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SOME THEORY FOR CONSTRUCTING GENERAL MINIMUM LOWER ORDER CONFOUNDING DESIGNS

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Abstract: General minimum lower order confounding (GMC) is a newly proposed design criterion that aims at keeping the lower order effects unaliased with one another to the extent possible. This paper shows that for $5N/16 < n \leq N/2$, $9N/32 < n \leq 5N/16$, and $17N/64 < n \leq 9N/32$, all GMC designs with N runs and n two-level factors are projections of maximal designs with $N/2$, $5N/16$, and $9N/32$ factors, respectively. Furthermore, it provides immediate approaches to constructing these GMC designs from the respective maximal designs; these approaches can produce many more GMC designs than the existing computer search method.

Key words and phrases: Alias set, general minimum lower order confounding, maximal design, minimum aberration.

1. Introduction

Regular fractional factorial designs are arguably the mostly widely used designs in experimental investigations. There are quite a few optimality criteria for choosing such designs. Among them, the minimum aberration (MA) criterion introduced by Fries and Hunter (1980) that treats factorial effects of the same order as equally important and lower order effects as more important than higher order ones, and the clear effects criterion of Wu and Chen (1992) that concerns effects not aliased with main effects and two-factor interactions. Recently, Zhang et al. (2008) introduced the aliased effect-number pattern and, based on this pattern, they proposed a new criterion of general minimum lower order confounding (GMC) that aims at keeping the lower order effects unaliased with one another in an explicit manner. Zhang and Mukerjee (2009) characterized the GMC criterion via complementary sets and listed the complementary designs of some GMC designs.

There has been some work on constructing GMC designs. Zhang et al. (2008) used computer search when run size is small, but this approach is time consuming and even inefficient when the run size is as large as 128. Zhang and Mukerjee (2009) presented some GMC designs based on the complementary design theory.

In this paper, we resort to maximal designs to construct GMC designs. A regular design of resolution IV or higher is called maximal if its resolution reduces to III whenever an extra factor is added. It is obvious that every regular design of resolution IV or higher is a projection of a maximal regular design of resolution IV or higher. Some recent results in the literature of finite projective geometry essentially have that, for $n \geq N/4+1$ with $N = 2^t$ $(t \geq 4)$ the run size and n the number of factors, all maximal regular designs of resolution IV or higher must have (Chen and Cheng (2006))

$$n \in \left\{ \frac{N}{2}, \frac{5N}{16}, \frac{9N}{32}, \frac{17N}{64}, \frac{33N}{128}, \dots \right\}.$$

Obviously, all regular designs of resolution IV or higher with $5N/16 < n \leq N/2$ must be projections of the maximal regular design with $N/2$ factors, all regular designs of resolution IV or higher with $9N/32 < n \leq 5N/16$ must be projections of the maximal regular design with either $N/2$ or $5N/16$ factors and, for $17N/64 < n \leq 9N/32$, all regular designs of resolution IV or higher must be projections of the maximal regular design with $N/2$, $5N/16$, or $9N/32$ factors, and so on. Even designs were defined by Draper and Mitchell (1967), in which all defining words have even lengths. Note that the maximal regular designs with $N/2$ factors are the only even designs that are maximal (Chen and Cheng (2006)). We call these maximal even designs.

To explore two-level regular designs several studies, including Chen and Hedayat (1996), Tang and Wu (1996), and Mukerjee and Wu (2006), have developed design theory based on projective geometry. Butler (2003) and Chen and Cheng (2009) developed a complementary design theory for the maximal regular design with $N/2$ factors. Chen and Cheng (2006) discussed the method of doubling for constructing two-level regular designs of resolution IV. They showed that all MA designs with $9N/32 < n \leq 5N/16$ factors are projections of the maximal design with $5N/16$ factors. In addition, they provided some interesting properties of doubling that play an important role here. Xu and Cheng (2008) developed a general complementary design theory for doubling and showed that all MA designs with $17N/64 < n \leq 5N/16$ factors are projections of the maximal design with $5N/16$ factors. Almost all these studies have been based on the MA criterion.

This paper investigates methods for constructing GMC designs, with $17N/64 < n \leq N/2$ two-level factors, from maximal designs. Section 2 reviews some basic concepts and background information. In Section 3, we show that all GMC designs with $5N/16 < n \leq N/2$ factors are projections of the maximal even design, that all GMC designs with $9N/32 < n \leq 5N/16$ and $17N/64 < n \leq 9N/32$ factors are projections of the maximal designs with $5N/16$ and $9N/32$ factors, respectively. What is more, all GMC designs with $5N/16 < n \leq N/2$, $9N/32 < n \leq 5N/16$, and $17N/64 < n \leq 9N/32$ factors can be immediately obtained by the methods of Section 3. Proofs are in Section 4. In addition, some GMC designs with small run sizes are tabulated in the Appendix.

2. Basic Concepts and Background Information

A regular 2^{n-p} design with $N = 2^{n-p}$ runs and n factors is specified by p independent defining words. The group generated by the p independent words is the defining contrast subgroup. The *resolution* is the length of the shortest word in the defining contrast subgroup. The vector $(A_1, \ldots, A_n)$ is called the wordlength pattern, where A_i is the number of defining words of length i in the defining contrast subgroup. The MA criterion proposed by Fries and Hunter (1980) chooses a design by sequentially minimizing $A_1, \ldots, A_n$.

Now we introduce the GMC criterion proposed by Zhang et al. (2008). An ith order effect is said to be aliased with jth order effects at degree k if it is aliased with k jth order effects. The notation ${}_i^{\#}C_j^{(k)}$ is used to denote the number of ith order effects which are aliased with jth order effects at degree k, and the vector $({}_i^{\#}C_j^{(0)}, {}_i^{\#}C_j^{(1)}, \ldots, {}_i^{\#}C_j^{(K_j)})$ is denoted as ${}_i^{\#}C_j$, where $K_j = \binom{n}{j}$. The sequence

$$^{\#}C = ({}_1^{\#}C_2,\ {}_2^{\#}C_1, {}_2^{\#}C_2,\ {}_0^{\#}C_3,\ {}_1^{\#}C_3,\ {}_2^{\#}C_3,\ {}_3^{\#}C_1,\ {}_3^{\#}C_2,\ {}_3^{\#}C_3, \ldots) \tag{2.1}$$

is called the *aliased effect-number pattern* of the design. The general rule that ranks the vector ${}_i^{\#}C_j$ ahead of the vector ${}_s^{\#}C_t$ in (2.1) is (i) $\max(i, j) < \max(s, t)$ or (ii) $\max(i, j) = \max(s, t)$ and $i < s$, or (iii) $\max(i, j) = \max(s, t)$, $i = s$ and $j < t$. The *general minimum lower order confounding* (GMC) criterion aims at sequential maximization of the elements of $^{\#}C$ from left to right. According to the fact that some terms in (2.1) are uniquely determined by others that precede them, Zhang and Mukerjee (2009) defined the GMC criterion more simply; Zhang et al. (2008) showed that a GMC design must have maximum resolution; for $n \le N/2$, it must also be a design of resolution IV or higher, such as a projection of the maximal even design. Thus if we want to find a GMC design with $n \le N/2$ factors, we only need to seek a regular design of resolution IV or higher that sequentially maximizes the simplified aliased effect-number pattern $^{\#}C = ({}_2^{\#}C_2,\ {}_1^{\#}C_3,\ {}_2^{\#}C_3,\ {}_3^{\#}C_2,\ {}_3^{\#}C_3, \ldots)$. Obviously, what we care the most about is ${}_2^{\#}C_2$ for resolution IV designs, and we show that ${}_2^{\#}C_2$ is enough for us to find GMC designs in the next section.

Meanwhile, for a resolution IV design, we know that $A_1 = A_2 = A_3 = 0$, and we can easily obtain that

$$A_4 = \frac{1}{6}\sum_{k=1}^{K_2} {}_2^{\#}C_2^{(k)} k, \tag{2.2}$$

$$A_5 = \frac{1}{10}\Big(\sum_{k=1}^{K_3} {}_2^{\#}C_3^{(k)} k - \sum_{k=1}^{K_2} {}_1^{\#}C_2^{(k)} k(n-3)\Big) = \frac{1}{10}\sum_{k=1}^{K_3} {}_2^{\#}C_3^{(k)} k,$$

$$A_6 = \frac{1}{20}\Big(\sum_{k=1}^{K_3} {}_3^{\#}C_3^{(k)} k - \sum_{k=1}^{K_2} {}_2^{\#}C_2^{(k)} k(n-4)\Big), \text{ where } K_2 = \binom{n}{2}, K_3 = \binom{n}{3}.$$

To get more relationships between the aliased effect-number pattern and the wordlength pattern of a design, refer to Zhang et al. (2008).

3. GMC Dsigns for $17N/64 < n \leq N/2$

In this section, we provide some direct approaches to constructing GMC designs with N runs and $17N/64 < n \leq N/2$ factors. Note that, for resolution IV designs, we need to sequentially maximize ${}_2^{\#}C_2^{(k)}$ for $k = 1, \ldots, \binom{n}{2}$ in order to obtain a GMC design, while minimizing A_4 first in order to achieve MA. From (2.2), we know that A_4 is a weighted sum of ${}_2^{\#}C_2^{(k)}$ for $k = 1, \ldots, \binom{n}{2}$, thus GMC designs may not agree with the MA designs of the same size, as is evidenced below.

3.1. GMC designs for $5N/16 < n \leq N/2$

In this subsection, we denote the $N \times N/2$ maximal even design by X. For a two-level regular $N_0 \times n_0$ design X_0 with entries 0 and 1, its double is defined as

$$\begin{pmatrix} X_0 & X_0 \\ X_0 & X_0 + 1 \end{pmatrix} \pmod 2.$$

Chen and Cheng (2006) and Xu and Cheng (2008) showed that the $N \times N/2$ maximal even design X can be obtained by repeatedly doubling $\binom{0}{1}$, $t-1$ times. For the $N \times N/2$ maximal even design, there are $N/2 - 1$ alias sets which do not contain main effects and each of them contains $N/4$ two-factor interactions; each factor of this design must be contained in a two-factor interaction in every alias set not containing main effects.

From Zhang et al. (2008) we know that a GMC design must have maximum resolution. Since all regular designs of resolution IV or higher with $5N/16 < n \leq N/2$ factors must be projections of the maximal even design with $N/2$ factors, a GMC design with $5N/16 < n \leq N/2$ factors is a projection of the maximal even design.

Theorem 1. *For $5N/16 < n \leq N/2$, the GMC design is a projection of the $N \times N/2$ maximal even design; it can be the design that consists of the last n factors of the maximal even design obtained by repeatedly doubling $\binom{0}{1}$.*

Butler (2003), Chen and Cheng (2006), and Xu and Cheng (2008) showed that the MA designs with N runs and $5N/16 < n \leq N/2$ factors are projections of the $N \times N/2$ maximal even design. They also found that an MA design D with N runs and $5N/16 < n \leq N/2$ factors should sequentially minimize $A_4, A_6, \ldots$ of $\overline{D}$, where $D \bigcup \overline{D} = X$ and X is the $N \times N/2$ maximal even design. It is easy to obtain that any two projection designs with $N/2 - 3$ factors of the $N \times N/2$ maximal even design are isomorphic, the same conclusion holds for projection designs of the maximal even design with $N/2-2$ and $N/2-1$ factors, respectively. Thus for $N/2 - 3 \leq n \leq N/2$, GMC and MA designs are same. For $5N/16 < n < N/2 - 3$, GMC and MA designs may be different.

Example 1. For $N = 64$ and $n = 26$, we obtain a GMC design by deleting the first six columns from the 64×32 maximal even design, while the design constructed by deleting the 1st, 2nd, 3rd, 5th, 9th, and 17th columns is an MA design. The word length patterns of these two designs are $(0, 0, 0, 518, 0, 7032, \ldots)$ and $(0, 0, 0, 515, 0, 7062, \ldots)$, respectively, the GMC design has ${}_2^{\#}C_2 = (\underbrace{0, \ldots, 0}_{9}, 240, 0, 72, 13, 0, \ldots, 0)$, and the MA design has ${}_2^{\#}C_2 = (\underbrace{0, \ldots, 0}_{9}, 160, 165, 0, \ldots, 0)$.

3.2. GMC designs for $9N/32 < n \leq 5N/16$

The maximal design of resolution IV with $N = 16 \cdot 2^r$ runs and $n = 5N/16$ factors can be found by repeatedly doubling the 2^{5-1} design X_0 defined by $I = ABCDE$, r times. As illustrated by Chen and Cheng (2006), there are $10 \cdot 2^r$ alias sets containing 2^r between-group two-factor interactions and $2^r - 1$ alias sets containing $5 \cdot 2^{r-1}$ within-group two-factor interactions, where between-group two-factor interactions are interactions of factors generated from two different factors of X_0, and within-group two-factor interactions are interactions of factors generated from a single factor of X_0.

All regular designs of resolution IV or higher with $9N/32 < n \leq 5N/16$ factors must be projections of the maximal regular design with either $N/2$ or $5N/16$ factors. We further develop the construction of GMC designs with $9N/32 < n \leq 5N/16$ factors.

Theorem 2. *For $N = 16 \cdot 2^r$ and $9N/32 < n \leq 5N/16$, all GMC designs are projections of the design X constructed by repeatedly doubling the 2^{5-1} design X_0 defined by $I = ABCDE$, r times. In addition, the deleted factors can be the first $5N/16 - n$ factors generated from a single factor of X_0.*

It is worth noting that, for $9N/32 < n \le 5N/16$, MA designs are also projections of the maximal design with $5N/16$ factors, see Theorems 3 and 4 of Xu and Cheng (2008). For $n = 5N/16$ and $n = 5N/16 - 1$ ($N \neq 32$), GMC and MA designs are same. However, for $9N/32 < n \le 5N/16 - 2$, GMC and MA designs are different, since the GMC designs are constructed by deleting factors generated from a single factor, and some of the pairwise differences of the numbers f_A, f_B, f_C, f_D, and f_E exceed one; for obtaining MA designs from X, such pairwise differences should not exceed one (Xu and Cheng (2008)). Here f_A, f_B, f_C, f_D, and f_E are the numbers of deleted factors that are generated from A, B, C, D, and E, respectively.

Example 2. For $N = 128$ and $n = 37$, we can obtain a GMC design by deleting the 1st, 6th, and 11th columns from the 128×40 maximal design, while the design obtained by deleting the first three columns from the 128×40 maximal design is an MA design. The word length patterns of these two designs are $(0, 0, 0, 889, \ldots)$ and $(0, 0, 0, 854, \ldots)$, respectively, the GMC design has ${}_2^{\#}C_2 = (0, 0, 0, 0, 160, 0, 0, 384, \underbrace{0, \ldots, 0}_{8}, 68, 54, 0, \ldots, 0)$ and the MA design has ${}_2^{\#}C_2 = (\underbrace{0, \ldots, 0}_{5}, 126, 357, 64, \underbrace{0, \ldots, 0}_{8}, 119, 0, \ldots, 0)$.

3.3. GMC designs for $17N/64 < n \le 9N/32$

The maximal design of resolution IV with $N = 32 \cdot 2^m$ runs and $n = 9N/32$ factors can be constructed by repeatedly doubling a regular 2^{9-4} design with defining contrast subgroup

$$\begin{aligned} I &= 1235 = 2346 = 3457 = 1456 = 1247 = 2567 = 1367 = 123456789 \\ &= 46789 = 15789 = 12689 = 23789 = 35689 = 13489 = 24589. \qquad (3.1) \end{aligned}$$

There are 15 alias sets containing one two-factor interaction and 7 alias sets containing three two-factor interactions in this 2^{9-4} design.

Clearly, in the $N \times 9N/32$ maximal design, there are $15 \cdot 2^m$ alias sets containing 2^m two-factor interactions, $7 \cdot 2^m$ alias sets containing $3 \cdot 2^m$ two-factor interactions, and $2^m - 1$ alias sets containing $9 \cdot 2^{m-1}$ two-factor interactions (Chen and Cheng (2006, Thm. 2.2)).

We have that, for $17N/64 < n \le 9N/32$, all regular designs of resolution IV or higher are projections of the maximal regular design with $N/2$, $5N/16$, or $9N/32$ factors. Since a GMC design has maximum resolution, the GMC design with N runs and $17N/64 < n \le 9N/32$ factors is also a projection of the maximal regular design with $N/2$, $5N/16$, or $9N/32$ factors.

Theorem 3. *For $N = 32 \cdot 2^m$ and $17N/64 < n \leq 9N/32$, the GMC design is a projection of the maximal design obtained by repeatedly doubling the 2^{9-4} design defined by* (3.1). *In addition, deleted factors can be the first $9N/32 - n$ factors generated from the single factor* 8 *or* 9.

It is worth noting that for $17N/64 < n \leq 9N/32$, GMC designs are projections of the maximal design with $9N/32$ factors, whereas MA designs are projections of the maximal design with $5N/16$ factors according to Theorem 3 of Xu and Cheng (2008). Thus these two kinds of designs are totally different.

Based on Theorems 1, 2, and 3, many GMC designs can be easily constructed; some thus constructed for $N \leq 128$ are tabulated in the Appendix.

We now use an example to illustrate the application of GMC designs.

Example 3. Suppose we wish to investigate some 18 factors in a 64-run experiment; besides the main effects, we also wish to estimate some two-factor interactions if certain two-factor interactions are negligible. For the 64×18 GMC design that is obtained by doubling the 32×9 design defined by (3.1), where the 18 factors are labeled by $1, \ldots, 9, t_1, \ldots, t_9$, respectively, it can be easily checked that ${}_2^{\#}C_2^{(0)} = 0$ and ${}_2^{\#}C_2^{(1)} = 60$, i.e., there are no clear two-factor interactions and 30 alias sets each containing two two-factor interactions. The alias relationships among these 60 two-factor interactions are:

$$i8 = t_i t_8,\ t_i 8 = i t_8,\ \text{for } i = 1, \ldots, 7, \text{ and}$$
$$i9 = t_i t_9,\ t_i 9 = i t_9,\ \text{for } i = 1, \ldots, 8.$$

So, under the assumption that three-factor and higher order interactions are negligible, if one of the two-factor interactions in each alias set can be ignored, then we can estimate all the main effects and the other 30 two-factor interactions in these aliased sets. For a 64×18 MA design, if we wish to estimate 30 two-factor interactions, it can be calculated that at least $3 + 25 \cdot 2 + 2 \cdot 3 = 59$ two-factor interactions need to be ignored, since ${}_2^{\#}C_2^{(0)} = 0$, ${}_2^{\#}C_2^{(1)} = 6$, ${}_2^{\#}C_2^{(2)} = 75$, and ${}_2^{\#}C_2^{(3)} = 96$ (cf., Zhang et al. (2008)).

When we have some prior knowledge about the effect hierarchy in practice, the GMC criterion chooses designs that are better than dose the MA criterion, though we recommend the MA designs in the absence of prior information.

4. Proofs

4.1. Some lemmas

First, some notation. There are $2^t - 1$ distinct points of the form $(x_1, \ldots x_t)'$ in the projective geometry $PG(t-1, 2)$ of dimension $t-1$ over $GF(2)$, where

$x_i = 0$ or 1 $(1 \le i \le t)$ and not all of $x_1, \ldots, x_t$ are zeros. In the following, we represent the point $(x_1, \ldots x_t)'$ of $PG(t-1,2)$ by $1^{x_1} \cdots t^{x_t}$, where i^{x_i} is dropped if $x_i = 0$. We denote the set that consists of all the points in $PG(t-1,2)$ with odd numbers of i's satisfying $x_i = 1$ $(1 \le i \le t)$ by O_t, and the set that consists of all the points in $PG(t-1,2)$ with even numbers of i's satisfying $x_i = 1$ $(1 \le i \le t)$ by E_t; there are $\sum_{1 \le 2i+1 \le t} \binom{t}{2i+1} = 2^{t-1}$ points in O_t and $\sum_{2 \le 2i \le t} \binom{t}{2i} = 2^{t-1} - 1$ points in E_t. For example, for $t = 4$, $O_4 = \{1, 2, 3, 123, 4, 124, 134, 234\}$ and $E_4 = \{12, 13, 14, 23, 34, 1234\}$. Furthermore, as in the Yates order, we can order the elements in O_t as

$$\begin{gathered} 1, 2, 3, 123, 4, 124, 134, 234, 5, 125, 135, 235, 145, 245, 345, 12345, \ldots, \\ t, 12t, \ldots, 123 \cdots t \text{ (for odd } t) \text{ or } 23 \cdots t \text{ (for even } t), \end{gathered} \tag{4.1}$$

where the elements $i, \ldots, 123 \cdots i$ (for odd i) or $23 \cdots i$ (for even i) can be viewed as triple sums $1+1+i, \ldots, 1+23 \cdots (i-1)+i$ (for odd i) or $1+123 \cdots (i-1)+i$ (for even i) for $i \ge 4$. In fact, for $t > 3$, O_t can be represented as $O_{t-1} \cup \{h_0+b+t | b \in O_{t-1}\}$ with $h_0 \in O_{t-1}$ and $O_3 = \{1, 2, 3, 123\}$. By taking $h_0 = 1$, ordering the elements of O_3 as 1, 2, 3, 123, and ordering the elements of $\{1+b+t | b \in O_{t-1}\}$ in accordance with that of O_{t-1}, then we have (4.1) directly. Note that the sum of any two points from O_t is a point from E_t. Meanwhile, for any point b from E_t, there are 2^{t-2} distinct pairs of points from O_t, such that the sum of each pair is equal to b and every point of O_t appears in one of these pairs, see for example, $12 = 1+2 = 3+123 = 4+124 = 134+234$ for $t = 4$.

Tang and Wu (1996) introduced the concept of isomorphism, we use it to reduce the number of searches for optimal solutions. An isomorphism ϕ is a one-one mapping from $PG(t-1,2)$ to $PG(t-1,2)$ such that $\phi(c_{i_1} + c_{i_2}) = \phi(c_{i_1}) + \phi(c_{i_2})$ for any c_{i_1} and c_{i_2} from $PG(t-1,2)$. Two regular designs, one consisting of the columns corresponding to the points $c_1, \ldots, c_n$ from $PG(t-1,2)$, and the other of the columns corresponding to the points $d_1, \ldots, d_n$ from $PG(t-1,2)$, are said to be isomorphic if there is an isomorphic mapping ϕ that maps c_i to d_i, $i = 1, \ldots, n$. Isomorphic designs are treated as the same.

Lemma 1. *Two designs, one consisting of the columns corresponding to the points $b_1, \ldots, b_v$ and the other consisting of the columns corresponding to the points $h_0+a+b_1, \ldots, h_0+a+b_v$, are isomorphic, where $h_0, a, b_1, \ldots, b_v \in O_t$.*

Proof. Let

$$\phi(b) = \begin{cases} h_0 + a + b, & \text{if } b \in O_t, \\ b, & \text{if } b \in E_t. \end{cases}$$

It is easy to verify that $\phi(b)$ is an isomorphic mapping from $PG(t-1,2)$ to $PG(t-1,2)$ that maps b_i to $h_0 + a + b_i$ for $h_0, a, b_i \in O_t$, $i = 1, \ldots, v$.

Let M be an m-subset of O_t, the rank of M, denoted by rank(M), is the maximal number of independent points in M. Note that any three points in O_t are independent, so if $m \geq 3$, rank$(M) \geq 3$. If M is a subset with rank $p+1$ $(3 \leq p+1 \leq t)$, then M can be represented as

$$M = H \cup \{h + a + b_1, \ldots, h + a + b_v\},$$

where H is a subset of O_p, embedded in O_t, with rank p, $h \in H$, $a \in O_t \backslash O_p$, and $b_1, \ldots, b_v \in O_p$.

Among the m points of M, each pair determines a point as their sum, but these $\binom{m}{2}$ points are not all distinct. With $S = \{a + b : a, b \in M\}$, the next lemma investigates the number of distinct points in S.

Lemma 2. *Let M be an m-subset of O_t, $m = 2^{k-1} + q \geq 3$, $0 < q \leq 2^{k-1}$, and $k + 1 \leq t$. Suppose rank $(M) = p + 1$, then $p \geq k$ and*

(i) *if $p = k$, there are $2^k - 1$ distinct points in S,*

(ii) *if $p > k$, there are more than $2^k - 1$ distinct points in S.*

The proof is carried out mainly by induction, it is long and omitted here. Interested readers can obtain it from the authors.

There are t independent factors in the $N \times N/2$ maximal even design with $N = 2^t$, and any factor of this design is a product of the t independent factors. Since there is no odd length word in an even design, any factor can only be the product of an odd number of these independent factors, i.e., each factor can be regarded as a point in O_t. As there are $N/2 = 2^{t-1}$ factors in the $N \times N/2$ maximal even design, each of the 2^{t-1} points in O_t corresponds to a factor of this design. Let X and X_{t-2} be the maximal even designs that are obtained by repeatedly doubling $\binom{0}{1}$, $t-1$ and $t-2$ times, respectively. Then the first 2^{t-2} factors of X form $\binom{X_{t-2}}{X_{t-2}}$, and the last 2^{t-2} factors of X form $\binom{X_{t-2}}{X_{t-2}+1}$ (mod 2). It can be easily observed that the jth column of X corresponds to the jth point in (4.1). In addition, the interaction of any two factors can be viewed as the sum of their corresponding points in O_t.

Let design $\overline{D}$ and D be a pair of complementary designs of the maximal even design X, i.e., $D \bigcup \overline{D} = X$, where design D has n factors and $\overline{D}$ has f factors with $n + f = N/2$. The n factors of D can be viewed as n points of O_t, which can be obtained by deleting f points that correspond to the f factors of $\overline{D}$ from O_t.

A simple group-theoretic argument gives the next result.

Lemma 3. *Suppose $D_1 \bigcup \overline{D}_1 = D_2 \bigcup \overline{D}_2 = X$. If $\overline{D}_1$ and $\overline{D}_2$ are isomorphic, then D_1 and D_2 are isomorphic.*

Let D be a projection design with n factors of the $N \times N/2$ maximal even design X. Order the numbers of two-factor interactions among the n factors in the alias sets of X nondecreasingly, and let $\delta_i(D)$ be the ith number. Then the GMC design among projections of the maximal even design should sequentially minimize $\delta_i(D)$. Actually, for $D \cup \overline{D} = X$,

$$\delta_i(D) = \frac{N}{4} - f + \delta_i(\overline{D}), \quad i = 1, \ldots, \frac{N}{2} - 1. \tag{4.2}$$

Let $\delta(D) = (\delta_1(D), \ldots, \delta_{N/2-1}(D))'$ and $\delta(\overline{D}) = (\delta_1(\overline{D}), \ldots, \delta_{N/2-1}(\overline{D}))'$.

Lemma 4. *Let $D_1 \bigcup \overline{D}_1 = D_2 \bigcup \overline{D}_2 = X$, where both D_1 and D_2 have $N/4 < n < N/2$ factors. Suppose $\delta(\overline{D}_1) \neq \delta(\overline{D}_2)$, and s is the smallest integer such that $\delta_s(\overline{D}_1) \neq \delta_s(\overline{D}_2)$. Then $\delta_s(\overline{D}_1) < \delta_s(\overline{D}_2)$ if and only if D_1 dominates D_2 under the GMC criterion.*

Lemma 5. *A design D with $N/4 < n < N/2$ factors has GMC, among projections of the $N \times N/2$ maximal even design, only if its complementary design $\overline{D}$ has $k+1$ independent factors, where k satisfies $2^{k-1} < f = N/2 - n \leq 2^k$.*

Proof. Since $2^{k-1} < f \leq 2^k$, from Lemma 2 we know that $\overline{D}$ has at least $k+1$ independent factors. In addition, there are $N/2 - 1 - (2^k - 1)$ zero elements in $\delta(\overline{D})$ if there are $k+1$ independent factors in $\overline{D}$, and there are less than $N/2 - 1 - (2^k - 1)$ zero elements in $\delta(\overline{D})$ if there are more than $k+1$ independent factors in $\overline{D}$. The result follows from Lemma 4.

Let $\overline{M}$ be the set consisting of the f points corresponding to the factors of $\overline{D}$. From Lemma 5, we have rank($\overline{M}$)$= k+1$ if D has GMC among all projections of the maximal even design, $2^{k-1} < f \leq 2^k$. Without loss of generality, we assume $\overline{M} \subseteq O_{k+1}$. If $f_c = 2^k - f$, then $0 \leq f_c < 2^{k-1}$ and $\overline{M}$ can be obtained by deleting f_c points from O_{k+1}. As in Lemma 5, it is easy to verify that if design D has GMC and $0 < f_c < 2^{k-1}$, the f_c deleted points from O_{k+1} must have $k_c + 1$ independent points, $2^{k_c - 1} < f_c \leq 2^{k_c}$. From Lemma 1, we know O_k and $O_{k+1} \backslash O_k$ are isomorphic, each with k independent points. Without loss of generality, assume that the f_c deleted points belong to $O_{k+1} \backslash O_k$. Thus, for a GMC design among all projections of the maximal even design, $O_k \subset \overline{M}$ and $\overline{M}$ can be represented as

$$\overline{M} = O_k \cup \{h + a + b_1, \ldots, h + a + b_v\},$$

where $h, b_1, \ldots, b_v \in O_k$, $a \in O_{k+1} \backslash O_k$ and $v = f - 2^{k-1}$.

Let $M' = \{b_1, \ldots, b_v\}$, and D' be the design whose factors correspond to the points of M'. It is easy to verify that

$$\delta_i(\overline{D}) = \begin{cases} 0, & \text{if } 1 \leq i \leq \frac{N}{2} - 2^k, \\ v, & \text{if } \frac{N}{2} - 2^k < i \leq \frac{N}{2} - 2^{k-1}, \\ \delta_i(D') + 2^{k-2} \geq v, & \text{if } \frac{N}{2} - 2^{k-1} < i \leq \frac{N}{2} - 1. \end{cases} \tag{4.3}$$

Clearly, if D'_0 is the only design to sequentially minimize $\delta(D')$, then $\overline{D}_0$ is the only design to sequentially minimize $\delta(\overline{D})$ and D_0 has GMC. Actually, as $0 < f < N/4$ and it can be expressed as $f = 2^{k_1-1} + \cdots + 2^{k_w-1} + f_0$, where $k_1 > \cdots > k_w$ and $0 \leq f_0 \leq 3$, to sequentially minimize $\delta(\overline{D})$, $\overline{M}$ can be represented as

$$\begin{aligned} \overline{M} = O_{k_1} &\cup \{h_1 + a_1 + b_1 | b_1 \in O_{k_2}\} \cup \cdots \\ &\cup \{h_1 + a_1 + \cdots + h_{w-1} + a_{w-1} + b_{w-1} | b_{w-1} \in O_{k_w}\} \\ &\cup \{h_1 + a_1 + \cdots + h_w + a_w + b_0 | b_0 \in M_0\}, \end{aligned}$$

where $h_i \in O_{k_i}$ and $a_i \in O_{k_i+1} \backslash O_{k_i}$ for $i = 1, \ldots, w$, and M_0 is an f_0-subset of $\{1, 2, 3\}$. Without loss of generality, we take $h_i = 1$ and $a_i = k_i + 1$ for $i = 1, \ldots, w$, i.e., M consists of the first f points given in (4.1).

Lemma 6. *Let D and $\overline{D}$ be complementary projection designs of the maximal even design X with $N = 2^t$ runs. For $N/4 < n \leq N/2$, D has GMC among all possible n-factor projections of the maximal even design if $\overline{D}$ consists of the first $N/2 - n$ factors of X obtained by doubling $\binom{0}{1}$, $t - 1$ times.*

From (4.2) and (4.3), the GMC design D with $N/4 < n \leq N/2$, among all n-factor projections of the maximal even design, has

$$\delta_i(D) = \begin{cases} \frac{N}{4} - f, & \text{if } 1 \leq i \leq \frac{N}{2} - 2^k, \\ \frac{N}{4} - 2^{k-1}, & \text{if } \frac{N}{2} - 2^k < i \leq \frac{N}{2} - 2^{k-1}, \\ \frac{N}{4} - f + \delta_i(D') + 2^{k-2} \geq \frac{N}{4} - 2^{k-1}, & \text{if } \frac{N}{2} - 2^{k-1} < i \leq \frac{N}{2} - 1. \end{cases} \tag{4.4}$$

The next lemma studies GMC designs, among all possible projections of the maximal design with $5N/16$ factors.

Lemma 7. *Let X be the $N \times 5N/16$ maximal design with $N = 16 \cdot 2^r$ that is generated by repeatedly doubling the 2^{5-1} design X_0 defined by $I = ABCDE$, r times. The design with $N/4 < n \leq 5N/16$ factors that has GMC among all n-factor projections of X and the $N \times N/2$ maximal even design is a projection of X, and the deleted factors are generated from a single factor of X_0.*

Proof. Let $u = 5N/16 - n$. Since $5N/16 - N/4 = 2^r$, we have $0 \le u < 2^r$ and $2^{r+1} < N/2 - n = 3 \cdot 2^r + u < 2^{r+2}$. It is easy to obtain that the result holds for $u = 0$. As for $u > 0$, we first show that the necessary condition for a design to have GMC among all n-factor projections of X is that the deleted factors must be generated from a single factor of X_0. For X, there are $10 \cdot 2^r$ alias sets containing 2^r between-group two-factor interactions and $2^r - 1$ alias sets containing $5 \cdot 2^{r-1}$ within-group two-factor interactions. If a design is obtained by deleting u factors from X, then the minimum number of two-factor interactions in the alias sets containing two-factor interactions is $2^r - u$, since $u < 2^r$ and $2^r < 5 \cdot 2^{r-1}$. To get a GMC design, we should maximize the number of those alias sets containing $2^r - u$ two-factor interactions. If the deleted factors are generated from a single factor of X_0, there are 2^{r+2} alias sets containing $2^r - u$ two-factor interactions; if the deleted u factors are generated from two different factors of X_0, there are less than 2^r alias sets containing $2^r - u$ two-factor interactions. If the deleted u factors are generated from more than two original factors, the design has no alias set containing $2^r - u$ two-factor interactions, i.e., any of the $11 \cdot 2^r - 1$ alias sets containing two-factor interactions has more than $2^r - u$ of them. So the deleted factors must be generated from a single factor of X_0 in order to produce a GMC design. In such a resulting design, the $10 \cdot 2^r$ alias sets that contain between-group two-factor interactions can be classified into two types: $4 \cdot 2^r$ alias sets containing $2^r - u$ two-factor interactions, and $6 \cdot 2^r$ alias sets containing 2^r two-factor interactions. Hence, the GMC design among all n-factor projections of X has ${}_2^{\#}C_2^{(2^r-u-1)} = 4 \cdot 2^r \cdot (2^r - u)$, ${}_2^{\#}C_2^{(2^r-1)} = 6 \cdot 2^r \cdot 2^r$, and ${}_2^{\#}C_2^{(l)} = 0$, for $0 \le l < 2^r - u - 1$ or $2^r - u - 1 < l < 2^r - 1$.

Next, we pay attention to the GMC design among all n-factor projections of the maximal even design. To get this design, $3 \cdot 2^r + u$ factors are deleted from the $16 \cdot 2^r \times 8 \cdot 2^r$ maximal even design. Thus from (4.4), we have ${}_2^{\#}C_2^{(2^r-u-1)} = 4 \cdot 2^r \cdot (2^r - u)$, ${}_2^{\#}C_2^{(2^{r+1}-1)} \le 2^{2r+3} - 2^{r+1}$, and ${}_2^{\#}C_2^{(l)} = 0$, for $0 \le l < 2^r - u - 1$ or $2^r - u - 1 < l < 2^{r+1} - 1$.

Now, it is obvious that the GMC design among all n-factor projections of X and the maximal even design is a design obtained by deleting $5N/16 - n$ factors from X, with the deleted factors generated from a single factor of X_0.

The following lemma studies GMC designs among all possible projections of the maximal design with $9N/32$ factors.

Lemma 8. *Let X be the $N \times 9N/32$ maximal design with $N = 32 \cdot 2^m$ that is obtained by repeatedly doubling the 2^{9-4} design defined by* (3.1)*. The design with $N/4 < n \le 9N/32$ factors that has GMC among all n-factor projections of X, the $N \times N/2$ maximal even design, and the $N \times 5N/16$ maximal design is a projection of X, and the deleted factors are generated from the single factor* 8 *or* 9.

Proof. Let $u = 9N/32 - n$. Since $9N/32 - N/4 < 2^m$, we have $0 \leq u < 2^m$. Similarly, as with Lemma 7, the GMC design among all n-factor projections of X is the design obtained by deleting u factors generated from the single factor 8 or 9. It is easily shown that there are $8 \cdot 2^m$ alias sets containing $2^m - u$ two-factor interactions and $7 \cdot 2^m$ alias sets containing 2^m two-factor interactions in such a design. So ${}_2^{\#}C_2^{(2^m-u-1)} = 8 \cdot 2^m \cdot (2^m - u)$, ${}_2^{\#}C_2^{(2^m-1)} = 7 \cdot 2^m \cdot 2^m$, and ${}_2^{\#}C_2^{(l)} = 0$, for $0 \leq l < 2^m - u - 1$ or $2^m - u < l < 2^m - 1$.

From Lemma 7, the GMC design with $N/4 < n \leq 9N/32$ factors among all n-factor projections of the $N \times N/2$ maximal even design and the $N \times 5N/16$ maximal design is a projection of the latter one. Here then is the design obtained by deleting $u + 2^m$ factors generated from a single original factor from the $N \times 5N/16$ maximal design. From the proof of Lemma 7, we know that the design has $8 \cdot 2^m$ alias sets containing $2^{m+1} - (2^m + u) = 2^m - u$ two-factor interactions, and $12 \cdot 2^m$ alias sets containing 2^{m+1} two-factor interactions. So ${}_2^{\#}C_2^{(2^m-u-1)} = 8 \cdot 2^m \cdot (2^m - u)$, ${}_2^{\#}C_2^{(2^{m+1}-1)} = 12 \cdot 2^m \cdot 2^{m+1}$, and ${}_2^{\#}C_2^{(l)} = 0$, for $0 \leq l < 2^m - u - 1$ or $2^m - u < l < 2^{m+1} - 1$.

Obviously, the projection design of X dominates the other one under the GMC criterion, so the proof is complete.

4.2. Proofs of theorems

Proof of Theorem 1. This is easily obtained from Lemma 6 and the fact that all GMC designs with $5N/16 < n \leq N/2$ are projections of the maximal even design.

Proof of Theorem 2. Let $u = 5N/16 - n$. Since $5N/16 - 9N/32 = 2^{r-1}$, we have $0 \leq u \leq 2^{r-1}$. Based on Lemma 7, we only need to investigate which factors generated from a single factor of X_0 are to be deleted in order to obtain GMC designs for $9N/32 < n \leq 5N/16$. Without loss of generality, let the deleted factors be generated from factor A. By deleting different u factors generated from A, the resulting designs have the same numbers of two-factor interactions in the alias sets containing between-group two-factor interactions. Thus, we only need to check the numbers of two-factor interactions in the alias sets containing within-group two-factor interactions generated from A in the resulting design. Let D_A be the design that is obtained by repeatedly doubling the design consisting of only one factor, A, r times. Then D_A has the same alias sets as that of the $2^{r+1} \times 2^r$ maximal even design. In addition, if there is an alias set that has w two-factor interactions in the design obtained by deleting u factors from D_A, there must be an alias set which has $2^{r+1} + \omega$ two-factor interactions in the design obtained by deleting the same u factors from X. Based on Lemma 6, as $u < 2^{r-1}$, the projection of X has GMC if the deleted factors are the first u factors generated from A. This completes the proof.

Proof of Theorem 3. This is easily obtained from Lemma 8.

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Appendix. Some GMC designs with small run sizes

N	n	GMC designs*
32	9	D_{32}^3 (the 2^{9-4} design defined by (3.1))
32	10	D_{32}^2 (the design yielded by doubling the design defined by $I = ABCDE$)
32	11–16	the design consisting of the last n columns of D_{32}^1*
64	18	D_{64}^3 (the design yielded by doubling D_{32}^3)
64	19–20	the design consisting of the last n columns of D_{64}^2†
64	21–32	the design consisting of the last n columns of D_{64}^1*
128	35–36	the design consisting of the last n columns of D_{128}^3‡
128	37	the design obtained by deleting the 1st, 6th and 11th columns from D_{128}^2†
128	38	the design obtained by deleting the 1st and 6th columns from D_{128}^2†
128	39–40	the design consisting of the last n columns of D_{128}^2†
128	40–64	the design consisting of the last n columns of D_{128}^1*

* D_{32}^1, D_{64}^1 and D_{128}^1 are the maximal even designs with 32, 64, and 128 runs, respectively;
† D_{64}^2 and D_{128}^2 are the designs yielded by doubling D_{32}^2 and D_{64}^2, respectively;
‡ D_{128}^3 is the design yielded by doubling D_{64}^3.

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