2	Rank
4-1.1	7	3	0 0	0 2 0 1	4, 4	3
	1					
4-1.2	6	3	0 0	0 1 2 0	4, 5	1
	1					
5-2.1	3 5	6	0 1 0	0 2 4 0	5, 4	1
	1 1					
6-3.1	3 5 6	7	0 3 0 0	0 3 8 0	6, 0	1
	1 1 1					

Table 3. Optimal blocking and foldover plans for 16-run $(2^{n-p}:2)$ designs.

Design	Additional Col. Foldover plan	Blocking Col.	TWP $(A_3^t,\ldots,A_7^t)$	BWP $(A_1^b,\ldots,A_4^b)$	cc_1, cc_2	Rank
5-1.1	15 1	3	0 0 0	0 1 1 0	5, 9	3
5-1.2	7 1	11	0 0 0	0 0 2 1	5, 10	1
6-2.1	7 11 1 0	13	0 1 0 0	0 0 4 2	6, 9	1
6-2.2	3 13 1 1	6	0 0 1 0	0 1 3 2	6, 14	4
6-2.3	3 12 1 1	5	0 0 0 1	0 1 4 1	6, 14	3
7-3.1	7 11 13 1 0 0	14	0 3 0 0 0	0 0 7 4	7, 6	1
7-3.2	3 5 14 1 1 1	9	0 1 2 0 0	0 1 6 4	7, 14	4
7-3.3	3 5 10 1 1 1	12	0 2 0 1 0	0 1 6 3	7, 8	5
8-4.1	7 11 13 14 1 1 0 0	3	0 6 0 0 0	0 4 0 16	8, 0	6
8-4.2	3 5 9 14 1 1 1 1	15	0 3 4 0 0	0 1 10 16	8, 12	1
8-4.5	3 5 6 9 1 1 1 1	14	0 5 0 2 0	0 1 10 6	8, 3	2
8-4.6	3 5 6 7 1 1 1 0	9	0 7 0 0 0	0 1 10 4	8, 6	3
9-5.1	3 5 9 14 15 1 1 1 1 0	6	0 6 8 0 0	0 4 8 16	9, 8	5
9-5.3	3 5 6 9 14 1 1 1 1 0	15	0 10 0 4 0	0 2 14 8	9, 2	2
9-5.4	3 5 6 9 10 1 1 1 1 1	13	0 9 0 6 0	0 2 14 9	9, 0	1
10-6.1	3 5 6 9 14 15 1 1 1 1 0 1	10	0 18 0 8 0	0 4 16 12	10, 0	4
10-6.2	3 5 6 9 10 13 1 1 1 1 1 0	14	0 16 0 12 0	0 3 19 13	10, 0	2
10-6.3	3 5 6 9 10 12 1 1 1 1 1 1	7	0 15 0 15 0	0 3 20 13	10, 0	1
11-7.1	3 5 6 9 10 13 14 1 1 1 1 1 0 0	15	0 26 0 24 0	0 4 25 20	11, 0	2
11-7.2	3 5 6 7 9 10 12 1 1 1 0 1 1 1	11	0 25 0 27 0	0 4 26 19	11, 0	1
12-8.2	3 5 6 7 9 10 11 12 1 1 1 0 1 1 0 1	13	0 38 0 52 0	0 5 34 28	12, 0	1

Table 4. Optimal blocking and foldover plans for 16-run $(2^{n-p}:2^2)$ designs.

Design	Additional Col. Foldover plan	Blocking Col.	TWP $(A_3^t,\ldots,A_7^t)$	BWP $(A_1^b,\ldots,A_4^b)$	cc_1, cc_2	Rank
5-1.1	15 1	3 5	0 0 0	0 3 3 0	5, 7	3
5-1.2	7 1	3 13	0 0 0	0 2 4 1	5, 8	1
6-2.1	7 11 1 0	3 13	0 1 0 0	0 3 8 2	6, 8	2
6-2.3	3 12 1 1	5 10	0 0 0 1	0 3 8 3	6, 12	1
7-3.1	7 11 13 1 0 0	3 5	0 3 0 0 0	0 9 0 16	7, 3	5
7-3.2	3 5 14 1 1 1	6 9	0 1 2 0 0	0 5 12 6	7, 12	1
7-3.3	3 5 10 1 1 1	6 9	0 2 0 1 0	0 5 12 5	7, 8	2
7-3.5	3 5 6 1 1 1	7 9	0 3 0 0 0	0 5 12 4	7, 4	3
8-4.1	7 11 13 14 1 1 0 0	3 5	0 6 0 0 0	0 12 0 32	8, 0	5
8-4.2	3 5 9 14 1 1 1 1	6 10	0 3 4 0 0	0 9 12 16	8, 10	4
8-4.3	3 5 10 12 1 1 1 1	6 11	0 5 0 2 0	0 7 18 10	8, 4	1
8-4.4	3 5 6 15 1 1 1 0	7 9	0 3 4 0 0	0 8 16 11	8, 8	3
9-5.1	3 5 9 14 15 1 1 1 1 0	6 10	0 6 8 0 0	0 12 16 32	9, 8	4
9-5.2	3 5 10 12 15 1 1 1 1 1	6 11	0 9 0 6 0	0 9 27 18	9, 0	1
9-5.3	3 5 6 9 14 1 1 1 1 0	7 10	0 10 0 4 0	0 10 24 18	9, 2	3
10-6.1	3 5 6 9 14 15 1 1 1 1 0 1	7 10	0 18 0 8 0	0 13 32 32	10, 0	2
10-6.2	3 5 6 9 10 13 1 1 1 1 1 0	7 11	0 16 0 12 0	0 12 36 30	10, 0	1
10-6.3	3 5 6 9 10 12 1 1 1 1 1 1	7 11	0 15 0 15 0	1 9 36 39	9, 0	3
11-7.1	3 5 6 9 10 13 14 1 1 1 1 1 0 0	7 11	0 26 0 24 0	0 15 48 48	11, 0	1
11-7.2	3 5 6 7 9 10 12 1 1 1 0 1 1 1	11 13	0 25 0 27 0	1 12 48 57	10, 0	2
12-8.1	3 5 6 9 10 13 14 15 1 1 1 1 1 0 0 1	7 11	0 39 0 48 0	0 18 64 72	12, 0	1
12-8.2	3 5 6 7 9 10 11 12 1 1 1 0 1 1 0 1	13 14	0 38 0 52 0	1 15 63 84	11, 0	2

Table 5. Optimal blocking and foldover plans for 32-run ($2^{n-p}:2$) designs.

Design	Additional Col. Foldover plan	Blocking Col.	TWP $(A_3^t,\ldots,A_7^t)$	BWP $(A_1^b,\ldots,A_4^b)$	cc_1, cc_2	Rank
6-1.1	31 1	7	0 0 0 0 0	0 0 2 0	6, 15	1
7-2.1	7 27 1 0	13	0 0 1 0 0	0 0 2 3	7, 21	5
7-2.2	7 25 1 1	11	0 0 0 1 0	0 0 3 2	7, 21	2
7-2.3	7 11 1 0	29	0 1 0 0 0	0 0 0 6	7, 15	7
7-2.5	3 28 1 1	13	0 0 0 0 1	0 0 3 3	7, 21	1
8-3.2	7 11 21 0 1 1	25	0 1 0 2 0	0 0 4 4	8, 22	3
8-3.3	7 11 19 1 1 0	29	0 2 0 0 0	0 0 0 12	8, 16	9
8-3.4	7 11 13 1 0 0	30	0 3 0 0 0	0 0 0 11	8, 13	10
8-3.5	3 13 22 1 1 1	25	0 0 2 1 0	0 0 4 5	8, 28	1
8-3.6	3 5 30 1 1 1	15	0 1 0 2 0	0 0 5 4	8, 22	4
8-3.7	3 13 21 1 1 0	26	0 1 1 0 1	0 0 3 6	8, 22	5
9-4.1	7 11 19 29 1 1 0 0	30	0 2 4 0 0	0 0 4 12	9, 24	4
9-4.5	7 11 13 14 1 1 0 0	19	0 6 0 0 0	0 0 4 8	9, 8	9
9-4.8	3 12 21 26 1 1 1 0	31	0 1 4 2 0	0 0 6 8	9, 30	1
10-5.1	7 11 19 29 30 1 1 0 0 0	5	0 4 8 0 0	0 2 4 12	10, 22	10
10-5.4	7 11 13 14 19 0 0 1 1 1	21	0 6 0 8 0	0 0 8 12	10, 17	4
10-5.8	3 5 14 22 25 1 1 1 1 1	31	0 2 8 4 0	0 0 8 16	10, 33	1
11-6.1	7 11 13 19 21 25 1 0 1 1 0 0	14	0 10 0 16 0	0 0 13 15	11, 10	5
11-6.2	7 11 13 14 19 21 0 1 1 0 1 0	25	0 10 0 16 0	0 0 12 16	11, 9	4
11-6.3	3 5 14 22 25 31 1 1 1 1 1 1	10	0 4 14 8 0	0 2 8 20	11, 32	10
11-6.6	3 5 10 23 27 28 1 1 1 0 0 0	13	0 5 12 7 4	0 1 10 21	11, 27	6
11-6.7	3 5 9 22 26 29 1 1 1 1 0 0	14	0 7 8 7 8	0 0 13 18	11, 22	1
11-6.10	3 5 9 14 18 29 1 1 1 1 1 1	31	0 6 10 8 4	0 1 10 20	11, 24	7
12-7.1	7 11 13 14 19 21 25 1 0 1 0 1 0 0	22	0 16 0 30 0	0 0 17 22	12, 5	4
12-7.2	7 11 13 14 19 21 22 0 1 0 1 1 1 0	25	0 15 0 32 0	0 0 16 24	12, 0	3
12-7.3	3 5 9 14 22 26 29 1 1 1 1 1 0 0	17	0 11 14 15 12	0 1 16 23	12, 20	6
12-7.5	3 5 10 12 22 27 29 1 1 1 1 1 1 0	15	0 10 18 10 12	0 2 12 29	12, 20	9
12-7.6	3 5 10 12 22 25 31 1 1 1 1 1 1 0	17	0 10 16 12 16	0 2 12 28	12, 18	8
12-7.7	3 5 6 15 23 25 30 1 1 1 0 0 0 0	10	0 10 20 8 8	0 2 12 29	12, 28	10
12-7.8	3 5 9 14 17 22 26 1 1 1 1 1 1 1	28	0 12 13 12 15	0 0 17 26	12, 17	1
12-7.9	3 5 9 14 15 22 26 1 1 1 1 0 0 0	29	0 12 14 12 12	0 0 16 27	12, 17	2
12-7.10	3 5 9 14 18 20 31 1 1 1 1 1 1 1	24	0 10 18 10 12	0 1 15 26	12, 19	5

Table 5. (Cont'd).

Design	Additional Col. Foldover plan	Blocking Col.	TWP $(A_3^t, \ldots, A_7^t)$	BWP $(A_1^b, \ldots, A_4^b)$	cc_1, cc_2	Rank
13-8.1	7 11 13 14 19 21 22 25 0 1 0 1 1 1 0 0	26	0 23 0 56 0	0 0 22 32	13, 0	2
13-8.2	3 5 9 14 17 22 26 28 1 1 1 1 1 1 1 1	15	0 18 20 24 28	0 1 21 33	13, 11	6
13-8.3	3 5 9 14 15 22 26 29 1 1 1 1 0 1 0 0	17	0 18 20 28 24	0 1 21 32	13, 19	7
13-8.4	3 5 9 14 15 22 26 28 1 1 1 1 0 1 0 0	17	0 18 20 24 28	0 1 20 34	13, 11	5
13-8.5	3 5 9 14 15 17 22 26 1 1 1 1 0 1 0 0	28	0 20 18 22 30	0 0 22 35	13, 16	1
13-8.9	3 5 9 15 18 20 24 31 1 1 1 0 1 1 1 1	14	0 15 27 21 27	0 1 21 36	13, 15	3
13-8.10	3 5 6 9 14 17 26 29 1 1 1 1 0 1 1 0	22	0 17 24 18 32	0 1 21 34	13, 13	4
14-9.1	7 11 13 14 15 17 22 26 28 1 1 1 1 1 0 0 0 0	28	0 33 0 96 0	0 0 28 44	14, 0	1
14-9.2	3 5 9 14 15 17 22 26 28 1 1 1 1 0 1 1 0 0	23	0 28 27 42 54	0 1 27 44	14, 12	5
14-9.3	3 5 9 14 15 17 22 23 26 1 1 1 1 0 1 1 0 0	28	0 31 24 40 56	0 0 28 46	14, 15	2
14-9.4	3 5 9 15 18 20 24 30 31 1 1 1 0 1 1 1 0 1	14	0 22 40 36 56	0 2 26 46	14, 8	7
14-9.5	3 5 9 14 15 18 20 24 31 1 1 1 1 0 1 1 1 1	19	0 22 41 36 52	0 2 26 45	14, 8	8
14-9.7	3 5 9 14 15 18 20 24 30 1 1 1 1 0 1 1 1 0	19	0 22 40 36 56	0 2 25 47	14, 8	6
14-9.9	3 5 6 9 14 17 22 26 27 1 1 1 1 0 1 0 1 0	28	0 24 36 36 60	0 1 27 48	14, 11	3
14-9.10	3 5 6 9 14 15 17 26 29 1 1 1 1 0 1 1 1 0	22	0 27 32 32 64	0 1 27 45	14, 7	4
15-10.1	7 11 13 14 19 21 22 25 26 28 1 1 1 1 0 0 1 1 0 0	31	0 45 0 160 0	0 0 35 60	15, 0	1
15-10.2	3 5 9 14 15 17 22 23 26 28 1 1 1 1 0 1 1 0 0 0	27	0 41 36 72 96	0 1 34 58	15, 13	5
15-10.3	3 5 9 14 15 17 22 23 26 27 1 1 1 1 0 1 1 0 1 0	28	0 45 32 72 96	0 0 35 60	15, 14	2
15-10.4	3 5 6 9 14 17 22 26 27 28 1 1 1 1 0 1 0 1 0 1	15	0 33 54 60 108	0 2 33 61	15, 6	6
15-10.5	3 5 6 9 14 15 17 22 26 29 1 1 1 1 0 1 1 0 1 0	27	0 37 48 56 112	0 2 32 58	15, 4	7
15-10.6	3 5 6 9 14 15 17 22 26 27 1 1 1 1 0 1 1 0 1 0	28	0 36 48 61 112	0 1 34 63	15, 7	4
15-10.7	3 5 9 14 18 20 23 24 27 29 1 1 1 1 1 1 1 1 1 1	31	0 33 44 96 72	0 3 28 64	15, 14	8
16-11.1	7 11 13 14 19 21 22 25 26 28 31 1 1 1 1 0 0 1 1 0 0 0	3	0 60 0 256 0	0 8 0 192	16, 0	10
16-11.2	3 5 9 14 15 17 22 23 26 27 28 1 1 1 0 1 1 1 0 1 0 0	29	0 57 48 120 160	0 1 42 76	16, 14	2
16-11.3	3 5 6 9 14 15 17 22 26 27 28 1 1 1 1 0 1 1 0 1 0 1	23	0 47 72 98 192	0 2 41 80	16, 4	3
16-11.5	3 5 9 14 18 20 23 24 27 29 31 1 1 1 1 0 1 1 0 1 1 0	28	0 51 64 102 192	0 1 42 82	16, 3	1
16-11.6	3 5 9 14 18 20 23 24 27 29 31 1 1 1 1 1 1 1 1 1 1 0	6	0 45 60 160 120	0 5 34 72	16, 14	9
16-11.9	3 5 6 9 10 14 15 17 22 26 29 1 1 1 1 1 0 1 1 0 0 0	23	0 53 52 136 144	0 3 38 72	16, 17	5
17-12.1	3 5 9 14 15 17 22 23 26 27 28 29 1 1 1 1 0 1 1 0 1 0 1 0	6	0 76 64 192 256	0 8 16 176	17, 16	5
17-12.2	3 5 6 9 14 15 17 22 23 26 27 28 1 1 1 1 0 1 1 0 1 1 0 1	29	0 64 96 156 320	0 2 50 104	17, 2	1
17-12.5	3 5 6 9 10 14 15 17 22 23 26 29 1 1 1 1 1 0 1 1 0 1 0 1	27	0 96 0 348 0	0 3 48 67	17, 2	4
18-13.1	3 5 6 9 14 15 17 22 23 26 27 28 29 1 1 1 1 0 1 1 0 1 1 0 1 0	10	0 84 128 240 512	0 8 32 184	18, 0	5
18-13.2	3 5 6 9 10 14 15 17 22 23 26 27 28 1 1 1 1 1 0 1 1 0 1 0 1 0	29	0 126 0 532 0	0 3 59 85	18, 0	2
18-13.3	3 5 6 7 9 10 11 17 18 19 28 29 30 1 1 1 0 1 1 0 1 1 0 1 0 0	31	0 78 144 228 528	0 3 60 132	18, 0	1

Table 6. Optimal blocking and foldover plans for 32-run ($2^{n-p}:2^2$) designs.

Design	Additional Col. Foldover plan	Blocking Col.	TWP ($A_3^t,\ldots,A_7^t$)	BWP ($A_1^b,\ldots,A_4^b$)	cc_1, cc_2	Rank
6-1.1	31 1	3 13	0 0 0 0	0 1 4 1	6, 14	1
7-2.1	7 27 1 0	11 21	0 0 1 0 0	0 1 6 5	7, 20	7
7-2.3	7 11 1 0	13 19	0 1 0 0 0	0 0 7 6	7, 15	2
7-2.5	3 28 1 1	13 22	0 0 0 0 1	0 0 7 7	7, 21	1
8-3.1	7 11 29 1 0 0	13 19	0 1 2 0 0	0 1 10 10	8, 21	5
8-3.2	7 11 21 0 1 1	13 19	0 1 0 2 0	0 1 10 10	8, 21	1
8-3.4	7 11 13 1 0 0	14 17	0 3 0 0 0	0 1 10 8	8, 12	6
8-3.5	3 13 22 1 1 1	7 24	0 0 2 1 0	0 2 8 10	8, 26	8
8-3.8	3 12 21 1 1 1	5 26	0 0 2 1 0	0 2 8 10	8, 26	8
9-4.1	7 11 19 29 1 1 0 0	5 27	0 2 4 0 0	0 4 8 20	9, 20	9
9-4.3	7 11 21 25 0 1 1 0	6 26	0 3 0 4 0	0 2 14 15	9, 18	3
9-4.6	3 13 21 26 1 1 1 0	6 25	0 1 4 2 0	0 2 14 17	9, 28	1
9-4.9	3 5 9 30 1 1 1 1	15 18	0 3 0 4 0	0 2 14 15	9, 19	3
9-4.10	3 5 10 28 1 1 1 1	12 23	0 2 3 1 1	0 2 14 16	9, 22	2
10-5.1	7 11 19 29 30 1 1 0 0 0	5 9	0 4 8 0 0	0 6 12 24	10, 20	9
10-5.3	7 11 13 19 21 0 0 1 1 0	9 22	0 6 0 8 0	0 3 19 23	10, 14	4
10-5.6	3 13 21 25 30 1 1 1 1 1	6 26	0 3 7 4 0	0 4 18 23	10, 26	6
10-5.8	3 5 14 22 25 1 1 1 1 1	10 21	0 2 8 4 0	0 4 16 28	10, 29	5
10-5.9	3 5 14 23 26 1 1 1 0 0	9 21	0 3 6 4 2	0 3 19 26	10, 24	1
11-6.1	7 11 13 19 21 25 1 0 1 1 0 0	3 28	0 10 0 16 0	0 4 26 34	11, 10	5
11-6.2	7 11 13 14 19 21 0 1 1 0 1 0	9 22	0 10 0 16 0	0 4 25 36	11, 9	4
11-6.3	3 5 14 22 25 31 1 1 1 1 1 0	7 10	0 4 14 8 0	0 6 23 36	11, 30	9
11-6.4	3 5 14 22 26 29 1 1 1 1 0 0	9 18	0 6 10 8 4	0 5 24 36	11, 22	7
11-6.6	3 5 10 23 27 28 1 1 1 0 0 0	15 17	0 5 12 7 4	0 5 24 37	11, 23	6
11-6.7	3 5 9 22 26 29 1 1 1 1 0 0	14 17	0 7 8 7 8	0 4 26 37	11, 20	1
11-6.8	3 5 9 22 26 28 1 1 1 1 1 0	14 17	0 8 8 4 8	0 4 25 38	11, 16	3
11-6.9	3 5 9 14 22 26 1 1 1 1 1 0	15 18	0 7 9 6 6	0 4 25 39	11, 20	2
12-7.1	7 11 13 14 19 21 25 1 0 1 0 1 0 0	3 28	0 16 0 30 0	0 5 34 50	12, 5	3
12-7.3	3 5 9 14 22 26 29 1 1 1 1 1 0 0	15 17	0 11 14 15 12	0 6 32 52	12, 18	5
12-7.5	3 5 10 12 22 27 29 1 1 1 1 1 1 0	7 18	0 10 18 10 12	0 7 30 52	12, 17	8
12-7.6	3 5 10 12 22 25 31 1 1 1 1 1 1 0	15 17	0 10 16 12 16	0 8 24 61	12, 16	9
12-7.7	3 5 6 15 23 25 30 1 1 1 0 0 0 0	9 18	0 10 20 8 8	0 8 24 61	12, 25	10
12-7.8	3 5 9 14 17 22 26 1 1 1 1 1 1 1	6 27	0 12 13 12 15	0 5 34 54	12, 16	1
12-7.10	3 5 9 14 18 20 31 1 1 1 1 1 1 1	15 23	0 10 18 10 12	0 6 31 56	12, 18	4

Table 6. (Cont'd).

Design	Additional Col. Foldover plan	Blocking Col.	TWP $(A_3^t, \ldots, A_7^t)$	BWP $(A_1^b, \ldots, A_4^b)$	cc_1, cc_2	Rank
13-8.1	7 11 13 14 19 21 22 25 0 1 0 1 1 1 0 0	3 28	0 23 0 56 0	0 6 44 72	13, 0	2
13-8.2	3 5 9 14 17 22 26 28 1 1 1 1 1 1 1 1	6 27	0 18 20 24 28	0 7 42 74	13, 10	5
13-8.3	3 5 9 14 15 22 26 29 1 1 1 1 0 1 0 0	6 17	0 18 20 28 24	0 7 43 72	13, 18	6
13-8.5	3 5 9 14 15 17 22 26 1 1 1 1 0 1 0 0	6 27	0 20 18 22 30	0 6 44 75	13, 15	1
13-8.9	3 5 9 15 18 20 24 31 1 1 1 0 1 1 1 1	14 19	0 15 27 21 27	0 7 42 77	13, 15	3
13-8.10	3 5 6 9 14 17 26 29 1 1 1 1 0 1 1 0	10 22	0 17 24 18 32	0 7 42 75	13, 12	4
14-9.1	7 11 13 14 15 17 22 26 28 1 1 1 1 1 0 0 0 0	3 28	0 33 0 96 0	0 7 56 100	14, 0	2
14-9.2	3 5 9 14 15 17 22 26 28 1 1 1 1 0 1 1 0 0	6 27	0 28 27 42 54	0 8 54 101	14, 11	5
14-9.3	3 5 9 14 15 17 22 23 26 1 1 1 1 0 1 1 0 0	6 27	0 31 24 40 56	0 7 56 102	14, 14	1
14-9.4	3 5 9 15 18 20 24 30 31 1 1 1 0 1 1 1 0 1	6 19	0 22 40 36 56	0 9 52 105	14, 8	6
14-9.9	3 5 6 9 14 17 22 26 27 1 1 1 1 0 1 0 1 0	10 23	0 24 36 36 60	0 8 54 105	14, 11	3
14-9.10	3 5 6 9 14 15 17 26 29 1 1 1 1 0 1 1 1 0	10 22	0 27 32 32 64	0 8 54 102	14, 6	4
15-10.1	7 11 13 14 19 21 22 25 26 28 1 1 1 1 0 0 1 1 0 0	3 5	0 45 0 160 0	0 21 0 312	15, 0	10
15-10.2	3 5 9 14 15 17 22 23 26 28 1 1 1 1 0 1 1 0 0 0	6 27	0 41 36 72 96	0 9 68 136	15, 12	2
15-10.3	3 5 9 14 15 17 22 23 26 27 1 1 1 1 0 1 1 0 1 0	6 10	0 45 32 72 96	0 18 25 218	15, 14	9
15-10.4	3 5 6 9 14 17 22 26 27 28 1 1 1 1 0 1 0 1 0 1	10 23	0 33 54 60 108	0 10 66 140	15, 6	4
15-10.6	3 5 6 9 14 15 17 22 26 27 1 1 1 1 0 1 1 0 1 0	10 23	0 36 48 61 112	0 9 68 141	15, 7	1
15-10.7	3 5 9 14 18 20 23 24 27 29 1 1 1 1 1 1 1 1 1 1	7 17	0 33 44 96 72	0 12 63 132	15, 12	7
15-10.10	3 5 6 9 14 15 17 22 23 26 1 1 1 1 0 1 1 0 1 1	7 27	0 45 32 72 96	0 9 68 132	15, 12	3
16-11.1	7 11 13 14 19 21 22 25 26 28 31 1 1 1 1 0 0 1 1 0 0 0	3 5	0 60 0 256 0	0 24 0 416	16, 0	10
16-11.3	3 5 6 9 14 15 17 22 26 27 28 1 1 1 1 0 1 1 0 1 0 1	10 23	0 47 72 98 192	0 11 82 186	16, 4	1
16-11.6	3 5 9 14 18 20 23 24 27 29 31 1 1 1 1 1 1 1 1 1 1 0	6 10	0 45 60 160 120	0 15 72 176	16, 12	7
16-11.9	3 5 6 9 10 14 15 17 22 26 29 1 1 1 1 1 0 1 1 0 0 0	7 27	0 53 52 136 144	0 13 76 176	16, 15	5
17-12.1	3 5 9 14 15 17 22 23 26 27 28 29 1 1 1 1 0 1 1 0 1 0 1 0	6 10	0 76 64 192 256	0 24 32 400	17, 16	5
17-12.2	3 5 6 9 14 15 17 22 23 26 27 28 1 1 1 1 0 1 1 0 1 1 0 1	10 18	0 64 96 156 320	0 21 56 321	17, 2	4
17-12.3	3 5 6 9 10 14 17 22 23 26 27 28 1 1 1 1 1 0 1 0 1 0 1 0	15 18	0 95 0 354 0	0 13 98 208	17, 0	1
17-12.5	3 5 6 9 10 14 15 17 22 23 26 29 1 1 1 1 1 0 1 1 0 1 0 1	7 27	0 96 0 348 0	0 14 96 202	17, 2	3
18-13.1	3 5 6 9 14 15 17 22 23 26 27 28 29 1 1 1 1 0 1 1 0 1 1 0 1 0	10 18	0 84 128 240 512	0 24 64 424	18, 0	5
18-13.3	3 5 6 7 9 10 11 17 18 19 28 29 30 1 1 1 0 1 1 0 1 1 0 1 0 0	12 20	0 78 144 228 528	0 21 85 363	18, 0	3
18-13.4	3 5 6 9 14 15 18 21 23 24 27 28 31 1 1 1 1 0 1 1 0 1 1 1 0 0	10 19	0 108 0 552 0	0 18 108 270	18, 0	2
18-13.5	3 5 6 9 10 14 17 22 23 24 27 28 29 1 1 1 1 1 0 1 0 1 1 1 0 1	15 21	0 113 0 547 0	0 15 116 277	18, 0	1
19-14.1	3 5 6 9 10 14 15 17 22 23 26 27 28 29 1 1 1 1 1 0 1 1 0 1 0 1 0 1	13 18	0 164 0 748 0	0 24 97 408	19, 0	5
19-14.2	3 5 6 7 9 10 11 17 18 19 28 29 30 31 1 1 1 0 1 1 0 1 1 0 1 0 0 1	12 20	0 100 192 336 832	0 24 97 472	19, 0	4
19-14.5	3 5 6 9 10 13 14 17 22 23 24 26 29 31 1 1 1 1 1 0 0 1 0 1 1 0 1 0	15 19	0 136 0 816 0	0 18 135 350	19, 0	1
20-15.1	3 5 6 9 10 14 15 17 18 22 23 26 27 28 29 1 1 1 1 1 0 1 1 1 0 1 0 1 0 1	7 25	0 188 0 1128 0	0 25 130 472	20, 0	4
20-15.4	3 5 6 9 10 14 15 17 18 22 23 26 27 28 31 1 1 1 1 1 0 1 1 1 0 1 0 1 0 0	7 11	0 175 0 1155 0	0 24 143 441	20, 0	2
20-15.5	3 5 6 9 10 13 14 15 17 18 22 23 26 27 28 1 1 1 1 1 0 0 1 1 1 0 1 0 1 0	7 25	0 176 0 1148 0	0 23 144 448	20, 0	1
21-16.1	3 5 6 9 10 14 15 17 18 22 23 26 27 28 29 31 1 1 1 1 1 0 1 1 1 0 1 0 1 0 1 0	7 11	0 220 0 1608 0	0 27 163 552	21, 0	5
21-16.4	3 5 6 9 10 13 14 17 19 22 23 24 26 28 29 31 1 1 1 1 1 0 0 1 0 0 1 1 0 0 1 0	15 20	0 210 0 1638 0	0 21 189 546	21, 0	1
21-16.5	3 5 6 9 10 13 14 15 17 18 21 22 23 24 26 29 1 1 1 1 1 0 0 1 1 1 0 0 1 1 0 1	7 27	0 209 0 1644 0	0 24 180 534	21, 0	2

Table 7. Optimal blocking and foldover plans for 32-run ($2^{n-p}:2^3$) designs.

Design	Additional Col. Foldover plan	Blocking Col.	TWP $(A_3^t,\ldots,A_7^t)$	BWP $(A_1^b,\ldots,A_4^b)$	cc_1, cc_2	Rank
6-1.1	31 1	3 12 21	0 0 0 0	0 3 8 3	6, 12	1
7-2.1	7 27 1 0	5 11 19	0 0 1 0 0	0 5 12 7	7, 16	4
7-2.2	7 25 1 1	3 9 21	0 0 0 1 0	0 5 12 7	7, 16	1
7-2.3	7 11 1 0	3 13 17	0 1 0 0 0	0 5 12 6	7, 12	6
7-2.5	3 28 1 1	5 9 18	0 0 0 0 1	0 6 9 9	7, 15	8
8-3.1	7 11 29 1 0 0	3 13 17	0 1 2 0 0	0 8 16 13	8, 16	7
8-3.2	7 11 21 0 1 1	3 13 17	0 1 0 2 0	0 7 18 14	8, 17	1
8-3.5	3 13 22 1 1 1	5 11 17	0 0 2 1 0	0 7 18 15	8, 21	5
8-3.7	3 13 21 1 1 0	5 10 19	0 1 1 0 1	0 7 18 14	8, 17	3
9-4.1	7 11 19 29 1 1 0 0	5 9 18	0 2 4 0 0	0 12 16 36	9, 16	7
9-4.8	3 12 21 26 1 1 1 0	6 11 18	0 1 4 2 0	0 9 27 26	9, 23	1
9-4.10	3 5 10 28 1 1 1 1	6 9 17	0 2 3 1 1	0 9 27 25	9, 19	2
10-5.1	7 11 19 29 30 1 1 0 0 0	3 5 9	0 4 8 0 0	0 17 24 42	10, 16	9
10-5.8	3 5 14 22 25 1 1 1 1 1	7 10 18	0 2 8 4 0	0 13 32 48	10, 24	3
10-5.9	3 5 14 23 26 1 1 1 0 0	6 9 18	0 3 6 4 2	0 12 36 43	10, 21	1
11-6.1	7 11 13 19 21 25 1 0 1 1 0 0	3 5 24	0 10 0 16 0	0 25 0 160	11, 6	10
11-6.3	3 5 14 22 25 31 1 1 1 1 1 0	6 9 18	0 4 14 8 0	0 19 36 74	11, 24	8
11-6.4	3 5 14 22 26 29 1 1 1 1 0 0	6 9 17	0 6 10 8 4	0 16 45 68	11, 19	3
11-6.6	3 5 10 23 27 28 1 1 1 0 0 0	6 11 18	0 5 12 7 4	0 17 40 77	11, 21	7
11-6.9	3 5 9 14 22 26 1 1 1 1 1 0	6 10 17	0 7 9 6 6	0 15 48 67	11, 18	1
12-7.1	7 11 13 14 19 21 25 1 0 1 0 1 0 0	3 12 17	0 16 0 30 0	0 30 0 239	12, 3	10
12-7.2	7 11 13 14 19 21 22 0 1 0 1 1 1 0	3 5 25	0 15 0 32 0	0 18 64 96	12, 0	2
12-7.4	3 5 9 14 22 26 28 1 1 1 1 1 0 0	6 10 17	0 12 13 12 15	0 19 60 102	12, 13	4
12-7.6	3 5 10 12 22 25 31 1 1 1 1 1 1 0	6 11 17	0 10 16 12 16	0 21 50 120	12, 16	5
12-7.7	3 5 6 15 23 25 30 1 1 1 0 0 0 0	7 9 18	0 10 20 8 8	0 22 48 117	12, 20	7
12-7.9	3 5 9 14 15 22 26 1 1 1 1 0 0 0	6 10 17	0 12 14 12 12	0 18 64 99	12, 14	1
12-7.10	3 5 9 14 18 20 31 1 1 1 1 1 1 1	6 11 19	0 10 18 10 12	0 19 60 104	12, 18	3
13-8.1	7 11 13 14 19 21 22 25 0 1 0 1 1 1 0 0	3 9 17	0 23 0 56 0	0 36 0 342	13, 0	10
13-8.3	3 5 9 14 15 22 26 29 1 1 1 1 0 1 0 0	6 10 18	0 18 20 28 24	0 31 35 232	13, 16	9
13-8.4	3 5 9 14 15 22 26 28 1 1 1 1 0 1 0 0	6 10 17	0 18 20 24 28	0 22 80 145	13, 8	1
13-8.5	3 5 9 14 15 17 22 26 1 1 1 1 0 1 0 0	6 11 18	0 20 18 22 30	0 30 36 235	13, 14	8
13-8.8	3 5 9 15 18 20 24 30 1 1 1 0 1 1 1 0	6 10 19	0 16 26 18 30	0 23 75 154	13, 12	2
13-8.9	3 5 9 15 18 20 24 31 1 1 1 0 1 1 1 1	6 11 17	0 15 27 21 27	0 26 60 180	13, 13	3

Table 7. (Cont'd).

Design	Additional Col. Foldover plan	Blocking Col.	TWP $(A_3^t, \ldots, A_7^t)$	BWP $(A_1^b, \ldots, A_4^b)$	cc_1, cc_2	Rank
14-9.1	7 11 13 14 15 17 22 26 28 1 1 1 1 1 0 0 0 0	5 9 17	0 33 0 96 0	0 42 0 478	14, 0	10
14-9.2	3 5 9 14 15 17 22 26 28 1 1 1 1 0 1 1 0 0	6 11 18	0 28 27 42 54	0 36 42 337	14, 13	8
14-9.7	3 5 9 14 15 18 20 24 30 1 1 1 1 0 1 1 1 0	6 10 19	0 22 40 36 56	0 26 100 209	14, 8	1
14-9.9	3 5 6 9 14 17 22 26 27 1 1 1 1 0 1 0 1 0	7 10 19	0 24 36 36 60	0 31 72 261	14, 10	4
15-10.1	7 11 13 14 19 21 22 25 26 28 1 1 1 1 0 0 1 1 0 0	3 5 9	0 45 0 160 0	0 49 0 648	15, 0	10
15-10.2	3 5 9 14 15 17 22 23 26 28 1 1 1 1 0 1 1 0 0 0	7 10 18	0 41 36 72 96	0 42 49 470	15, 14	8
15-10.3	3 5 9 14 15 17 22 23 26 27 1 1 1 1 0 1 1 0 1 0	6 10 18	0 45 32 72 96	0 42 49 466	15, 14	9
15-10.7	3 5 9 14 18 20 23 24 27 29 1 1 1 1 1 1 1 1 1 1	6 10 19	0 33 44 96 72	0 33 112 292	15, 8	2
15-10.9	3 5 9 14 15 18 20 23 24 30 1 1 1 0 1 1 1 1 1 1	6 10 19	0 51 0 144 0	0 30 125 264	15, 0	1
16-11.1	7 11 13 14 19 21 22 25 26 28 31 1 1 1 1 0 0 1 1 0 0 0	3 5 9	0 60 0 256 0	0 56 0 864	16, 0	9
16-11.2	3 5 9 14 15 17 22 23 26 27 28 1 1 1 0 1 1 1 0 1 0 0	6 10 18	0 57 48 120 160	0 49 56 636	16, 15	8
16-11.3	3 5 6 9 14 15 17 22 26 27 28 1 1 1 1 0 1 1 0 1 0 1	7 11 18	0 47 72 98 192	0 43 98 506	16, 4	5
16-11.5	3 5 9 14 18 20 23 24 27 29 31 1 1 1 1 0 1 1 0 1 1 0	7 10 18	0 51 64 102 192	0 43 98 502	16, 5	6
16-11.6	3 5 9 14 18 20 23 24 27 29 31 1 1 1 1 1 1 1 1 1 1 0	6 7 10	0 45 60 160 120	1 33 145 392	15, 9	10
16-11.8	3 5 6 9 10 14 17 22 23 26 29 1 1 1 1 1 0 1 0 1 0 1	7 11 18	0 71 0 226 0	0 39 126 402	16, 2	3
16-11.9	3 5 6 9 10 14 15 17 22 26 29 1 1 1 1 1 0 1 1 0 0 0	7 11 18	0 53 52 136 144	0 41 120 416	16, 11	4
16-11.10	3 5 6 9 10 14 17 22 26 29 31 1 1 1 1 1 0 1 0 0 1 0	7 11 18	0 65 0 236 0	0 37 140 372	16, 1	1
17-12.1	3 5 9 14 15 17 22 23 26 27 28 29 1 1 1 1 0 1 1 0 1 0 1 0	6 10 18	0 76 64 192 256	0 56 64 848	17, 16	5
17-12.2	3 5 6 9 14 15 17 22 23 26 27 28 1 1 1 1 0 1 1 0 1 1 0 1	7 10 18	0 64 96 156 320	0 50 112 678	17, 2	4
17-12.3	3 5 6 9 10 14 17 22 23 26 27 28 1 1 1 1 1 0 1 0 1 0 1 0	7 11 19	0 95 0 354 0	0 45 147 542	17, 0	1
17-12.5	3 5 6 9 10 14 15 17 22 23 26 29 1 1 1 1 1 0 1 1 0 1 0 1	7 11 18	0 96 0 348 0	0 46 144 538	17, 2	3
18-13.1	3 5 6 9 14 15 17 22 23 26 27 28 29 1 1 1 1 0 1 1 0 1 1 0 1 0	7 10 18	0 84 128 240 512	0 57 128 896	18, 0	5
18-13.3	3 5 6 7 9 10 11 17 18 19 28 29 30 1 1 1 0 1 1 0 1 1 0 1 0 0	12 13 20	0 78 144 228 528	1 50 162 776	17, 0	3
18-13.4	3 5 6 9 14 15 18 21 23 24 27 28 31 1 1 1 1 0 1 1 0 1 1 1 0 0	7 10 19	0 108 0 552 0	0 45 216 612	18, 0	1
19-14.1	3 5 6 9 10 14 15 17 22 23 26 27 28 29 1 1 1 1 1 0 1 1 0 1 0 1 0 1	7 11 18	0 164 0 748 0	0 59 192 928	19, 0	4
19-14.2	3 5 6 7 9 10 11 17 18 19 28 29 30 31 1 1 1 0 1 1 0 1 1 0 1 0 0 1	12 13 20	0 100 192 336 832	1 57 185 1008	18, 0	5
19-14.5	3 5 6 9 10 13 14 17 22 23 24 26 29 31 1 1 1 1 1 0 0 1 0 1 1 0 1 0	7 11 18	0 136 0 816 0	0 51 252 784	19, 0	1
20-15.1	3 5 6 9 10 14 15 17 18 22 23 26 27 28 29 1 1 1 1 1 0 1 1 1 0 1 0 1 0 1	7 11 19	0 188 0 1128 0	0 62 256 1073	20, 0	2
20-15.4	3 5 6 9 10 14 15 17 18 22 23 26 27 28 31 1 1 1 1 1 0 1 1 1 0 1 0 1 0 0	7 11 19	0 175 0 1155 0	1 55 279 1037	19, 0	4
20-15.5	3 5 6 9 10 13 14 15 17 18 22 23 26 27 28 1 1 1 1 1 0 0 1 1 1 0 1 0 1 0	7 11 19	0 176 0 1148 0	0 59 280 1012	20, 0	1
21-16.1	3 5 6 9 10 14 15 17 18 22 23 26 27 28 29 31 1 1 1 1 1 0 1 1 1 0 1 0 1 0 1 0	7 11 19	0 220 0 1608 0	1 62 318 1297	20, 0	4
21-16.4	3 5 6 9 10 13 14 17 19 22 23 24 26 28 29 31 1 1 1 1 1 0 0 1 0 0 1 1 0 0 1 0	7 11 18	0 210 0 1638 0	0 63 343 1218	21, 0	1
21-16.5	3 5 6 9 10 13 14 15 17 18 21 22 23 24 26 29 1 1 1 1 1 0 0 1 1 1 0 0 1 1 0 1	7 11 19	0 209 0 1644 0	1 58 346 1244	20, 0	3

Table 8. Optimal blocking and foldover plans for 64-run $(2^{n-p}:2)$ designs.

Design	Additional Col. Foldover plan	Blocking Col.	TWP $(A_3^t,\ldots,A_7^t)$	BWP $(A_1^b,\ldots,A_4^b)$	cc_1, cc_2	Rank
7-1.1	63 1	7	0 0 0 0 0	0 0 1 1	7, 21	1
8-2.1	15 51 1 1	21	0 0 0 1 0	0 0 1 2	8, 28	1
9-3.1	7 27 45 1 0 0	51	0 0 2 1 0	0 0 1 5	9, 36	4
9-3.3	7 27 43 1 1 0	53	0 0 2 0 0	0 0 0 6	9, 36	3
9-3.5	7 25 42 1 1 1	53	0 0 0 3 0	0 0 0 9	9, 36	1
10-4.1	7 27 43 53 1 1 0 0	13	0 0 4 2 0	0 0 2 8	10, 45	2
10-4.2	7 25 42 53 1 1 1 0	62	0 0 3 3 1	0 0 2 7	10, 45	1
10-4.6	7 11 29 45 1 0 1 0	51	0 1 4 0 0	0 0 0 9	10, 39	8
10-4.7	7 25 42 52 1 1 1 0	63	0 1 0 6 0	0 0 0 14	10, 39	3
11-5.1	7 11 29 45 51 1 0 1 0 0	62	0 1 7 4 0	0 0 2 13	11, 49	5
11-5.3	7 11 29 46 49 1 0 0 0 1	60	0 1 6 4 2	0 0 3 12	11, 49	3
11-5.4	7 11 21 46 56 0 1 1 0 1	54	0 1 5 6 2	0 0 2 15	11, 49	4
11-5.5	7 11 29 45 49 1 1 0 0 1	62	0 2 4 4 4	0 0 2 12	11, 43	6
11-5.7	7 11 21 38 57 0 1 1 1 0	58	0 1 4 6 4	0 0 3 14	11, 49	1
11-5.10	7 11 13 30 46 1 0 0 1 0	49	0 3 7 0 0	0 0 2 11	11, 40	10
12-6.1	7 11 29 45 51 62 1 0 1 0 0 0	14	0 2 12 8 0	0 0 8 13	12, 54	4
12-6.6	7 11 19 37 57 63 0 1 1 1 1 0	29	0 2 8 10 8	0 0 4 20	12, 54	1
12-6.9	7 11 21 25 38 58 0 1 1 0 1 0	60	0 3 7 10 6	0 0 3 22	12, 48	5
13-7.1	7 11 21 25 38 58 60 0 1 1 0 1 1 0	31	0 4 14 16 12	0 0 8 22	13, 54	2
13-7.3	7 11 19 29 37 59 62 0 1 1 0 1 0 1	41	0 3 12 24 8	0 0 8 20	13, 60	1
13-7.4	7 11 19 29 37 41 60 1 0 1 0 1 1 0	50	0 5 13 13 15	0 0 7 23	13, 48	6
13-7.8	7 11 19 37 41 60 63 0 0 1 1 1 1 0	29	0 6 12 10 18	0 0 6 24	13, 48	9
13-7.9	7 11 19 29 37 41 47 1 1 0 0 1 0 0	30	0 6 10 18 12	0 0 4 32	13, 42	8
14-8.1	7 11 19 30 37 41 49 60 1 0 1 0 1 1 0 0	63	0 8 20 22 28	0 0 8 34	14, 46	5
14-8.3	7 11 13 19 21 25 35 60 0 1 1 0 1 0 1 0	63	0 11 12 28 28	0 0 5 44	14, 40	8
14-8.4	7 11 13 14 19 21 25 54 0 1 0 1 1 1 0 0	41	0 16 7 30 26	0 0 5 39	14, 30	10
14-8.6	7 11 19 29 30 37 41 49 1 0 1 0 0 1 1 0	60	0 8 19 22 30	0 0 8 34	14, 46	4
14-8.7	7 11 19 30 37 41 52 56 1 1 0 0 0 1 1 0	47	0 7 16 36 20	0 0 10 28	14, 49	1
14-8.8	7 11 13 19 21 41 54 63 1 0 0 1 1 1 1 0	25	0 8 18 24 30	0 0 11 29	14, 52	3
14-8.10	7 11 19 29 37 41 47 49 1 0 1 0 1 1 0 0	55	0 8 15 34 22	0 0 9 29	14, 43	2

Table 8. (Cont'd).

Design	Additional Col. / Foldover plan	Blocking Col.	TWP $(A_3^t, \ldots, A_7^t)$	BWP $(A_1^b, \ldots, A_4^b)$	cc_1, cc_2	Rank
15-9.1	7 11 19 30 37 41 49 60 63 1 1 0 1 1 0 0 0 0	29	0 12 30 34 51	0 0 13 39	15, 45	4
15-9.3	7 11 19 29 37 41 47 49 55 1 0 1 0 1 1 0 0 0	59	0 11 22 60 36	0 0 12 38	15, 39	1
15-9.5	7 11 13 14 19 21 22 25 58 0 1 0 1 1 1 0 0 0	37	0 23 10 56 40	0 0 6 54	15, 27	10
15-9.6	7 11 13 19 21 35 37 57 58 0 1 1 0 1 1 0 1 0	60	0 12 27 38 54	0 0 10 48	15, 45	2
15-9.10	7 11 13 14 19 21 25 35 60 0 0 1 1 1 1 0 1 0	63	0 17 16 48 48	0 0 6 60	15, 38	9
16-10.1	7 11 13 19 21 35 37 57 58 60 0 1 1 0 1 1 0 1 0 0	14	0 17 40 56 96	0 0 16 53	16, 41	3
16-10.2	7 11 19 29 37 41 47 49 55 59 0 1 1 0 1 0 1 1 0 0	62	0 15 30 100 60	0 0 15 50	16, 30	1
16-10.5	7 11 13 14 19 21 22 25 35 60 0 1 0 1 1 1 0 1 0 0	63	0 25 20 80 80	0 0 7 80	16, 33	14
16-10.6	7 11 13 14 19 21 22 25 26 60 1 1 1 1 0 0 1 1 0 0	35	0 33 12 96 64	0 0 7 72	16, 29	15
16-10.7	7 11 13 14 19 21 35 37 57 58 0 0 1 1 1 1 1 0 1 0	60	0 17 36 64 96	0 0 12 66	16, 41	2
16-10.8	7 11 13 14 19 21 25 35 60 63 0 0 1 1 1 0 0 1 1 0	22	0 19 32 66 96	0 0 19 50	16, 44	6
16-10.10	7 11 13 14 19 21 22 35 37 57 0 0 1 1 1 1 0 1 0 0	58	0 21 24 80 80	0 0 7 84	16, 27	8
16-10.14	7 11 19 37 41 47 49 55 59 62 0 1 1 1 0 1 1 0 0 1	29	0 20 0 160 0	0 0 0 120	16, 0	7
17-11.1	7 11 13 14 19 21 35 37 57 58 60 0 0 1 1 1 0 1 0 1 0 0	22	0 23 54 90 162	0 0 19 72	17, 34	1
17-11.2	7 11 19 29 37 41 47 49 55 59 62 0 1 1 0 1 0 1 1 0 0 1	13	0 20 40 160 96	0 1 20 55	17, 15	6
17-11.3	7 11 13 19 21 25 35 37 41 49 63 0 0 1 1 0 1 1 1 0 0 1	62	0 25 35 136 120	0 1 15 60	17, 30	7
17-11.7	7 11 13 14 19 21 22 35 37 38 57 0 0 1 1 1 0 1 1 1 0 0	58	0 28 32 128 128	0 0 8 112	17, 16	2
17-11.10	7 11 13 14 19 21 22 25 26 35 60 1 1 1 1 0 0 1 1 0 0 0	63	0 36 24 128 128	0 0 8 104	17, 32	5
18-12.1	7 11 13 14 19 21 22 35 37 57 58 60 0 0 1 1 1 0 1 1 1 1 0 0	38	0 30 72 140 264	0 0 22 96	18, 24	1
18-12.2	7 11 13 14 19 21 22 35 37 38 57 58 0 0 1 1 1 0 1 1 1 0 0 0	60	0 32 64 152 256	0 0 16 116	18, 16	2
18-12.3	7 11 13 14 19 21 22 25 26 35 60 63 1 1 1 1 0 0 1 1 0 0 0 0	33	0 40 48 160 256	0 2 12 98	18, 32	5
19-13.1	7 11 13 14 19 21 22 35 37 38 57 58 60 0 0 1 1 1 0 1 1 1 0 0 0 0	63	0 40 96 200 416	0 0 25 124	19, 12	1

Table 9. Optimal blocking and foldover plans for 64-run $(2^{n-p} : 2^2)$ designs.

Design	Additional Col. Foldover plan	Blocking Col.	TWP $(A_3^t, \ldots, A_7^t)$	BWP $(A_1^b, \ldots, A_4^b)$	cc_1, cc_2	Rank
7-1.1	63 1	7 25	0 0 0 0 0	0 0 3 3	7, 21	1
8-2.1	15 51 1 1	21 42	0 0 0 1 0	0 0 4 5	8, 28	1
9-3.1	7 27 45 1 0 0	14 49	0 0 2 1 0	0 0 6 9	9, 36	4
9-3.3	7 27 43 1 1 0	13 51	0 0 2 0 0	0 0 4 14	9, 36	3
9-3.5	7 25 42 1 1 1	11 52	0 0 0 3 0	0 0 6 9	9, 36	1
9-3.6	7 11 53 0 1 1	22 45	0 1 0 0 2	0 0 4 13	9, 30	5
10-4.1	7 27 43 53 1 1 0 0	13 51	0 0 4 2 0	0 0 8 18	10, 45	2
10-4.2	7 25 42 53 1 1 1 0	13 54	0 0 3 3 1	0 0 9 16	10, 45	1
10-4.5	7 11 29 49 1 1 0 1	19 45	0 1 2 2 2	0 0 8 17	10, 39	4
11-5.1	7 11 29 45 51 1 0 1 0 0	14 49	0 1 7 4 0	0 1 12 21	11, 48	7
11-5.7	7 11 21 38 57 0 1 1 1 0	13 55	0 1 4 6 4	0 0 13 24	11, 49	1
11-5.8	7 11 21 41 51 0 1 1 0 0	25 47	0 2 4 4 4	0 0 12 24	11, 43	3
12-6.1	7 11 29 45 51 62 1 0 1 0 0 0	14 17	0 2 12 8 0	0 2 16 29	12, 52	9
12-6.4	7 11 21 41 54 56 1 0 0 1 1 0	26 37	0 3 7 10 6	0 1 15 33	12, 47	4
12-6.6	7 11 19 37 57 63 0 1 1 1 1 0	13 35	0 2 8 10 8	0 1 17 32	12, 53	1
12-6.7	7 11 19 29 37 59 0 1 1 0 1 0	30 41	0 2 8 12 4	0 1 16 32	12, 53	2
12-6.10	7 11 13 19 46 49 0 0 1 1 1 0	21 42	0 4 6 8 8	0 1 15 32	12, 44	8
13-7.1	7 11 21 25 38 58 60 0 1 1 0 1 1 0	31 40	0 4 14 16 12	0 2 18 46	13, 52	7
13-7.3	7 11 19 29 37 59 62 0 1 1 0 1 0 1	6 41	0 3 12 24 8	0 2 20 44	13, 60	6
13-7.6	7 11 19 30 37 41 52 1 1 0 0 1 0 0	29 47	0 5 11 19 11	0 1 20 47	13, 47	1
13-7.7	7 11 13 19 37 57 63 0 1 0 1 1 1 0	21 35	0 5 12 15 16	0 1 22 45	13, 50	2
14-8.1	7 11 19 30 37 41 49 60 1 0 1 0 1 1 0 0	29 35	0 8 20 22 28	0 2 25 61	14, 46	8
14-8.3	7 11 13 19 21 25 35 60 0 1 1 0 1 0 1 0	14 49	0 11 12 28 28	0 1 27 61	14, 39	2
14-8.4	7 11 13 14 19 21 25 54 0 1 0 1 1 1 0 0	22 41	0 16 7 30 26	0 1 27 56	14, 29	3
14-8.5	7 11 13 14 19 21 22 57 0 1 0 1 1 1 0 0	26 37	0 15 8 32 24	0 0 28 62	14, 25	1
14-8.7	7 11 19 30 37 41 52 56 1 1 0 0 0 1 1 0	15 50	0 7 16 36 20	0 2 24 64	14, 49	4
14-8.8	7 11 13 19 21 41 54 63 1 0 0 1 1 1 1 0	25 35	0 8 18 24 30	0 2 25 61	14, 50	6
15-9.1	7 11 19 30 37 41 49 60 63 1 1 0 1 1 0 0 0 0	5 35	0 12 30 34 51	0 3 32 78	15, 44	9
15-9.3	7 11 19 29 37 41 47 49 55 1 0 1 0 1 1 0 0 0	5 59	0 11 22 60 36	0 3 28 86	15, 38	7
15-9.6	7 11 13 19 21 35 37 57 58 0 1 1 0 1 1 0 1 0	14 49	0 12 27 38 54	0 2 33 82	15, 45	4
16-10.1	7 11 13 19 21 35 37 57 58 60 0 1 1 0 1 1 0 1 0 0	14 22	0 17 40 56 96	0 3 41 104	16, 40	8
16-10.2	7 11 19 29 37 41 47 49 55 59 0 1 1 0 1 0 1 1 0 0	5 35	0 15 30 100 60	0 4 40 101	16, 28	15
16-10.5	7 11 13 14 19 21 22 25 35 60 0 1 0 1 1 1 0 1 0 0	26 37	0 25 20 80 80	0 1 42 108	16, 32	3
16-10.6	7 11 13 14 19 21 22 25 26 60 1 1 1 1 0 0 1 1 0 0	28 35	0 33 12 96 64	0 1 42 100	16, 28	4
16-10.7	7 11 13 14 19 21 35 37 57 58 0 0 1 1 1 1 1 0 1 0	22 41	0 17 36 64 96	0 2 41 110	16, 41	5
16-10.8	7 11 13 14 19 21 25 35 60 63 0 0 1 1 1 0 0 1 1 0	22 37	0 19 32 66 96	0 3 38 106	16, 43	10
16-10.9	7 11 13 14 19 21 22 35 57 60 0 0 1 1 1 1 0 1 0 0	25 38	0 19 32 66 96	0 1 42 114	16, 35	1
16-10.10	7 11 13 14 19 21 22 35 37 57 0 0 1 1 1 1 0 1 0 0	25 38	0 21 24 80 80	0 1 42 112	16, 26	2

Table 10. Optimal blocking and foldover plans for 64-run $(2^{n-p} : 2^3)$ designs.

Design	Additional Col. Foldover plan	Blocking Col.	TWP $(A_3^t, \ldots, A_7^t)$	BWP $(A_1^b, \ldots, A_4^b)$	cc_1, cc_2	Rank
7-1.1	63 1	7 25 42	0 0 0 0 0	0 0 7 7	7, 21	1
8-2.1	15 51 1 1	3 20 41	0 0 0 1 0	0 2 8 10	8, 26	1
9-3.1	7 27 45 1 0 0	11 22 35	0 0 2 1 0	0 2 14 18	9, 34	4
9-3.5	7 25 42 1 1 1	3 13 49	0 0 0 3 0	0 2 14 18	9, 34	1
9-3.6	7 11 53 0 1 1	5 19 41	0 1 0 0 2	0 2 14 17	9, 30	5
10-4.1	7 27 43 53 1 1 0 0	13 17 34	0 0 4 2 0	0 4 16 30	10, 41	8
10-4.2	7 25 42 53 1 1 1 0	11 22 38	0 0 3 3 1	0 3 19 29	10, 42	1
10-4.7	7 25 42 52 1 1 1 0	11 22 39	0 1 0 6 0	0 3 19 28	10, 36	2
10-4.9	7 11 21 45 0 1 1 0	6 25 35	0 1 2 2 0	0 3 19 28	10, 38	3
11-5.1	7 11 29 45 51 1 0 1 0 0	14 17 36	0 1 7 4 0	0 5 24 41	11, 44	7
11-5.2	7 25 42 52 63 1 1 1 0 0	11 22 39	0 1 4 6 4	0 4 25 45	11, 45	1
11-5.4	7 11 21 46 56 0 1 1 0 1	6 25 43	0 1 5 6 2	0 4 26 43	11, 47	3
11-5.7	7 11 21 38 57 0 1 1 1 0	9 19 37	0 1 4 6 4	0 4 26 43	11, 45	2
11-5.8	7 11 21 41 51 0 1 1 0 0	6 25 35	0 2 4 4 4	0 4 25 44	11, 41	5
12-6.1	7 11 29 45 51 62 1 0 1 0 0 0	14 17 36	0 2 12 8 0	0 6 32 61	12, 48	7
12-6.4	7 11 21 41 54 56 1 0 0 1 1 0	6 25 42	0 3 7 10 6	0 5 34 63	12, 43	4
12-6.6	7 11 19 37 57 63 0 1 1 1 1 0	9 21 38	0 2 8 10 8	0 5 34 64	12, 49	1
12-6.7	7 11 19 29 37 59 0 1 1 0 1 0	6 24 41	0 2 8 12 4	0 5 34 64	12, 53	2
13-7.1	7 11 21 25 38 58 60 0 1 1 0 1 1 0	13 19 33	0 4 14 16 12	0 7 42 88	13, 49	4
13-7.3	7 11 19 29 37 59 62 0 1 1 0 1 0 1	6 24 41	0 3 12 24 8	0 6 44 92	13, 60	1
13-7.6	7 11 19 30 37 41 52 1 1 0 0 1 0 0	10 23 38	0 5 11 19 11	0 6 44 90	13, 42	2
13-7.7	7 11 13 19 37 57 63 0 1 0 1 1 1 0	9 21 38	0 5 12 15 16	0 6 44 90	13, 47	3
14-8.1	7 11 19 30 37 41 49 60 1 0 1 0 1 1 0 0	5 25 35	0 8 20 22 28	0 9 52 119	14, 45	8
14-8.5	7 11 13 14 19 21 22 57 0 1 0 1 1 1 0 0	3 25 37	0 15 8 32 24	0 7 56 118	14, 24	2
14-8.7	7 11 19 30 37 41 52 56 1 1 0 0 0 1 1 0	15 18 35	0 7 16 36 20	0 7 56 126	14, 48	1
15-9.1	7 11 19 30 37 41 49 60 63 1 1 0 1 1 0 0 0 0	6 25 42	0 12 30 34 51	0 12 55 177	15, 37	10
15-9.3	7 11 19 29 37 41 47 49 55 1 0 1 0 1 1 0 0 0	5 25 35	0 11 22 60 36	0 9 68 166	15, 36	1
15-9.4	7 11 13 14 19 21 35 41 63 0 0 1 1 1 1 1 0 0	12 22 38	0 15 18 48 48	0 9 68 162	15, 32	2
15-9.5	7 11 13 14 19 21 22 25 58 0 1 0 1 1 1 0 0 0	5 26 35	0 23 10 56 40	0 9 68 154	15, 24	4
15-9.6	7 11 13 19 21 35 37 57 58 0 1 1 0 1 1 0 1 0	12 22 38	0 12 27 38 54	0 10 66 161	15, 41	5
15-9.10	7 11 13 14 19 21 25 35 60 0 0 1 1 1 1 0 1 0	12 22 37	0 17 16 48 48	0 9 68 160	15, 36	3
16-10.1	7 11 13 19 21 35 37 57 58 60 0 1 1 0 1 1 0 1 0 0	14 22 38	0 17 40 56 96	0 12 81 210	16, 37	5
16-10.7	7 11 13 14 19 21 35 37 57 58 0 0 1 1 1 1 1 0 1 0	12 22 38	0 17 36 64 96	0 11 82 216	16, 40	1
16-10.11	7 11 13 19 21 35 41 50 61 62 1 1 0 0 0 0 0 1 1 0	5 27 42	0 24 0 143 0	0 11 82 209	16, 16	3
16-10.15	7 11 13 19 21 25 35 44 55 61 1 1 0 0 1 0 1 1 0 0	3 28 37	0 24 0 141 0	0 11 82 209	16, 13	2

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