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GENERAL MINIMUM LOWER ORDER CONFOUNDING IN BLOCK DESIGNS USING COMPLEMENTARY SETS

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Abstract: We consider regular fractions of s-level factorials arranged in block designs. Optimal designs are explored under the criterion of general minimum lower order confounding which aims, in an elaborate manner, at keeping the lower order factorial effects unaliased with one another and unconfounded with blocks. A finite projective geometric formulation, that identifies the alias sets with the points and the blocking system with a flat of the geometry, forms the mathematical basis of our approach. Theoretical results and tables are obtained in terms of complementary sets and an idea of double complementation is found to be useful in some situations.

Key words and phrases: Alias set, double complementation, effect hierarchy principle, flat, projective geometry, wordlength pattern.

1. Introduction

Optimal selection of regular fractional factorial plans, under model uncertainty, has been a subject of considerable recent interest. Blocking makes the problem significantly more complex because it warrants simultaneous handling of two wordlength patterns (WLPs), one arising from the choice of the fraction and the other due to blocking. Design criteria, involving either interpenetration of the two WLPs (Sitter, Chen and Feder (1997), and Cheng and Wu (2002)) or their combination (Chen and Cheng (1999), and Cheng and Tang (2005)) have been proposed and extensively studied; see Mukerjee and Wu (2006, Chap. 7) for a review. These criteria, based on the effect hierarchy principle (Wu and Hamada (2000, p.112)), are inspired by that of minimum aberration (MA) in the unblocked case and are all motivated, in various senses, by the objectives of (a) keeping the lower order factorial effects unaliased with one another and (b) avoiding their being confounded with blocks. For unblocked two-level factorials, Zhang, Li, Zhao and Ai (2008) introduced a new criterion of general minimum lower order confounding (GMLOC or GMC for short) that aims at achieving (a) in a very elaborate manner. Zhang and Mukerjee (2009) obtained, for general s-level factorials and in the unblocked case, results on the GMC criterion in terms of complementary sets.

The purpose of the present paper is to develop, in the presence of blocks, a theory for the GMC criterion via the use of complementary sets. This is done for general s-level factorials. The GMC criterion, as adapted to block designs, is motivated by the twin objectives (a) and (b) above and attempts to realize these objectives via explicit consideration of the alias sets in addition to the two WLPs. A finite projective geometric formulation, as well as the findings in Zhang and Mukerjee (2009) in the unblocked case, form the foundation of our results. Substantial additional work is, however, required because in the blocked case one needs to deal with three sets, one signifying the fraction, one for blocking and a third one which is the complement of the union of the first two. Our final results and tables are expected to be particularly useful in the practically important nearly saturated situation where the complementary set is relatively small in size and hence easy to handle. This advantage of considering complementary sets has been well recognized for other design criteria as well, both in the blocked and unblocked cases; see Tang and Wu (1996), Suen, Chen and Wu (1997), Chen and Cheng (1999) and Cheng and Tang (2005), among others. An auxiliary approach of double complementation is seen to further simplify the derivation in some situations.

2. GMC Criterion for Block Designs

Consider an s^n factorial involving n factors each at s levels, s (≥ 2) being a prime or prime power. A typical pencil $b = (b_1, \ldots, b_n)'$ is a nonnull n-vector over the finite field $GF(s)$, and pencils with proportional elements are considered identical. A pencil with i nonzero elements represents a factorial effect involving i factors and is called an ith order pencil ($1 \leq i \leq n$). The case $i = 1$ gives a *main effect* (ME) while the case $i > 1$ corresponds to an *interaction.*

With reference to an s^n factorial, we are interested in regular $1/s^m$ fractions arranged in s^r blocks, where $1 \leq m < n$, $1 \leq r < n - m$ and, to avoid trivialities, $n \geq 4$. Such a design will be simply referred to as an (s^{n-m}, s^r) design. For $1 \leq i \leq n$, let A_i be the number of ith order pencils appearing in the defining relation of the design and B_i be the number of ith order pencils that are confounded with its blocks. Given the utmost importance of the MEs, we consider only those designs where no ME pencil appears in the defining equation or is confounded with blocks, and no two ME pencils are aliased with each other. Then $A_1 = A_2 = B_1 = 0$ and, as per the terminology introduced in Mukerjee and Wu (1999), the design has blocked resolution three or higher.

The sequences $(A_3, \ldots, A_n)$ and $(B_2, \ldots, B_n)$, arising respectively from the choice of the fraction and blocking, are called the WLPs of the design. As hinted in the introduction, blocked MA criteria, involving either interpenetration or combination of these two WLPs have received attention in the literature. These include criteria based on sequential minimization of

$$W^{(1)} = (A_3, B_2, A_4, B_3, A_5, B_4, \ldots), \qquad W^{(2)} = (A_3, A_4, B_2, A_5, A_6, B_3, A_7, \ldots),$$
$$W^{(3)} = (A_3, B_2, A_4, A_5, B_3, A_6, A_7, \ldots), W^{(4)} = (3A_3 + B_2, A_4, 10A_5 + B_3, \ldots).$$

Sitter, Chen and Feder (1997) proposed $W^{(1)}$, Zhang and Park (2000) considered $W^{(3)}$, while Cheng and Wu (2002) suggested both $W^{(2)}$ and $W^{(3)}$. Chen and Cheng (1999) also mentioned $W^{(3)}$. For two-level factorials, Chen and Cheng (1999) considered $W^{(4)}$ and Cheng and Tang (2005) discussed variations thereof, one of which has the same leading term as $W^{(4)}$.

We now introduce the GMC criterion for (s^{n-m}, s^r) designs of blocked resolution three or higher. For $1 \le i \le n$, there are K_i $[= \binom{n}{i}(s-1)^{i-1}]$ ith order pencils, of which $K_i - A_i - B_i$ $(= {}_i^{\#}C_0$, say) neither appear in the defining relation nor are confounded with blocks, and hence remain estimable unless they are aliased with other potentially important pencils. Suppose among these $K_i - A_i - B_i$ pencils, there are ${}_i^{\#}C_j^{(k)}$, each of which is aliased with k jth order pencils (excluding itself, if $j = i$), and write ${}_i^{\#}C_j$ for the vector $({}_i^{\#}C_j^{(0)}, {}_i^{\#}C_j^{(1)}, \ldots, {}_i^{\#}C_j^{(K_j)})$, $1 \le i, j \le n$. The sequence

$$^{\#}C = \left({}_1^{\#}C_2, {}_2^{\#}C_0, {}_2^{\#}C_1, {}_2^{\#}C_2, {}_1^{\#}C_3, {}_2^{\#}C_3, {}_3^{\#}C_0, {}_3^{\#}C_1, {}_3^{\#}C_2, {}_3^{\#}C_3, \ldots\right) \tag{2.1}$$

is called the *aliased effect-number pattern* (AENP) of the design. The AENP incorporates the loss of information on pencils due to appearance in the defining equation or confounding with blocks via the terms ${}_i^{\#}C_0$, and captures the nature of aliasing explicitly via the terms ${}_i^{\#}C_j^{(k)}$ $(i, j \ge 1)$.

The effect hierarchy principle helps in explaining why it is meaningful to consider the terms in (2.1) sequentially from left to right. Suppose, in addition to the MEs, two-factor interactions (2fis) are possibly present. Then the first priority is estimating the MEs, i.e., keeping them unaliased with the 2fi pencils to the extent possible, and this makes ${}_1^{\#}C_2$ the first term in (2.1). Turning next to the estimation of any 2fi pencil, note that confounding with blocks would render this impossible while, for an unconfounded 2fi pencil, aliasing with a ME pencil would pose a more serious challenge than that with another 2fi pencil. Since $A_2 = 0$, we thus get the three subsequent terms ${}_2^{\#}C_0$, ${}_2^{\#}C_1$ and ${}_2^{\#}C_2$ in that order. Similarly, if three-factor interactions (3fis) are also possibly present, then estimation of the MEs and the 2fis leads to the next two terms ${}_1^{\#}C_3$ and ${}_2^{\#}C_3$ in (2.1) and, thereafter, estimation of the 3fis themselves entails the terms ${}_3^{\#}C_0$, ${}_3^{\#}C_1$, ${}_3^{\#}C_2$ and ${}_3^{\#}C_3$ in that order.

We also observe that for any fixed i and j $(i, j \ge 1)$, the first element ${}_i^{\#}C_j^{(0)}$ of ${}_i^{\#}C_j$ signifies no aliasing, while the subsequent elements ${}_i^{\#}C_j^{(k)}$ signify progressively more severe aliasing as k increases. Hence, in addition to considering the terms in (2.1) from left to right as indicated above, upon reaching a particular ${}^{\#}_{i}C_j$ $(i, j \geq 1)$, it makes sense to maximize its elements sequentially from left to right; similarly when any ${}^{\#}_{i}C_0$ $(= K_i - A_i - B_i)$ is reached, one should try to make it as large as possible. These points, when summed up, amount to sequential maximization of the elements of ${}^{\#}C$ in (2.1), from left to right, which is precisely how we define the GMC criterion for block designs. As a consequence of this definition, the terms ${}^{\#}_{j}C_1$ $(j \geq 2)$ can be dropped from (2.1) because analogously to the unblocked case (Zhang and Mukerjee (2009)), they are uniquely determined by some of the preceding terms. Thus, in effect, the blocked GMC criterion aims at sequential maximization, from left to right, of the elements of a reduced version of (2.1) given by

$$ {}^{\#}C = \left({}^{\#}_{1}C_2, {}^{\#}_{2}C_0, {}^{\#}_{2}C_2, {}^{\#}_{1}C_3, {}^{\#}_{2}C_3, {}^{\#}_{3}C_0, {}^{\#}_{3}C_2, {}^{\#}_{3}C_3, \ldots \right). \tag{2.2} $$

We remark that this can as well be looked upon as a very elaborate version of the clear effects criterion, e.g., the first element of (2.2), namely ${}^{\#}_{1}C_2^{(0)}$, represents the number of clear ME pencils.

3. Geometric Formulation and Preliminary Results

Let P be the set of points of the finite projective geometry $PG(n-m-1, s)$. As usual, points with proportional coordinates are identical. A $(w-1)$-flat of P is an L_w-subset of P that is closed, up to proportionality, under the formation of nonnull linear combinations. Here $L_w = (s^w - 1)/(s-1)$ $(w = 1, 2, \ldots)$. For any nonempty subset Q of P, let $V(Q)$ be the matrix given by the points of Q as columns. Then the following well-known result holds (see e.g., Mukerjee and Wu (2006, Chap. 7)).

Lemma 1. *Any (s^{n-m}, s^r) design d of blocked resolution three or higher is represented by an ordered pair of disjoint subsets (T_0, T) of P such that T_0 is an $(r-1)$-flat, T has cardinality n, the matrix $V(T)$ has full row rank, and*

(a) *any pencil b appears in the defining relation of d if and only if $V(T)b = 0$,*

(b) *any pencil b is confounded with blocks in d if and only if $V(T)b$ is nonnull and proportional to some point of T_0,*

(c) *any two pencils $b^{(1)}$ and $b^{(2)}$, neither of which is a defining pencil or confounded with blocks, are aliased with each other in d if and only if $V(T)b^{(1)}$ and $V(T)b^{(2)}$ are proportional to the same point of $P \setminus T_0$.*

In view of Lemma 1, an (s^{n-m}, s^r) design d of blocked resolution three or higher will be denoted simply by the corresponding pair of sets (T_0, T). Lemma

1 also implies that such a design can exist only if $L_r + n \leq L_{n-m}$, a condition which is assumed to hold. Lemma 1(c) shows a useful one to one correspondence between the unconfounded alias sets and the points of $P \setminus T_0$. Consider any $(r-1)$-flat T_0, any point π, and any nonempty subset Q, of P. Let $q = \#Q$, where $\#$ denotes cardinality, and Ω_{iq} be the set of q-vectors over $GF(s)$ having i nonzero elements. For $i \geq 1$, define

$$R_i(Q,\pi) = (s-1)^{-1}\#\{\lambda : \lambda \in \Omega_{iq}, V(Q)\lambda \text{ is nonnull and proportional to } \pi\}, \tag{3.1}$$

$$B_i(T_0,Q) = \sum_{\pi \in T_0} R_i(Q,\pi). \tag{3.2}$$

In particular, for an (s^{n-m}, s^r) design (T_0,T), Lemma 1(b), (c) and (3.1), (3.2) show that $B_i(T_0,T)$ is the same as B_i in the blocking WLP of the design, while $R_i(T,\pi)$ equals the number of ith order pencils appearing in the alias set corresponding to π, for every $\pi \in P \setminus T_0$. Consequently,

$${}^{\#}_{i}C_i^{(k)} = (k+1)\#\{\pi : \pi \in P \setminus T_0, R_i(T,\pi) = k+1\},\ 0 \leq k \leq K_i,\ 1 \leq i \leq n, \tag{3.3}$$

$${}^{\#}_{i}C_j^{(k)} = \sum_{jk} R_i(T,\pi),\ 0 \leq k \leq K_j,\ 1 \leq i \neq j \leq n, \tag{3.4}$$

where $\sum_{jk}$ is sum over π such that $\pi \in P \setminus T_0$ and $R_j(T,\pi) = k$. Let $\widetilde{T}$ be the complement of T in $P \setminus T_0$ and $U = T_0 \cup \widetilde{T}$. Write $f = \#\widetilde{T}$. Then $\#U = L_r + f$ ($= u$, say). Lemma 2 below connects the leading terms of the AENP of the design (T_0,T) with either the complementary set $\widetilde{T}$ or the set U containing $\widetilde{T}$. Parts (a) and (c)−(e) of the lemma follow from Zhang and Mukerjee (2009) using (3.3), (3.4) and the fact that U is the complement of T in P. Part (b) follows from Lemmas 6.3.3 and 6.3.4 of Mukerjee and Wu (2006) noting that ${}^{\#}_{2}C_0 = K_2 - B_2(T_0,T)$. In Lemma 2, c_0, c_1 and c_2 are constants which may depend on s, n, m and r, but not on the specific choice of T_0 and T. The details on these constants will not be needed in the sequel.

Lemma 2. *For the design* (T_0,T)*,*

(a) ${}^{\#}_{1}C_2^{(k)} = \#\{\pi : \pi \in T, (1/2)(s-1)(n-u-1) + R_2(U,\pi) = k\}$, $0 \leq k \leq K_2$,

(b) ${}^{\#}_{2}C_0 = c_0 - B_2(T_0,\widetilde{T})$,

(c) ${}^{\#}_{2}C_2^{(k)} = (k+1)[\#\{\pi : \pi \in T, (1/2)(s-1)(n-u-1) + R_2(U,\pi) = k+1\} + \#\{\pi : \pi \in \widetilde{T}, (1/2)(s-1)(n-u+1) + R_2(U,\pi) = k+1\}]$, $0 \leq k \leq K_2$,

(d) ${}^{\#}_{1}C_3^{(k)} = \#\{\pi : \pi \in T, c_1 - (2s-3)R_2(U,\pi) - R_3(U,\pi) = k\}$, $0 \leq k \leq K_3$,

(e) ${}_2^{\#}C_3^{(k)} = \sum_{3k}^{(1)}\{(1/2)(s-1)(n-u-1)+R_2(U,\pi)\} + \sum_{3k}^{(2)}\{(1/2)(s-1)(n-u+1)+R_2(U,\pi)\}$, $0 \le k \le K_3$,

where $\sum_{3k}^{(1)}$ *is sum over* π *such that* $\pi \in T$ *and* $c_1-(2s-3)R_2(U,\pi)-R_3(U,\pi)=k$, *while* $\sum_{3k}^{(2)}$ *is sum over* π *such that* $\pi \in \widetilde{T}$ *and* $c_2-(2s-3)R_2(U,\pi)-R_3(U,\pi)=k$.

This section is concluded with another lemma that arises from Lemma 2(a) and which follows via the same arguments as used by Zhang and Mukerjee (2009) in the unblocked case. For a block design (T_0, T), let $\delta = (\delta_1, \ldots, \delta_n)$ be the vector with elements $R_2(U,\pi)$, $\pi \in T$, arranged in nondecreasing order. Also, write

$$g = \#\{\pi : \pi \in T, R_2(U,\pi) > 0\}. \tag{3.5}$$

Lemma 3.

(a) *Suppose the vectors* δ *for two designs are not identical, and let* j *be the smallest integer such that the quantities* δ_j *for the two designs differ. Then the design with a smaller* δ_j *dominates the other under the GMC criterion.*

(b) *A design can have GMC only if it minimizes* g.

(c) *If* u $(= L_r + f)$ *equals* L_w, *with* $w \ge r$, *then a design has GMC if and only if the corresponding set* U *is a* $(w-1)$*-flat.*

(d) *Let* $s = 2$. *If* $4 \le u \le 6$ *or* $8 \le u \le 14$, *then a design can have GMC only if* U *is contained in a* 2*-flat or a* 3*-flat, respectively.*

(e) *Let* $s = 3$. *If* $5 \le u \le 12$, *then a design can have GMC only if* U *is contained in a* 2*-flat.*

4. Some Results under the Blocked GMC Criterion

We now build upon the ideas and preliminary results presented in Section 3 to obtain some theoretical results on optimal block designs under the GMC criterion. These will also be useful in preparing the tables that follow in the next section. To avoid trivialities, hereafter we assume that the set $\widetilde{T}$ is nonempty, i.e., $f > 0$, which implies that $u > L_r$.

The first result, shown in Theorem 1 below and proved in the Appendix, extends 3(c), in the two level case, to the situation where u is not exactly equal but close to the cardinality of a flat. Let $\pi_1, \ldots, \pi_w$ be linearly independent points of the projective geometry P, where $w > r$. Denote the $(r-1)$-flat spanned by $\pi_1, \ldots, \pi_r$ by Δ_r, and the $(w-1)$-flat spanned by $\pi_1 \ldots, \pi_w$ by Δ_w. Let $\overline{\Delta} = \Delta_w \setminus \Delta_r$.

Theorem 1. *Let* $s = 2$. *For* $u = 2^w - j$, *where* $w > r$ *and* $j = 2, 3, 4$, *the design given by* $T_0 = \Delta_r$ *and* $\widetilde{T}$ *as shown below has GMC:*

(a) $\widetilde{T} = \overline{\Delta} \setminus \{\pi_{r+1}\}$, *if* $j = 2$ *and* $w > r$;

(b1) $\widetilde{T} = \overline{\Delta} \setminus \{\pi_{r+1}, \pi_1 + \pi_{r+1}\}$, *if* $j = 3$ *and* $w = r + 1$;

(b2) $\widetilde{T} = \overline{\Delta} \setminus \{\pi_{r+1}, \pi_{r+2}\}$, *if* $j = 3$ *and* $w \geq r + 2$;

(c1) $\widetilde{T} = \overline{\Delta} \setminus \{\pi_{r+1}, \pi_1 + \pi_{r+1}, \pi_2 + \pi_{r+1}\}$, *if* $j = 4$ *and* $w = r + 1$;

(c2) $\widetilde{T} = \overline{\Delta} \setminus \{\pi_{r+1}, \pi_{r+2}, \pi_1 + \pi_{r+1} + \pi_{r+2}\}$, *if* $j = 4$ *and* $w \geq r + 2$.

Part (c2) of Theorem 1 springs a little surprise because, in analogy with (b2), one would rather expect the choice $\widetilde{T} = \overline{\Delta} \setminus \{\pi_{r+1}, \pi_{r+2}, \pi_{r+3}\}$ to entail a GMC design for $w \geq r + 3$. The proof in the Appendix will clarify why this is not really the case. Turning to general s, we have Theorem 2 below pertaining to the case $f \leq s^r$, where $f = \#\widetilde{T}$. For $s = 2$, Chen and Cheng (1999) reported the same necessary condition as in Theorem 2 under their blocked MA criterion based on the combined WLP $W^{(4)}$ shown in Section 2. However, despite this identity of the necessary conditions, it will be seen in the next section that their criterion and GMC do not always yield the same optimal design.

Theorem 2. *If* $f \leq s^r$, *then a design can have GMC only if* U *is contained in an* r*-flat.*

Proof. If an $(r-1)$-flat is nested in an r-flat, then there are s^r points that belong to the latter but not to the former. Hence for $f \leq s^r$, one can always choose T_0 and $\widetilde{T}$ such that their union U is contained in an r-flat, say Δ. Then the points of U cannot span any point outside Δ, so that by (3.1) and (3.5), for the resulting design, we get

$$g \leq \#\Delta - \#U = s^r - f. \tag{4.1}$$

Next consider a design for which U is not contained in an r-flat. Then there exist linearly independent points π_j, $1 \leq j \leq r + 2$, such that $\pi_1, \ldots, \pi_r$ span T_0 while $\pi_{r+1}, \pi_{r+2} \in \widetilde{T}$. Let $\Delta^{(j)}$ be the r-flat spanned by $\pi_1, \ldots, \pi_r$ and π_{r+j} $(j = 1, 2)$. Every point of $\Delta^{(j)} \setminus T_0$, other than π_{r+j}, is spanned by π_{r+j} and one point of T_0. Hence by (3.1), $R_2(U, \pi) > 0$ for every π that belongs to $\Delta^{(1)} \cup \Delta^{(2)}$ but not to U. Since $\#\{\Delta^{(1)} \cup \Delta^{(2)}\} = 2s^r + L_r$, it follows from (3.5) that the design under consideration must satisfy $g \geq \#\{\Delta^{(1)} \cup \Delta^{(2)}\} - \#U = 2s^r - f$. Comparing this with (4.1), the result follows from Lemma 3(b).

Theorem 3 below shows how the class of competing designs, that satisfy the necessary condition in Theorem 2, can be reduced further in the two-level case, thereby facilitating the application of the GMC criterion. Part (b) of Theorem 3 formally resembles Lemma 3(a) but is actually much deeper because it is reached only after passing through Theorem 2 and Theorem 3(a). For $s = 2$ and $f \leq 2^r$, consider any design meeting the condition in Theorem 2, and write $\widetilde{U}$ for the complement of U in the r-flat that contains U. Then $\#\widetilde{U} = 2^r - f = p$, say; cf. (4.1). Let $\theta = (\theta_1, \ldots, \theta_p)$ be the vector with elements $R_3(\widetilde{U}, \pi)$, $\pi \in \widetilde{U}$, arranged in nondecreasing order. Since U $(= T_0 \cup \widetilde{T})$ is the complement of T in the entire projective geometry, we get $\widetilde{U} \subset T$, and consideration of $\widetilde{U}$ amounts to double complementation. Typically, $\widetilde{U}$ is much smaller than T and this helps.

Theorem 3. *Let $s = 2$ and $f \leq 2^r$. Consider designs for which U is contained in an r-flat.*

(a) *All such designs have the same ${}_1^{\#}C_2$, ${}_2^{\#}C_0$ and ${}_2^{\#}C_2$.*
(b) *Suppose the vectors θ for two such designs are not identical and let j be the smallest integer such that the quantities θ_j for the two designs differ. Then the design with a smaller θ_j dominates the other under the GMC criterion.*

Proof. (a) For every design under consideration, $\widetilde{T}$ is of the form

$$\widetilde{T} = \{\alpha, \alpha + \alpha_1, \ldots, \alpha + \alpha_{f-1}\}, \tag{4.2}$$

where $\alpha \notin T_0$ and $\alpha_1, \ldots, \alpha_{f-1}$ are distinct members of T_0. By (4.2), any two points of U can add up to a point outside T_0 if and only if one of these points is in T_0 and the other is in $\widetilde{T}$. Hence, from (3.1), $R_2(U, \pi)$ equals $f - 1$ for every $\pi \in \widetilde{T}$, is f for every $\pi \in \widetilde{U}$, and is 0 for every $\pi \in T \setminus \widetilde{U}$. Since $\widetilde{U} \subset T$, Lemma 2(a), (c) now show that all designs considered here have the same ${}_1^{\#}C_2$ and ${}_2^{\#}C_2$. Also, by (4.2) and Lemma 2(b), they all have the same $B_2(T_0, \widetilde{T})$ and hence the same ${}_2^{\#}C_0$.

(b) In view of part (a), under the GMC criterion, one needs to consider the next term ${}_1^{\#}C_3$ in the AENP (2.2) for discrimination among the designs that are being considered. As $\widetilde{U} \subset T$, by Lemma 2(d) with $s = 2$ we get, for $0 \leq k \leq K_3$,

$$\begin{aligned} {}_1^{\#}C_3^{(k)} &= \#\{\pi : \pi \in \widetilde{U}, c_1 - R_2(U, \pi) - R_3(U, \pi) = k\} \\ &\quad + \#\{\pi : \pi \in T \setminus \widetilde{U}, c_1 - R_2(U, \pi) - R_3(U, \pi) = k\}. \end{aligned} \tag{4.3}$$

Since the points of U cannot span a point outside the r-flat containing U, by (3.1) we further get $R_2(U, \pi) = R_3(U, \pi) = 0$ for every $\pi \in T \setminus \widetilde{U}$, so that the second term on the right-hand side, say $c^{(k)}$, does not depend on the specific design. Also, as noted in the proof of (a), $R_2(U,\ pi) = f$ for every $\pi \in \widetilde{U}$. Hence (4.3) yields

$${}_1^{\#}C_3^{(k)} = c^{(k)} + \#\{\pi : \pi \in \widetilde{U}, c_1 - f - R_3(U, \pi) = k\}. \tag{4.4}$$

Since U and $\widetilde{U}$ are complements of each other in an r-flat, which in itself is isomorphic to a finite projective geometry of dimension r, invoking a result from Zhang and Mukerjee (2009, Sec. 3) for the unblocked case, for any $\pi \in \widetilde{U}$, we obtain $R_3(U,\pi) = c - R_2(\widetilde{U},\pi) - R_3(\widetilde{U},\pi)$, where c is a constant that may depend on n, m and r, but not on the specific design. From (4.2) and the definition of $\widetilde{U}$, observe that $\widetilde{U} = \{\alpha + \overline{\alpha}_1, \alpha + \overline{\alpha}_2, \ldots\}$, where $\{\overline{\alpha}_1, \overline{\alpha}_2, \ldots\}$ is the complement of $\{\alpha_1, \ldots, \alpha_{f-1}\}$ in T_0. Then, any two points of $\widetilde{U}$ add up to a point of T_0, i.e., by (3.1), $R_2(\widetilde{U},\pi) = 0$, for every $\pi \in \widetilde{U}$. The facts just noted, together with (4.4), imply that

$$ {}_1^{\#}C_3^{(k)} = c^{(k)} + \#\{\pi : \pi \in \widetilde{U}, R_3(\widetilde{U},\pi) = k + f + c - c_1\}. $$

Recalling the definition of θ, part (b) of the theorem is now evident.

Curiously, Theorem 3 does not have a counterpart for $s \geq 3$, in which case designs meeting the necessary condition in Theorem 2 can be discriminated even on the basis of ${}_1^{\#}C_2$. Example 3 of the next section will serve as an illustration.

5. Examples and Design Tables

This section presents three tables, Tables 1, 2 and 3, showing optimal block designs under the GMC criterion for (i) $s = 2$, $u\ (= 2^r - 1 + f) \leq 15$, (ii) $s = 2$, $r = 4$, $f \leq 16$, and (iii) $s = 3$, $u\ [= (1/2)(3^r - 1) + f] \leq 13$. These tables have a reasonably wide coverage of designs that accommodate a relatively large number of factors vis-à-vis the number of runs. For example, with two-level factorials, in addition to completely settling the case of 16-run designs, Table 1 covers 32-run designs for $n \geq 16$, 64-run designs for $n \geq 48$ and 128-run designs for $n \geq 112$, where n is the number of factors. Furthermore, with 16 blocks, Table 2 covers 64-run designs for $32 \leq n \leq 47$ and 128-run designs for $96 \leq n \leq 111$. With three-level factorials, Table 3 completely settles the case of 27-run designs and covers 81-run designs for $n \geq 27$.

Some of the tabulated optimal designs, such as the ones corresponding to (a) $r = 1$ and $f = 3, 4, 5, 11, 12, 13$, or (b) $r = 2$ and $f = 1, 2, 3, 9, 10, 11$ or (c) $r = 3$ and $f = 5, 6, 7$, in the two level case, follow directly from Theorem 1. For the rest, use of Theorems 2, 3 and Lemma 3 significantly reduces the search. In particular, for each f considered in Table 2, Theorem 3(b) yields the solution. Three illustrative examples follow. We often represent any point $(x_1, x_2, \ldots, x_{n-m})'$ of the finite projective geometry using the simple notation $1^{x_1}2^{x_2}\cdots(n-m)^{x_{n-m}}$, with i^{x_i} dropped if $x_i = 0$.

Example 1. With reference to Table 1, let $s = 2$, $r = 2$, and $f = 7$. This is one of the more challenging situations because none of the results of Section 4 is applicable. But here $u = 10$ and Lemma 3(d) shows that for a design to have GMC, the set U must be contained in a 3-flat. But then U itself represents an unblocked 2^{10-6} design and, following Chen, Sun and Wu (1993), there are four nonisomorphic possibilities for U, namely

(a) $U = \{1, 2, 3, 4, 12, 13, 23, 14, 234, 1234\}$, (b) $U = \{1, 2, 3, 4, 12, 13, 23, 14, 24, 134\}$,
(c) $U = \{1, 2, 3, 4, 12, 13, 23, 14, 24, 34\}$, (d) $U = \{1, 2, 3, 4, 12, 13, 23, 123, 14, 24\}$.

The vector δ in Lemma 3(a) equals $(0^{n-5}, 4, 4, 4, 4, 5)$ and $(0^{n-5}, 3, 3, 4, 4, 4)$ for (a) and (b), while it equals $(0^{n-5}, 3, 3, 3, 3, 3)$ for both (c) and (d), where 0^j is the null row vector of order j. Thus (a) and (b) are eliminated by Lemma 3(a), i.e., via consideration of ${}^{\#}_{1}C_2$. Continuing with (c) and (d), we consider all possible ways of partitioning U in either case into T_0 and $\widetilde{T}$, keeping in mind that here $r = 2$ and hence T_0 has to be a 1-flat, i.e., a line. This reveals that, up to isomorphism, the unique minimizer of $B_2(T_0, \widetilde{T})$ is $T_0 = \{1, 4, 14\}$, $\widetilde{T} = \{2, 3, 12, 13, 23, 123, 24\}$, arising from (d). By Lemma 2(b), this gives the optimal deign under the GMC criterion. The design actually shown in Table 1 for $r = 2$ and $f = 7$ is isomorphic to the one so obtained.

Example 2. With reference to Table 2, let $s = 2$, $r = 4$ and $f = 10$. By Theorem 2, $\widetilde{T}$ must have the form (4.2). But then the subset $\{\alpha_1, \ldots, \alpha_9\}$, of the 3-flat T_0, represents an unblocked 2^{9-5} design. Hence, without loss of generality, if one takes T_0 as the 3-flat generated by the points 1, 2, 3 and 4, then following Chen, Sun and Wu (1993), at most five nonisomorphic possibilities for $\widetilde{T}$ emerge:

(a) $\widetilde{T} = \{5, 15, 25, 35, 45, 125, 135, 145, 2345, 12345\}$,
(b) $\widetilde{T} = \{5, 15, 25, 35, 45, 125, 135, 245, 345, 12345\}$,
(c) $\widetilde{T} = \{5, 15, 25, 35, 45, 125, 135, 235, 145, 2345\}$,
(d) $\widetilde{T} = \{5, 15, 25, 35, 45, 125, 135, 235, 145, 245\}$,
(e) $\widetilde{T} = \{5, 15, 25, 35, 45, 125, 135, 235, 1235, 145\}$.

When viewed as block designs in conjunction with T_0, however, (a) and (e) turn out to be isomorphic, and the same happens with (c) and (d). Thus it remains to compare (a), (b) and (c), for which the vector θ in Theorem 3(b) is found to be $(2, 2, 2, 2, 2, 2)$, $(0, 0, 0, 0, 0, 0)$, and $(0, 0, 1, 1, 1, 1)$, respectively. Hence the design with $\widetilde{T}$ as in (b) has GMC.

Example 3. With reference to Table 3, let $s = 3$, $r = 2$ and $f = 3$. Then $f < 3^r$ and, by Theorem 2, the set U for a GMC design must be contained in a 2-flat. Only two nonisomorphic possibilities arise, namely, $\widetilde{T} = \{3, 13, 13^2\}$ and $\widetilde{T} = \{3, 13, 23\}$, with $T_0 = \{1, 2, 12, 12^2\}$ in both cases. The vector δ considered in Lemma 3equals $(0^{n-6}, 3, 3, 3, 3, 3, 3)$ for the first design and $(0^{n-6}, 3, 3, 3, 4, 4, 4)$ for the second design. Hence the first design has GMC. Here one could discriminate between the two designs on the basis of δ, i.e., ${}^{\#}_{1}C_2$. This may be contrasted with what was seen in Theorem 3(a) for $s = 2$.

Table 1. Optimal designs under the blocked GMC criterion for $s = 2$ and $u \leq 15$.

r	f	$\widetilde{T}$
1	1	$\{2\}$
1	2	$\{2, 12\}$
1	3	$\{2, 3, 23\}$
1	4	$\{2, 12, 3, 23\}$
1	5	$\{2, 12, 3, 13, 23\}$
1	6	$\{2, 12, 3, 13, 23, 123\}$
1	7	$\{2, 3, 23, 4, 24, 34, 234\}$
1	8	$\{2, 3, 23, 4, 24, 34, 234, 12\}$
1	9	$\{2, 3, 23, 4, 24, 34, 234, 12, 13\}$
1	10	$\{2, 12, 3, 13, 23, 123, 4, 24, 34, 234\}$
1	11	$\{2, 3, 23, 4, 14, 24, 124, 34, 134, 234, 1234\}$
1	12	$\{2, 12, 3, 13, 23, 123, 4, 14, 24, 124, 134, 234\}$
1	13	$\{2, 12, 3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234\}$
1	14	$\{2, 12, 3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234, 1234\}$
2	1	$\{3\}$
2	2	$\{3, 13\}$
2	3	$\{3, 13, 23\}$
2	4	$\{3, 13, 23, 123\}$
2	5	$\{3, 13, 23, 123, 4\}$
2	6	$\{3, 13, 4, 14, 34, 134\}$
2	7	$\{3, 13, 23, 4, 14, 34, 134\}$
2	8	$\{3, 13, 23, 4, 14, 24, 34, 134\}$
2	9	$\{3, 13, 23, 4, 14, 24, 34, 134, 234\}$
2	10	$\{3, 13, 23, 123, 4, 14, 124, 34, 134, 234\}$
2	11	$\{3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234\}$
2	12	$\{3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234, 1234\}$
3	1	$\{4\}$
3	2	$\{4, 14\}$
3	3	$\{4, 14, 24\}$
3	4	$\{4, 14, 24, 34\}$
3	5	$\{4, 14, 24, 124, 34\}$
3	6	$\{4, 14, 24, 124, 134, 234\}$
3	7	$\{4, 14, 24, 124, 34, 134, 234\}$
3	8	$\{4, 14, 24, 124, 34, 134, 234, 1234\}$

Note: For $r=1$, $T_0=\{1\}$; for $r=2$, $T_0=\{1, 2, 12\}$; for $r=3$, $T_0=\{1, 2, 12, 3, 13, 23, 123\}$.

Analogously to the fact that in the unblocked case the clear effects criterion does not always agree with the MA criterion, the present blocked GMC criterion agrees to a considerable extent but not always with the criteria, mentioned in Section 2, that are solely based on the two WLPs. For example, in the setup of Table 1, the GMC criterion agrees with the three MA criteria arising from $W^{(1)}$, $W^{(2)}$, and $W^{(3)}$ except when (i) $r = 1$, $f = 9$, (ii) $r = 1$, $f = 10$, (iii) $r = 1$, $f = 11$, and (iv) $r = 2$, $f = 7$. Under (i) and (iv), the GMC criterion agrees with

$W^{(1)}$ and $W^{(3)}$ but differs from $W^{(2)}$, while under (ii) and (iii) it differs from all the three. Similarly, in the setup of Table 2, the GMC criterion agrees with the one arising from $W^{(4)}$ except when $5 \le f \le 9$.

Appendix: Proof of Theorem 1

Let ψ denote the number of lines contained in U, and ϕ_1 and ϕ_2 denote the numbers of pairs, arising out of the points of U, with sum falling inside U and outside U, respectively. Clearly,

$$\phi_1 + \phi_2 = \frac{u(u-1)}{2}, \qquad \psi = \frac{\phi_1}{3}. \tag{A.1}$$

Lemma 4. *If $2^{w-1} \le u < 2^w - 1$, then $\phi_2 \ge (u-h)(2h-u+1)$ where $h = 2^{w-1} - 1$, and the case of equality arises only if U is contained in a $(w-1)$-flat.*

Proof. For u as stated, following Chen and Hedayat (1996), $\psi \le (1/6)h(h-1) + (1/2)(u-h)(u-h-1)$, with equality only if U is contained in a $(w-1)$-flat. The result now follows by noting that $\phi_2 = (1/2)u(u-1) - 3\psi$, by (A.1).

Lemma 5. *If $u = 2^w - j$, where $w > r$ and $j = 2, 3, 4$, then a design can have GMC only if U is contained in a $(w-1)$-flat.*

Proof. For $w \le 4$, the result is either trivial or covered by Lemma 3(d). So, we consider only $w \ge 5$. Then the stated form of u satisfies $2^{w-1} \le u < 2^w - 1$, and hence the inequality in Lemma 4 holds. Moreover, with u as stated, one can always construct a design such that U is contained in a $(w-1)$-flat and hence, analogously to (4.1), for such a design one gets

$$g \le j - 1. \tag{A.2}$$

Let G be the set of points which lie in T (i.e., outside U) and equal the sum of any two points of U. By (3.1) and (3.5), $g = \#G$. Since no two lines can have more than one point in common, there are at most $[(1/2)u]$ pairs of points (here $[(1/2)u]$ is the largest integer in $(1/2)u$), arising out of U, with sum equal to any particular point of G. Thus

$$\phi_2 \le g\Big[\frac{u}{2}\Big]. \tag{A.3}$$

Together with Lemma 4, this implies that $(u-h)(2h-u+1) \le \phi_2 \le g[(1/2)u]$, i.e.,

$$(h+2-j)(j-1) \le \phi_2 \le g\Big[h+1-\frac{j}{2}\Big] \tag{A.4}$$

because $u = 2^w - j = 2h + 2 - j$.

If $j = 2$, then (A.4) yields $g \ge 1$, so that by (A.2) the minimum possible value of g is 1. Thus, by Lemma 3(b), a design can have GMC only if it has

Table 2. Optimal designs under the blocked GMC criterion for $s = 2$, $r = 4$, $f \leq 16$.

f	$\widetilde{T}$
1	$\{5\}$
2	$\{5, 15\}$
3	$\{5, 15, 25\}$
4	$\{5, 15, 25, 35\}$
5	$\{5, 15, 25, 35, 1235\}$
6	$\{5, 15, 25, 125, 35, 135\}$
7	$\{5, 15, 25, 125, 35, 135, 235\}$
8	$\{5, 15, 25, 35, 1235, 45, 1245, 1345\}$
9	$\{5, 15, 25, 125, 35, 135, 45, 145, 2345\}$
10	$\{5, 15, 25, 125, 35, 135, 45, 245, 345, 12345\}$
11	$\{5, 15, 25, 125, 35, 135, 235, 45, 145, 245, 345\}$
12	$\{5, 15, 25, 125, 35, 135, 235, 1235, 45, 145, 245, 345\}$
13	$\{5, 15, 25, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345\}$
14	$\{5, 15, 25, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345, 1345\}$
15	$\{5, 15, 25, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345, 1345, 2345\}$
16	$\{5, 15, 25, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345, 1345, 2345, 12345\}$

Note: $T_0 = \{1, 2, 12, 3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234, 1234\}$.

Table 3. Optimal designs under the blocked GMC criterion for $s = 3$ and $u \leq 13$.

r	f	$\widetilde{T}$
1	1	$\{2\}$
1	2	$\{2, 12\}$
1	3	$\{2, 12, 12^2\}$
1	4	$\{2, 3, 23, 23^2\}$
1	5	$\{2, 3, 23, 23^2, 13\}$
1	6	$\{2, 12, 12^2, 3, 23, 23^2\}$
1	7	$\{2, 3, 23, 23^2, 13, 123, 12^23\}$
1	8	$\{2, 12^2, 3, 13^2, 23^2, 123^2, 12^23, 12^23^2\}$
1	9	$\{2, 12, 12^2, 3, 13, 23, 23^2, 12^23, 12^23^2\}$
1	10	$\{2, 12, 12^2, 3, 13, 13^2, 23, 23^2, 123, 123^2\}$
1	11	$\{2, 12, 12^2, 3, 13, 13^2, 23, 23^2, 123, 123^2, 12^23\}$
1	12	$\{2, 12, 12^2, 3, 13, 13^2, 23, 23^2, 123, 123^2, 12^23, 12^23^2\}$
2	1	$\{3\}$
2	2	$\{3, 13\}$
2	3	$\{3, 13, 13^2\}$
2	4	$\{3, 13, 13^2, 23^2\}$
2	5	$\{3, 13, 13^2, 23, 23^2\}$
2	6	$\{3, 13, 13^2, 23, 23^2, 123\}$
2	7	$\{3, 13, 13^2, 23, 23^2, 123, 123^2\}$
2	8	$\{3, 13, 13^2, 23, 23^2, 123, 123^2, 12^23\}$
2	9	$\{3, 13, 13^2, 23, 23^2, 123, 123^2, 12^23, 12^23^2\}$

Note: For $r = 1$, $T_0 = \{1\}$; for $r = 2$, $T_0 = \{1, 2, 12, 12^2\}$.

$g = 1$, in which case equality holds throughout in (A.4) and hence in Lemma 4, and consequently U is contained in a $(w-1)$-flat. The same arguments prove the result for $j = 3$.

Finally, let $j = 4$. Then (A.4) yields $g \geq 3(h-2)/(h-1)$, i.e., $g \geq 3$, because g is an integer, and $h \geq 15$ as $w \geq 5$. Using (A.2) again, the minimum possible value of g is 3. Thus, as before, a design can have GMC only if it has $g = 3$, in which case (A.4) yields

$$3(h-2) \leq \phi_2 \leq 3(h-1). \tag{A.5}$$

Now, for $u = 2^w - 4$, it is not hard to see that $(1/2)u(u-1)$ is an integral multiple of 3. Hence, by (A.1), so is ϕ_2 because ψ is an integer. Therefore, (A.5) implies that ϕ_2 equals either $3(h-2)$ or $3(h-1)$. If $\phi_2 = 3(h-2)$, then equality holds in Lemma 4 and the result follows. Continuing with $g = 3$, next let $\phi_2 = 3(h-1)$. Then equality holds in (A.3), and hence writing $G = \{\beta_1, \beta_2, \beta_3\}$, for every i $(= 1, 2, 3)$ there exists a partitioning of U into $2^{w-1} - 2$ pairs of points such that the two points in each of these pairs add up to β_i. As a result, one can find points α_0, α_1 and α_2 in U such that

$$\alpha_0 + \alpha_1 = \beta_1, \qquad \alpha_0 + \alpha_2 = \beta_2. \tag{A.6}$$

Furthermore, if $\beta_1 + \beta_2 \in U$, then there exists $\alpha \in U$ such that $\alpha + (\beta_1 + \beta_2) = \beta_1$, i.e., $\alpha = \beta_2$, which is impossible as α belongs to U while β_2 does not. Thus $\beta_1 + \beta_2 \notin U$. On the other hand, by (A.6), $\beta_1 + \beta_2 = \alpha_1 + \alpha_2$, so that $\beta_1 + \beta_2 \in G$, by the definition of G. Since β_3 is the only point of G other than β_1 and β_2, we get $\beta_3 = \beta_1 + \beta_2$. It follows that the union of U and G, having cardinality $2^w - 1$, is closed under the addition of distinct elements, i.e., this union is a $(w-1)$-flat which contains U.

Proof of Theorem 1. We sketch a proof of only part (c2). The proofs of other parts are similar and simpler. In view of Lemma 5, it suffices to consider designs for which $T_0 = \Delta_r$ and U is contained in Δ_w, where Δ_r and Δ_w are as defined above the statement of the theorem. The idea of double complementation is again useful. Let $\widetilde{U}$ be the complement of U in Δ_w, or equivalently of $\widetilde{T}$ in $\overline{\Delta} = \Delta_w \setminus \Delta_r$. Note that $\widetilde{U} \subset T$. The only nonisomorphic possibilities for $\widetilde{U}$ are as follows:

(a) $\widetilde{U} = \{\pi_{r+1}, \pi_1 + \pi_{r+1}, \pi_2 + \pi_{r+1}\}$, (b) $\widetilde{U} = \{\pi_{r+1}, \pi_{r+2}, \pi_1 + \pi_{r+1}\}$,
(c) $\widetilde{U} = \{\pi_{r+1}, \pi_{r+2}, \pi_{r+1} + \pi_{r+2}\}$, (d) $\widetilde{U} = \{\pi_{r+1}, \pi_{r+2}, \pi_1 + \pi_{r+1} + \pi_{r+2}\}$,
(e) $\widetilde{U} = \{\pi_{r+1}, \pi_{r+2}, \pi_{r+3}\}$.

Of these, (a) cannot arise when $r = 1$, while (e) cannot arise when $w = r + 2$.

For discrimination among (a)−(e) under the GMC criterion, the following facts will be useful. Among these, (i)−(iv) follow from Zhang and Mukerjee (2009, Sec. 3), because U and $\widetilde{U}$ are complements of each other in Δ_w which in itself is isomorphic to a finite projective geometry of dimension $w-1$. Similarly, (v) follows from Mukerjee and Wu (2006, Chap. 6). Finally, (vi) arises because the points of U cannot span a point outside Δ_w.

(i) If $\pi \in U$ then $R_2(U,\pi) = l_1 + R_2(\widetilde{U},\pi)$.

(ii) If $\pi \in \widetilde{U}$ then $R_2(U,\pi) = l_2 + R_2(\widetilde{U},\pi)$.

(iii) If $\pi \in U$ then $R_3(U,\pi) = l_3 - R_2(\widetilde{U},\pi) - R_3(\widetilde{U},\pi)$.

(iv) If $\pi \in \widetilde{U}$ then $R_3(U,\pi) = l_4 - R_2(\widetilde{U},\pi) - R_3(\widetilde{U},\pi)$.

(v) $B_2(T_0,\widetilde{T}) = l_5 + B_2(T_0,\widetilde{U})$.

(vi) If $\pi \in T \setminus \widetilde{U}$ then $R_2(U,\pi) = R_3(U,\pi) = 0$.

Here $l_1,\ldots,l_5$ are constants not depending on the specific design. For instance, $l_1 = 2^{w-1} - 4$.

By (3.1), with $\widetilde{U}$ as in (c) above, $R_2(\widetilde{U},\pi) = 1$ for each $\pi \in \widetilde{U}$, while with $\widetilde{U}$ as in (a), (b), (d) or (e) above, $R_2(\widetilde{U},\pi) = 0$ for each $\pi \in \widetilde{U}$. Hence using the facts (ii) and (vi) along with Lemma 3(a), consideration of ${}_1^{\#}C_2$ eliminates (c). Next, by (3.2), with $\widetilde{U}$ as in (a), (b), (d), and (e), $B_2(T_0,\widetilde{U})$ equals $3, 1, 0,$ and 0 respectively. As a result, using (v) and Lemma 2(b), consideration of ${}_2^{\#}C_0$ eliminates (a) and (b). It remains to compare (d) and (e) on the basis of the subsequent terms of the AENP (2.2).

Recalling that $\widetilde{T} = \overline{\Delta} \setminus \widetilde{U}$, we observe two additional facts.

(vii) With both (d) and (e), $R_2(\widetilde{U},\pi) = R_3(\widetilde{U},\pi) = 0$ for every $\pi \in \widetilde{U}$, $R_2(\widetilde{U},\pi) = 1$ for three points π of $\widetilde{T}$, and $R_2(\widetilde{U},\pi) = 0$ for every other $\pi \in \widetilde{T}$.

(viii) With (d), $R_3(\widetilde{U},\pi) = 0$ for every $\pi \in \widetilde{T}$. On the other hand, with (e), $R_3(\widetilde{U},\pi) = 1$ for one point π of $\widetilde{T}$, this point being different from the three points of $\widetilde{T}$ with $R_2(\widetilde{U},\pi) = 1$, and $R_3(\widetilde{U},\pi) = 0$ for every other $\pi \in \widetilde{T}$.

The fact (vii), in conjunction with (i),(ii), (iv), and (vi), implies that the choices of $\widetilde{U}$ as in (d) and (e) entail the same ${}_2^{\#}C_2$ and ${}_1^{\#}C_3$, as one may verify using parts (c) and (d) of Lemma 2. Turning to the next term ${}_2^{\#}C_3$, let $\rho^{(k)} = {}_2^{\#}C_3^{(k)}(d) - {}_2^{\#}C_3^{(k)}(e)$, where ${}_2^{\#}C_3^{(k)}(d)$ and ${}_2^{\#}C_3^{(k)}(e)$ are the values of ${}_2^{\#}C_3^{(k)}$ for the choices (d) and (e), respectively, of $\widetilde{U}$. Some intricate algebra, based on Lemma 2(e) together with the facts (i)−(iv) and (vi)−(viii) and the expression for l_1 as mentioned above, shows the existence of an integer k_0 such that $\rho^{(k)} = 0$ if $k < k_0$ and $\rho^{(k)} = (1/2)(n-3)$ if $k = k_0$. Since $n \geq 4$, it follows that (d) dominates (e) under the GMC criterion, thus completing the proof.

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