Statistica Sinica **27** (2017), 907-930
doi: https://doi.org/10.5705/ss.202015.0460

EXTREME VERSIONS OF WANG RISK MEASURES AND THEIR ESTIMATION FOR HEAVY-TAILED DISTRIBUTIONS

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Abstract: In this paper, we build simple extreme analogues of Wang distortion risk measures and we show how this makes it possible to consider many standard measures of extreme risk, including the usual extreme Value-at-Risk or Tail-Value-at-Risk, as well as the recently introduced extreme Conditional Tail Moment, in a unified framework. We then introduce adapted estimators when the random variable of interest has a heavy-tailed distribution and we prove their asymptotic normality. The finite sample performance of our estimators is assessed in a simulation study and we showcase our techniques on two sets of data.

Key words and phrases: Asymptotic normality, conditional tail moment, distortion risk measure, extreme-value statistics, heavy-tailed distribution.

1. Introduction

Understanding the extremes of a random phenomenon is a major question in various areas of statistical application. For instance, a stimulating topic comes from the fact that extreme phenomena may have strong adverse effects on financial institutions or insurance companies, and the investigation of those effects on financial returns makes up a large part of the recent extreme value literature; see *e.g.* Drees (2003) and Rootzén and Tajvidi (1997). A further application in actuarial science is, for insurance companies operating in Europe, the computation of their own solvency capital so as to fulfill the European Union Solvency II directive requirement that an insurance company be able to survive the upcoming calendar year with a probability not less than 0.995.

A commonly encountered problem when analyzing the extremes of a random variable is that the straightforward empirical estimator of the quantile function is not consistent at extreme levels. In many of the aforementioned applications, this issue can be bypassed by modeling the problem using heavy-tailed distributions. Roughly speaking, a distribution is said to be heavy-tailed if and only if its related survival function decays like a power function at infinity; its so-called tail index is then the parameter that controls its rate of convergence to 0 at infinity.

A heuristic consequence of this is, if q denotes the underlying quantile function,

$$q(\delta) \approx \left(\frac{1-\beta}{1-\delta}\right)^{\gamma} q(\beta)$$

when β, δ are close to 1 and γ is the tail index of the distribution. This implies that the quantile function can be consistently estimated at an arbitrarily high extreme level δ provided it can be consistently estimated at a much smaller level β (usually by an empirical quantile) and a consistent estimator $\widehat{\gamma}$ of γ can be computed (see the examples in Section 3 of de Haan and Ferreira (2006)). This procedure, suggested by Weissman (1978), is arguably the simplest and most popular device as far as extreme quantile estimation is concerned.

Of course, the estimation of a single extreme quantile, or Value-at-Risk (VaR) as it is known in the actuarial and financial literature, only gives incomplete information on the extremes of a random variable. This is one of the reasons why other quantities, which take into account the whole right tail of the random variable of interest, were developed and studied. Examples of such indicators include the Tail Value-at-Risk (TVaR), also called Expected Shortfall, and the Stop-loss Premium for reinsurance problems, see Embrechts, Klüppelberg and Mikosch (1997) and McNeil, Frey and Embrechts (2005). When the related survival function is continuous, these measures can be obtained by combining the VaR and a Conditional Tail Moment (CTM), as introduced by El Methni, Gardes and Girard (2014).

In our opinion, a way to encompass all these indicators in a unified framework is to consider the flexible class of Wang distortion risk measures (DRMs), introduced by Wang (1996). The aforementioned VaR, TVaR and CTM actually are particular cases of Wang DRMs, and so are many other interesting risk measures such as the Wang transform (Wang (2000)), the tail standard deviation premium calculation principle (Furman and Landsman (2006)) and the newly introduced GlueVaR of Belles-Sampera, Guillén and Santolino (2014). In this paper, we show how a simple linear transformation allows one to construct an extreme analogue of a Wang DRM, and we consider its estimation under classical conditions in extreme value theory. Our method, it appears, provides a unified framework for the study of many frequently used extreme risk metrics, and we shall underline in particular that several results of the literature can be recovered from our results.

The outline of our paper is as follows. We first recall the definition of a Wang DRM in Section 2. In Section 3, we present a simple way to build extreme analogues of Wang DRMs and we consider their estimation. Section 4 is devoted to the study of the finite-sample performance of our estimators, and we showcase our method on two data sets in Section 5. Section 6 concludes the paper with a discussion of our results. The proofs of our results and some additional tables and figures are deferred to the Supplementary Material.

2. Wang Risk Measures

In this paper, $g : [0,1] \to [0,1]$ is a distortion function if it is right-continuous and nondecreasing with $g(0) = 0$ and $g(1) = 1$. The Wang DRM of a positive random variable X with distortion function g is then

$$R_g(X) := \int_0^\infty g(1 - F(x))dx,$$

where F is the cumulative distribution function (cdf) of X. An alternative, easily interpretable expression of $R_g(X)$ can be found. Denote by q the quantile function of X, $q(\alpha) = \inf\{x \geq 0 \,|\, F(x) \geq \alpha\}$ for all $\alpha \in (0,1)$. Let $m = \inf\{\alpha \in [0,1] \,|\, g(\alpha) > 0\}$ and $M = \sup\{\alpha \in [0,1] \,|\, g(\alpha) < 1\}$. Assume for the moment that q is continuous on $U \cap (0,1)$ with U an open interval containing $[1-M, 1-m]$. Noticing that F is the right-continuous inverse of q, a classical change-of-variables formula and an integration by parts then entail that $R_g(X)$, provided it is finite, can be written as a Lebesgue-Stieltjes integral:

$$R_g(X) = \int_0^1 g(\alpha)dq(1-\alpha) = \int_0^1 q(1-\alpha)dg(\alpha).$$

A Wang DRM can thus be understood as a weighted version of the expectation of the random variable X. Specific examples include the quantile at level β or VaR(β), obtained by setting $g(x) = \mathbb{I}\{x \geq 1 - \beta\}$, with $\mathbb{I}\{\cdot\}$ denoting the indicator function; the Tail Value-at-Risk TVaR(β) in the worst $100(1-\beta)\%$ of cases, the average of all quantiles exceeding VaR(β), is recovered by taking $g(x) = \min(x/(1-\beta), 1)$. In Table 1 of the Supplementary Material we give further examples of classical DRMs and their distortion functions. Broadly speaking, the class of Wang DRMs allows almost total flexibility as far as the weighting scheme is considered. Besides, any spectral risk measure of X (see Cotter and Dowd (2006)) is also a Wang DRM.

Furthermore, we note that if $h : [0,\infty) \to [0,\infty)$ is a strictly increasing, continuously differentiable function, then the Wang DRM of $h(X)$ with distortion function g is

$$R_g(h(X)) = \int_0^1 h \circ q(1-\alpha)dg(\alpha). \tag{2.1}$$

For instance, the choices $g(x) = \min(x/(1-\beta), 1)$, $\beta \in (0,1)$ and $h(x) = x^a$, with a a positive real number, yield, after integrating by parts,

$$R_g(X^a) = \mathrm{CTM}_a(\beta) := \mathbb{E}(X^a|X > q(\beta)),$$

provided F is continuous. This is the Conditional Tail Moment (CTM) of order a of the random variable X, as introduced in El Methni, Gardes and Girard (2014). Especially, when F is continuous, the TVaR coincides with the Conditional Tail Expectation of X. Table 2 in the Supplementary Material gives several examples of risk measures, such as the Stop-loss Premium (SP) that can be obtained by combining a finite number of CTMs and the VaR.

In an actuarial context, a DRM is a coherent risk measure (see Artzner et al. (1999)) if and only if the distortion function g is concave (Wirch and Hardy (2002)). Coherency of a risk measure reflects in particular on the diversification principle which asserts that aggregating two risks cannot be worse than handling them separately (Artzner et al. (1999)). Especially, while the VaR is not a coherent risk measure, the TVaR is, for instance, and this has already been noted several times in the recent literature. It should be acknowledged nonetheless that the VaR is subadditive (and therefore coherent) in the right tail under certain conditions, see Daníelsson et al. (2013).

The discussion about the relative merits of VaR, distortion risk measures, and TVaR is not limited to coherency: although the popular saying in insurance and finance is that TVaR is more conservative than VaR, Kou and Peng (2014) argue that TVaR should actually be compared to the median shortfall (see Kou, Peng and Heyde (2013)), and in this case the aforementioned conclusion is no longer necessarily true. Cont, Deguest and Scandolo (2010) show that VaR is more robust than TVaR against small departures from the model or from the data, although it might be less aggregation-robust, see Embrechts, Wang and Wang (2015). Linton and Xiao (2013) argue that the inference procedure for the extreme VaR is easier than for the extreme TVaR because it does not depend on tail heaviness (at least theoretically, for heavy-tailed data). There are concerns related to the practical use of the VaR: for instance, in the Basel II and III accords (see Basel Committee on Banking Supervision (2006, 2011)) the VaR-based risk measure used to compute capital requirements for trading books, whose relationship to the 99.9% VaR is studied in Gordy (2003), has been criticized for being procyclical (see Adrian and Brunnermeier (2008)), or, as Kou and Peng (2014) point out, for being low in booms and high in crises, which is of course a problem as far as regulation is concerned. Keppo, Kofman and Meng (2010) even show that the Basel accords capital requirements may sometimes increase the default probability of a bank, contrary to the regulators' original aim.

Yet another property of risk measures, namely elicitability (see Gneiting (2011) and Ziegel (2015)), has gained prominence in recent years since it has been argued to allow for correct forecast performance comparisons. A related concept is consistency, introduced by Davis (2013). While the VaR is an elicitable (and consistent) risk measure, the TVaR is not; more generally, it has been shown recently by Kou and Peng (2014) and Wang and Ziegel (2015) that Wang DRMs different from either the VaR or the simple expectation do not satisfy such a property. An example of a risk measure that is both coherent and elicitable is the expectile (Newey and Powell (1987); in a financial context, Kuan, Yeh and Hsu (2009)) when it is larger than the expectation. The estimation of extreme expectiles, which to the best of our knowledge cannot be written as a simple combination of extreme Wang DRMs of X, is beyond the scope of this paper.

3. Framework

3.1. Extreme versions of Wang DRMs and their estimation

Extreme versions of Wang risk measures may be obtained as follows. Let g be a distortion function and for every $\beta \in (0,1)$, consider the function g_β defined by

$$\forall y \in [0,1],\ g_\beta(y) := g\left(\min\left[1, \frac{y}{1-\beta}\right]\right) = \begin{cases} g\left(\dfrac{y}{1-\beta}\right) & \text{if} \quad y \le 1-\beta, \\ 1 & \text{otherwise.} \end{cases}$$

Such a function, which is deduced from g by a simple piecewise linear transform of its argument, is thus constantly 1 on $[1-\beta, 1]$. Especially, if g gives rise to a coherent Wang DRM, so does g_β. We now consider the Wang DRM of X with distortion function g_β:

$$R_{g,\beta}(X) := \int_0^\infty g_\beta(1 - F(x))dx.$$

Because the inequality $F(x) \ge \beta$ is equivalent to $x \ge q(\beta)$, we have:

$$R_{g,\beta}(X) = \int_0^\infty g(1 - F_\beta(x))dx \quad \text{with} \quad F_\beta(x) := \max\left[0, \frac{F(x) - \beta}{1-\beta}\right]. \tag{3.1}$$

When q is continuous and strictly increasing in a neighborhood of β, then

$$F_\beta(x) = \max\left[0, \frac{F(x) - F(q(\beta))}{1 - q(\beta)}\right] = \mathbb{P}(X \le x | X > q(\beta)),$$

which makes the interpretation of the risk measure $R_{g,\beta}(X)$ clear: it is the Wang

DRM of X given that it lies above the level $q(\beta)$. In other words, we have shown the following.

Proposition 1. *Assume that for some $t > 0$, the function q is continuous and strictly increasing on $[t, 1)$. Then for all $\beta > t$ and any strictly increasing and continuously differentiable function h on $(0, \infty)$,*

$$R_{g,\beta}(h(X)) = R_g(h(X_\beta)) \quad \textit{with} \quad \mathbb{P}(X_\beta \leq x) = \mathbb{P}(X \leq x | X > q(\beta)).$$

When $\beta \uparrow 1$, we may then think of this construction as a way to consider Wang DRMs of the extremes of X.

Choosing $h(x) = x$ makes it possible to recover some simple and widely used extreme risk measures: the usual extreme VaR is obtained by setting $g(x) = \mathbb{I}\{x = 1\}$, and an extreme version of the TVaR is obtained by taking $g(x) = x$. The same idea yields extreme analogues of the various risk measures shown in Table 1 of the Supplementary Material. Furthermore, as highlighted in Section 2, choosing $g(x) = x$ and $h(x) = x^a$, $a > 0$, yields an extreme version of a CTM of X, and therefore extreme versions of quantities such as those introduced in Table 2 of the Supplementary Material can be studied.

It is worth noting at this point that the construction presented in this paper is different from that of Vandewalle and Beirlant (2006). In the latter paper, the authors consider the Wang DRM R_g of $(X - R)\mathbb{I}\{X > R\} = \max(X - R, 0)$ for large R. Their construction is thus adapted to the examination of excess-of-loss reinsurance policies for extreme losses; their work is, by the way, restricted to the case of a concave function g satisfying a regular variation condition in a neighborhood of 0. It therefore excludes the simple VaR risk measure, for instance, as well as the Conditional-Value-at-Risk (CVaR) and the GlueVaR of Belles-Sampera, Guillén and Santolino (2014). Our idea is rather to consider a conditional construction in the sense that we look at the Wang DRMs of X given that it lies above a high level, with conditions as weak as possible on the function g, in an effort to be able to examine the extremes of X in as unified a way as possible.

3.2. Estimation using an asymptotic equivalent

We now give a first idea on how to estimate this type of extreme risk measure. Let $(X_1, \ldots, X_n)$ be a sample of independent and identically distributed copies of a random variable X having cdf F, and let (β_n) be a nondecreasing sequence of real numbers belonging to $(0, 1)$, which converges to 1. Assume for the time being that X is Pareto distributed,

$$\forall x > 1,\ \mathbb{P}(X \le x) = 1 - x^{-1/\gamma},$$

where $\gamma > 0$ is the so-called tail index of X. In this case, the quantile function of X is $q(\alpha) = (1-\alpha)^{-\gamma}$ for all $\alpha \in (0,1)$. Using (2.1) in Section 2 and a simple change of variables, we get:

$$\begin{aligned} R_{g,\beta_n}(h(X)) = \int_0^1 h \circ q(1-\alpha)dg_{\beta_n}(\alpha) &= \int_0^1 h \circ q(1-(1-\beta_n)s)dg(s) \\ &= \int_0^1 h(q(\beta_n)s^{-\gamma})dg(s). \end{aligned} \tag{3.2}$$

In this case, an estimator of $R_{g,\beta_n}(h(X))$ would then be obtained by plugging estimators of $q(\beta_n)$ and γ in the right-hand side of (3.2).

Of course, in general, a strong relationship such as (3.2) cannot be expected to hold, but it stays true to some extent when X has a heavy-tailed distribution, the rigorous definition of which we recall now. A function f is said to be regularly varying at infinity with index $b \in \mathbb{R}$ if f is nonnegative and for any $x > 0$, $f(tx)/f(t) \to x^b$ as $t \to \infty$; the distribution of X is then said to be heavy-tailed when $1 - F$ is regularly varying with index $-1/\gamma < 0$, the parameter γ being the so-called tail index of the cdf F. This condition, which is a usual restriction in extreme value theory (see de Haan and Ferreira (2006)), essentially says that $1-F(x)$ is in some sense close to $x^{-1/\gamma}$ when x is large. In the sequel, we therefore assume that X is heavy-tailed. We also suppose that the quantile function q of X is continuous and strictly increasing in a neighborhood of infinity, which makes possible the use of (2.1) for n large enough.

Finally, we assume that the function h is a positive power of x: $h(x) = x^a$, where $a > 0$. This choice allows us to consider estimators of a large class of risk measures of X, including the aforementioned CTM. In this case (see Lemma 3 in the Supplementary Material), it holds that

$$R_{g,\beta_n}(X^a) = [q(\beta_n)]^a \int_0^1 s^{-a\gamma} dg(s)(1+\mathrm{o}(1)) \quad \text{as} \quad n \to \infty,$$

provided $\int_0^1 s^{-a\gamma-\eta} dg(s) < \infty$ for some $\eta > 0$. This suggests that the above idea for the construction of the estimator can still be used provided n is large enough. Specifically, if $\widehat{q}_n(\alpha) = X_{\lceil n\alpha \rceil, n}$ denotes the empirical quantile function, in which $X_{1,n} \le \cdots \le X_{n,n}$ are the order statistics of the sample $(X_1, \ldots, X_n)$ and $\lceil \cdot \rceil$ is the ceiling function, we set

$$\widehat{R}^{AE}_{g,\beta_n}(X^a) := X^a_{\lceil n\beta_n \rceil, n} \int_0^1 s^{-a\widehat{\gamma}_n} dg(s), \tag{3.3}$$

where $\widehat{\gamma}_n$ is any consistent estimator of γ. This estimator is called the AE estimator in what follows; notice that the integrability condition $\int_0^1 s^{-a\gamma-\eta} dg(s) < \infty$, which should be thought of as a condition that guarantees the existence of the considered Wang DRM, makes the estimator introduced here well-defined with probability arbitrarily large when n is large enough, due to the consistency of $\widehat{\gamma}_n$. For a related but different idea, see Vandewalle and Beirlant (2006).

An appealing feature of the AE estimator is that it is easy to compute in many cases:

- in the case of the Conditional Tail moment of order a, *i.e.* $g(x) = x$, the estimator reads
$$\widehat{R}^{AE}_{g,\beta_n}(X^a) = X^a_{\lceil n\beta_n\rceil,n} \int_0^1 s^{-a\widehat{\gamma}_n} ds = \frac{X^a_{\lceil n\beta_n\rceil,n}}{1 - a\widehat{\gamma}_n}$$
when $a\widehat{\gamma}_n < 1$. In particular, this provides an estimator different from the sample average estimator of El Methni, Gardes and Girard (2014);

- in the case of the Dual Power risk measure, *i.e.* $g(x) = 1 - (1-x)^{1/\alpha}$ where $0 < \alpha < 1$ and $a = 1$, then when $r := 1/\alpha$ is an integer, the estimator is
$$\widehat{R}^{AE}_{g,\beta_n}(X) = X_{\lceil n\beta_n\rceil,n} \int_0^1 r s^{-\widehat{\gamma}_n}(1-s)^{r-1} ds = \frac{r!\Gamma(1-\widehat{\gamma}_n)}{\Gamma(1-\widehat{\gamma}_n + r)} X_{\lceil n\beta_n\rceil,n}$$
provided $\widehat{\gamma}_n < 1$. Here Γ is Euler's Gamma function, $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$;

- in the case of the Proportional Hazard transform, *i.e.* $g(x) = x^\alpha$ where $0 < \alpha < 1$ and $a = 1$, the estimator is
$$\widehat{R}^{AE}_{g,\beta_n}(X) = X_{\lceil n\beta_n\rceil,n} \int_0^1 \alpha s^{\alpha-\widehat{\gamma}_n-1} ds = \frac{\alpha X_{\lceil n\beta_n\rceil,n}}{\alpha - \widehat{\gamma}_n}$$
provided $\widehat{\gamma}_n < \alpha$.

In order to examine the asymptotic properties of our estimator, it is necessary to compute the order of magnitude of its asymptotic bias. To do so, it is convenient to use an assumption on the left-continuous inverse U of $1/(1-F)$, defined by $U(t) = q(1-t^{-1})$. Specifically, we assume that U is regularly varying with index γ and satisfies the following second-order condition (see de Haan and Ferreira (2006)).

Condition $\mathcal{C}_2(\gamma, \rho, A)$: for any $x > 0$, we have
$$\lim_{t\to\infty} \frac{1}{A(t)} \left(\frac{U(tx)}{U(t)} - x^\gamma \right) = x^\gamma \frac{x^\rho - 1}{\rho}$$

with $\gamma > 0$, $\rho \leq 0$, and A is a Borel measurable function that converges to 0 and has constant sign. When $\rho = 0$, the right-hand side is to be read as $x^\gamma \log x$.

We highlight that in condition $\mathcal{C}_2(\gamma, \rho, A)$, the function $|A|$ is necessarily regularly varying at infinity with index ρ (see Theorem 2.3.3 in de Haan and Ferreira (2006)). Such an assumption is classical when studying the rate of convergence of an estimator of a parameter describing the extremes of a random variable, and all standard examples of heavy-tailed distributions satisfy this condition (see *e.g.* the examples pp.61–62 in de Haan and Ferreira (2006)).

Theorem 1. *Assume that U is regularly varying with index $\gamma > 0$. Assume further that $\beta_n \to 1$ and $n(1-\beta_n) \to \infty$.*

1. *Pick a distortion function g and $a > 0$. If there is some $\eta > 0$ such that*
$$\int_0^1 s^{-a\gamma-\eta} dg(s) < \infty$$
and $\widehat{\gamma}_n$ is a consistent estimator of γ, then
$$\frac{\widehat{R}^{AE}_{g,\beta_n}(X^a)}{R_{g,\beta_n}(X^a)} - 1 \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty.$$

2. *Assume moreover that U satisfies condition $\mathcal{C}_2(\gamma, \rho, A)$ and $\sqrt{n(1-\beta_n)}A((1-\beta_n)^{-1}) \to \lambda \in \mathbb{R}$. Pick a $d-$tuple of distortion functions $(g_1, \ldots, g_d)$ and $a_1, \ldots, a_d > 0$. If for some $\eta > 0$,*
$$\forall j \in \{1, \ldots, d\}, \ \int_0^1 s^{-a_j\gamma-1/2-\eta} dg_j(s) < \infty,$$
then, provided we have the joint convergence
$$\sqrt{n(1-\beta_n)} \left(\widehat{\gamma}_n - \gamma, \frac{X_{\lceil n\beta_n \rceil, n}}{q(\beta_n)} - 1 \right) \xrightarrow{d} (\Gamma, \Theta)$$
it holds that the random vector
$$\sqrt{n(1-\beta_n)} \left(\frac{\widehat{R}^{AE}_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\beta_n}(X^{a_j})} - 1 \right)_{1 \leq j \leq d}$$
asymptotically has the joint distribution of
$$\left(a_j \left[-\lambda \frac{\int_0^1 s^{-a_j\gamma}((s^{-\rho}-1)/\rho) dg_j(s)}{\int_0^1 s^{-a_j\gamma} dg_j(s)} + \frac{\int_0^1 s^{-a_j\gamma} \log(1/s) dg_j(s)}{\int_0^1 s^{-a_j\gamma} dg_j(s)} \Gamma + \Theta \right] \right)_{1 \leq j \leq d}.$$

Because of the restriction $n(1-\beta_n) \to \infty$, Theorem 1 only ensures that the estimator consistently estimates so-called intermediate (*i.e.* not "too extreme") Wang DRMs. This restriction will be lifted in Section 3.4 by the introduction of an estimator adapted to the extreme-value framework.

3.3. Estimation using a functional plug-in estimator

Our idea here is to introduce an alternative estimator obtained by making a single approximation instead of the two successive ones

$$q(1-(1-\beta_n)s) \approx q(\beta_n)s^{-\gamma} \approx X_{\lceil n\beta_n\rceil,n} s^{-\widehat{\gamma}_n},$$

which we can then expect to perform better than the AE estimator. Recall that

$$R_{g,\beta_n}(h(X)) = \int_0^1 h \circ q(1-(1-\beta_n)s)dg(s).$$

We consider the statistic obtained by replacing the function $s \mapsto q(1-(1-\beta_n)s)$ by its empirical counterpart $s \mapsto \widehat{q}_n(1-(1-\beta_n)s) = X_{\lceil n(1-(1-\beta_n)s)\rceil,n}$. This yields the functional plug-in estimator

$$\widehat{R}^{PL}_{g,\beta_n}(h(X)) = \int_0^1 h \circ \widehat{q}_n(1-(1-\beta_n)s)dg(s) \tag{3.4}$$

which we call the PL estimator. Contrary to the AE estimator, the PL estimator is well-defined and finite with probability 1, and does not require an external estimator of γ. Its expression is a bit more involved though; in the case when $n(1-\beta_n)$ is actually a positive integer and g is continuous on $[0,1]$, it is easy to show that it takes the simpler form

$$\widehat{R}^{PL}_{g,\beta_n}(h(X)) = h(X_{n\beta_n+1,n}) + \sum_{i=1}^{n(1-\beta_n)-1} g\left(\frac{i}{n(1-\beta_n)}\right)[h(X_{n-i+1,n}) - h(X_{n-i,n})].$$

Our aim is now to examine the asymptotic properties of the PL estimator:

Theorem 2. *Assume that U satisfies condition $\mathcal{C}_2(\gamma,\rho,A)$. Assume further that $\beta_n \to 1$, $n(1-\beta_n) \to \infty$ and $\sqrt{n(1-\beta_n)}A((1-\beta_n)^{-1}) \to \lambda \in \mathbb{R}$. Pick a $d-$tuple of distortion functions $(g_1,\ldots,g_d)$ and $a_1,\ldots,a_d > 0$. If for some $\eta > 0$,*

$$\forall j \in \{1,\ldots,d\}, \ \int_0^1 s^{-a_j\gamma-1/2-\eta}dg_j(s) < \infty,$$

then

$$\sqrt{n(1-\beta_n)}\left(\frac{\widehat{R}^{PL}_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\beta_n}(X^{a_j})} - 1\right)_{1\le j\le d} \xrightarrow{d} \mathcal{N}(0,V)$$

with V being the $d \times d$ matrix whose $(i,j)-$th entry is

$$V_{i,j} = a_i a_j \gamma^2 \frac{\int_{[0,1]^2} \min(s,t) s^{-a_i\gamma-1} t^{-a_j\gamma-1} dg_i(s) dg_j(t)}{\int_0^1 s^{-a_i\gamma} dg_i(s) \int_0^1 t^{-a_j\gamma} dg_j(t)}.$$

This asymptotic normality result, unsurprisingly, is also restricted to the case $n(1-\beta_n) \to \infty$, as was Theorem 1. We can draw an interesting consequence from Theorem 2: for $b \in \mathbb{R}$, consider the class $\mathcal{E}_b([0,1])$ of those continuously differentiable functions on $(0,1)$ such that $s^{-b}|g'(s)|$ is bounded for s in a neighborhood of 0. For instance, any polynomial function belongs to $\mathcal{E}_0([0,1])$, and the Proportional Hazard (Wang (1996)) distortion function $g(s) = s^\alpha$, $\alpha \in (0,1)$, belongs to $\mathcal{E}_{\alpha-1}([0,1])$.

Corollary 1. *Assume that U satisfies condition $\mathcal{C}_2(\gamma,\rho,A)$. Assume further that $\beta_n \to 1$, $n(1-\beta_n) \to \infty$ and $\sqrt{n(1-\beta_n)}A((1-\beta_n)^{-1}) \to \lambda \in \mathbb{R}$. Pick a $d-$tuple of distortion functions $(g_1,\ldots,g_d)$ and $a_1,\ldots,a_d > 0$. Assume there are $b_1,\ldots,b_d \in \mathbb{R}$ such that for all $j \in \{1,\ldots,d\}$, we have $g_j \in \mathcal{E}_{b_j}([0,1])$. If*

$$\forall j \in \{1,\ldots,d\},\ \gamma < \frac{2b_j+1}{2a_j},$$

then

$$\sqrt{n(1-\beta_n)} \left(\frac{\widehat{R}^{PL}_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\beta_n}(X^{a_j})} - 1 \right)_{1\le j\le d} \xrightarrow{d} \mathcal{N}(0,V)$$

with V as in Theorem 2.

In particular, the condition on γ we get for the asymptotic normality of the CTM of order a, obtained with $g(x) = x$ and thus $g \in \mathcal{E}_0([0,1])$, is $\gamma < 1/2a$, which is the condition obtained by El Methni, Gardes and Girard (2014). One may also readily check that the asymptotic variance is the same as in Theorem 1 there.

Just like the AE estimator, the PL estimator is only consistent when (β_n) is an intermediate sequence. Our purpose is now to remove this restriction by using the extrapolation methodology of Weissman (1978).

3.4. Estimating extreme risk measures of arbitrary order

For any $s \in (0,1)$ and $a > 0$ we have

$$[q(1-(1-\delta_n)s)]^a = \left(\frac{1-\beta_n}{1-\delta_n}\right)^{a\gamma} [q(1-(1-\beta_n)s)]^a (1+\mathrm{o}(1))$$

as $n \to \infty$, as a consequence of the regular variation property of U, and provided that (β_n) is a sequence converging to 1 such that $(1-\delta_n)/(1-\beta_n)$ converges to a positive limit. Integrating this relationship with respect to the distortion measure dg therefore suggests that

$$R_{g,\delta_n}(X^a) = \left(\frac{1-\beta_n}{1-\delta_n}\right)^{a\gamma} R_{g,\beta_n}(X^a)(1+\mathrm{o}(1)),$$

see Lemma 5 in the Supplementary Material for a stronger and rigorous statement. A way to design an adapted estimator of the extreme risk measure $R_{g,\delta_n}(X^a)$, when $n(1-\delta_n) \to c < \infty$, is thus to take a sequence (β_n) such that $n(1-\beta_n) \to \infty$, and to plug in any relatively consistent estimator $\widehat{R}_{g,\beta_n}(X^a)$ of the intermediate Wang DRM $R_{g,\beta_n}(X^a)$. This yields a Weissman-type estimator of $R_{g,\delta_n}(X^a)$ (see Weissman (1978)):

$$\widehat{R}^W_{g,\delta_n}(X^a;\beta_n) := \left(\frac{1-\beta_n}{1-\delta_n}\right)^{a\widehat{\gamma}_n} \widehat{R}_{g,\beta_n}(X^a).$$

This principle can of course be applied to the AE and PL estimators to obtain two different extrapolated estimators. Our asymptotic result on this class of estimators is the following.

Theorem 3. *Assume that U satisfies condition $\mathcal{C}_2(\gamma,\rho,A)$, with $\rho<0$. Assume further that $\beta_n,\delta_n \to 1$, $n(1-\beta_n)\to\infty$, $(1-\delta_n)/(1-\beta_n)\to 0$, $\sqrt{n(1-\beta_n)}/\log[(1-\beta_n)/(1-\delta_n)]\to\infty$ and $\sqrt{n(1-\beta_n)}A((1-\beta_n)^{-1})\to\lambda\in\mathbb{R}$. Pick a $d-$tuple of distortion functions $(g_1,\ldots,g_d)$ and $a_1,\ldots,a_d>0$. If for some $\eta>0$,*

$$\forall j\in\{1,\ldots,d\},\ \int_0^1 s^{-a_j\gamma-1/2-\eta}dg_j(s)<\infty$$

and $\sqrt{n(1-\beta_n)}(\widehat{\gamma}_n-\gamma)\xrightarrow{d}\xi$, then provided

$$\forall j\in\{1,\ldots,d\},\ \sqrt{n(1-\beta_n)}\left(\frac{\widehat{R}_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\beta_n}(X^{a_j})}-1\right)=\mathrm{O}_{\mathbb{P}}(1)$$

we have that

$$\frac{\sqrt{n(1-\beta_n)}}{\log([1-\beta_n]/[1-\delta_n])}\left(\frac{\widehat{R}^W_{g_j,\delta_n}(X^{a_j};\beta_n)}{R_{g_j,\delta_n}(X^{a_j})}-1\right)_{1\le j\le d}\xrightarrow{d}\begin{pmatrix}a_1\xi\\ \vdots\\ a_d\xi\end{pmatrix}.$$

In the particular case $d=1$, $a=1$ and $g(x)=0$ if $x<1$, we recover the asymptotic result about Weissman's estimator, see Theorem 4.3.8 in de Haan and Ferreira (2006); for $g(x)=x$ and $d=1$, we recover a result similar to Theorem 2 of El Methni, Gardes and Girard (2014) if the intermediate estimator is the PL estimator.

In practical situations, the estimation of the parameter γ is of course a central question. Classical tail index estimators (see Section 3 of de Haan and Ferreira (2006)), when computed with the top $100(1-\beta_n)\%$ of the data, converge at the required rate $\sqrt{n(1-\beta_n)}$. The choice of the intermediate level β_n, which is crucial, is a difficult problem however, and we discuss a possible selection rule in our simulation study.

4. Simulation Study

The finite-sample performance of our estimators is illustrated in a simulation study, where we considered a couple of classical heavy-tailed distributions and three different distortion functions g. The distributions studied were: the Fréchet distribution: $F(x) = \exp(-x^{-1/\gamma})$, $x > 0$; and the Burr distribution: $F(x) = 1 - (1 + x^{-\rho/\gamma})^{1/\rho}$, $x > 0$ (here $\rho \le 0$). Both of these distributions have extreme value index γ and their respective second-order parameters are -1 and ρ, see *e.g.* Beirlant et al. (2004). We considered the following distortion functions: the Conditional Tail Expectation (CTE) function $g(x) = x$; the Dual Power (DP) function $g(x) = 1 - (1-x)^{1/\alpha}$ with $\alpha \in (0,1)$ (when $r := 1/\alpha$ is a positive integer, the related DRM is the expectation of $\max(X_1, \ldots, X_r)$ for independent copies $X_1, \ldots, X_r$ of X), and the Proportional Hazard (PH) transform function $g(x) = x^\alpha$, $\alpha \in (0,1)$.

We first discuss the choice of the level β_n. This step is crucial: choosing β_n too close to 1 increases the variance of the estimator dramatically, while choosing β_n too far from 1 results in biased estimates. In many cases, the analysis starts by drawing a plot of one or several tail index estimators, and then by selecting β_n in a region contained in the extremes of the sample where the estimation is "stable". Our purpose here is to suggest an automatic such choice. We work with the popular Hill estimator (Hill (1975)):

$$\widehat{\gamma}_{\beta_n} = H_n(\lceil n(1-\beta_n)\rceil) \quad \text{with} \quad H_n(k) = \frac{1}{k}\sum_{i=1}^{k} \log\left(X_{n-i+1,n}\right) - \log\left(X_{n-k,n}\right),$$

which we shall also use to estimate the extreme value index γ. Our idea is to detect the last stability region in the Hill plot $\beta \mapsto \widehat{\gamma}_\beta$; choosing β in this region most often realizes a decent bias-variance trade-off. Specifically:

- choose $\beta_0 > 0$ and a window parameter $h > 1/n$;
- for $\beta_0 < \beta < 1-h$, let $I(\beta, h) = [\beta, \beta + h]$ and compute the standard deviation $\sigma(\beta, h)$ of the set of estimates $\{\widehat{\gamma}_b,\, b \in I(\beta, h)\}$;
- if $\beta \mapsto \sigma(\beta, h)$ is monotonic, let β_{lm} be β_0 if it is increasing and $1-h$ if it is decreasing;

- otherwise, denote by β_{lm} the last value of β such that $\sigma(\beta,h)$ is locally minimal and its value is less than the average value of the function $\beta \mapsto \sigma(\beta,h)$;

- choose β^* such that $\widehat{\gamma}_{\beta^*}$ is the median of $\{\widehat{\gamma}_b, b \in I(\beta_{lm},h)\}$.

This procedure is somewhat related to others in the extreme value literature (see *e.g.* Resnick and Stărică (1997), Drees, de Haan and Resnick (2000), de Sousa and Michailidis (2004), Frahm, Junker and Schmidt (2005), Stupfler (2013), Gardes and Stupfler (2014) and Stupfler (2016)). An illustration of this technique on a simulated data set is given in Figure 1 of the Supplementary Material.

In each case, we carried out our computations on $N = 5,000$ independent samples of $n \in \{100, 300\}$ independent copies of X; our choice procedure was conducted with $\beta_0 = 0.5$ and $h = 0.1$. We recorded relative mean squared errors (MSEs):

$$\text{MSE}(\widehat{R}^W_{g,\delta}) = \frac{1}{N}\sum_{i=1}^N \left(\frac{\widehat{R}^W_{g,\delta}(X;\beta_i^*)}{R_{g,\delta}(X)} - 1 \right)^2$$

at $\delta = 0.99$, 0.995 and 0.999 (here β_i^* is the chosen intermediate level for the $i-$th sample). Our results are reported in Tables 1–3. It appears on these examples that the PL estimator performs at least as well as the AE estimator, as expected; besides, the PL estimator performs markedly better than the AE estimator for smaller samples or when the behavior of g' around 0 becomes more challenging, as can be seen by comparing the results obtained when $n = 100$ for the DP(1/3) or PH(2/3) risk measure. Results deteriorate when γ increases: a possible explanation lies in the fact that the asymptotic distribution of our estimator is essentially that of $\widehat{\gamma}_n$ by Theorem 3, which is a Gaussian distribution with variance proportional to γ^2 (see Theorem 3.2.5 in de Haan and Ferreira (2006)). Results however improve when $|\rho|$ increases, which was expected since the larger is $|\rho|$, the smaller is the bias in the estimation.

Besides, the PL estimator seems to be at least somewhat robust to a violation of the integrability condition in Theorems 2–3, as can be seen on the example of the PH(2/3) risk measure with $\gamma = 1/4$. When comparing the results for the CTE and PH(2/3) risk measures, it can also be seen that results deteriorate as the limit of $g'(s)$ as $s \downarrow 0$ increases. This likely comes from the fact that an increasing such limit amplifies the error made by the empirical quantile function, all the more so as the latter error itself increases when estimating quantiles whose order is very close to 1. The AE estimator, meanwhile, could have been thought

Table 1. Relative MSE for both estimators, case of the CTE.

Value of γ	δ	Estimator	Fréchet		Burr $\rho = -1$		Burr $\rho = -2$	
			$n = 100$	$n = 300$	$n = 100$	$n = 300$	$n = 100$	$n = 300$
$\gamma = 1/6$	0.99	AE	0.0325	0.0098	0.0374	0.0133	0.0291	0.0095
		PL	0.0317	0.0097	0.0357	0.0127	0.0286	0.0094
	0.995	AE	0.0457	0.0137	0.0540	0.0191	0.0401	0.0130
		PL	0.0446	0.0135	0.0518	0.0184	0.0395	0.0129
	0.999	AE	0.0891	0.0258	0.1115	0.0386	0.0752	0.0236
		PL	0.0871	0.0255	0.1073	0.0375	0.0741	0.0235
$\gamma = 1/5$	0.99	AE	0.0519	0.0164	0.0627	0.0199	0.0472	0.0140
		PL	0.0502	0.0161	0.0588	0.0191	0.0461	0.0138
	0.995	AE	0.0739	0.0229	0.0915	0.0289	0.0657	0.0191
		PL	0.0717	0.0225	0.0862	0.0277	0.0643	0.0189
	0.999	AE	0.1500	0.0437	0.1952	0.0589	0.1266	0.0349
		PL	0.1461	0.0430	0.1850	0.0569	0.1239	0.0344
$\gamma = 1/4$	0.99	AE	0.0973	0.0285	0.1028	0.0349	0.0834	0.0248
		PL	0.0900	0.0278	0.0944	0.0332	0.0835	0.0246
	0.995	AE	0.1411	0.0402	0.1515	0.0509	0.1190	0.0341
		PL	0.1305	0.0392	0.1395	0.0484	0.1202	0.0337
	0.999	AE	0.3039	0.0787	0.3350	0.1063	0.2492	0.0631
		PL	0.2807	0.0768	0.3102	0.1017	0.2604	0.0622

Table 2. Relative MSE for both estimators, case of the DP(1/3).

Value of γ	δ	Estimator	Fréchet		Burr $\rho = -1$		Burr $\rho = -2$	
			$n = 100$	$n = 300$	$n = 100$	$n = 300$	$n = 100$	$n = 300$
$\gamma = 1/6$	0.99	AE	0.0487	0.0169	0.0629	0.0215	0.0458	0.0140
		PL	0.0448	0.0160	0.0549	0.0194	0.0443	0.0142
	0.995	AE	0.0653	0.0225	0.0866	0.0295	0.0609	0.0182
		PL	0.0597	0.0212	0.0757	0.0267	0.0586	0.0184
	0.999	AE	0.1177	0.0394	0.1658	0.0549	0.1084	0.0307
		PL	0.1073	0.0371	0.1456	0.0499	0.1033	0.0306
$\gamma = 1/5$	0.99	AE	0.0808	0.0261	0.0988	0.0336	0.0680	0.0211
		PL	0.0743	0.0256	0.0852	0.0304	0.0652	0.0217
	0.995	AE	0.1100	0.0349	0.1376	0.0463	0.0907	0.0276
		PL	0.1004	0.0339	0.1187	0.0417	0.0862	0.0281
	0.999	AE	0.2078	0.0620	0.2723	0.0870	0.1630	0.0468
		PL	0.1879	0.0598	0.2362	0.0785	0.1535	0.0468
$\gamma = 1/4$	0.99	AE	0.1558	0.0449	0.2175	0.0570	0.1327	0.0376
		PL	0.1397	0.0439	0.1707	0.0501	0.1252	0.0388
	0.995	AE	0.2182	0.0602	0.3161	0.0787	0.1818	0.0494
		PL	0.1932	0.0582	0.2471	0.0690	0.1698	0.0503
	0.999	AE	0.4485	0.1086	0.7089	0.1508	0.3561	0.0854
		PL	0.3899	0.1038	0.5482	0.1323	0.3279	0.0852

Table 3. Relative MSE for both estimators, case of the PH(2/3).

Value of γ	δ	Estimator	Fréchet		Burr $\rho = -1$		Burr $\rho = -2$	
			$n = 100$	$n = 300$	$n = 100$	$n = 300$	$n = 100$	$n = 300$
$\gamma = 1/6$	0.99	AE	0.0517	0.0162	0.0618	0.0207	0.0487	0.0141
		PL	0.0395	0.0145	0.0421	0.0157	0.0382	0.0133
	0.995	AE	0.0699	0.0216	0.0848	0.0282	0.0654	0.0184
		PL	0.0534	0.0191	0.0584	0.0215	0.0511	0.0172
	0.999	AE	0.1290	0.0383	0.1612	0.0523	0.1196	0.0311
		PL	0.0993	0.0334	0.1143	0.0406	0.0932	0.0286
$\gamma = 1/5$	0.99	AE	0.0800	0.0272	0.1116	0.0335	0.0756	0.0204
		PL	0.0579	0.0221	0.0670	0.0240	0.0583	0.0186
	0.995	AE	0.1083	0.0363	0.1549	0.0455	0.1010	0.0267
		PL	0.0780	0.0291	0.0941	0.0327	0.0776	0.0239
	0.999	AE	0.2020	0.0644	0.3067	0.0843	0.1829	0.0454
		PL	0.1457	0.0515	0.1916	0.0619	0.1401	0.0397
$\gamma = 1/4$	0.99	AE	0.1920	0.0461	0.2432	0.0678	0.1516	0.0405
		PL	0.1008	0.0347	0.1122	0.0438	0.0927	0.0355
	0.995	AE	0.2669	0.0613	0.3421	0.0921	0.2055	0.0529
		PL	0.1384	0.0453	0.1595	0.0594	0.1242	0.0452
	0.999	AE	0.5454	0.1088	0.7137	0.1727	0.3928	0.0906
		PL	0.2760	0.0796	0.3409	0.1136	0.2330	0.0748

to provide additional robustness against this defect, since it only depends on a single intermediate order statistic and the Hill estimator, but it actually fails to improve upon the PL estimator, most likely because the multiplicative factor $(2/3 - \widehat{\gamma}_n)^{-1}$ (in the PH case) makes it severely underperform in some samples.

5. Data Application

5.1. Analysis of extreme swings of the results curve of a professional poker player

We apply our method to the study of the results of high-stakes poker player Tom Dwan. The original data, extracted from results publicly available at `http://www.highstakesdb.com`, consists in his cumulative results on the Internet, aggregated over all poker variants and recorded approximately every five days from mid-October 2008 to April 2011. In this study, we focused on the sub-parts of the results curve when the player was either consistently winning or losing. The analysis of such timeframes helps poker players understand their own behavior (and possibly that of their opponents as well) during winning and losing streaks.

To this end, we recorded the values of the local minima and maxima of the results curve and we constructed the differences between two such consecutive points. The data was made of $n = 68$ observations, which represented the aggregated results during alternative winning and losing streaks. Our aim was to analyze the extreme such streaks (also called “swings” in poker parlance). Our data X_t, represented in Figure 1 (see also Figure 2 in the Supplementary Material), was the absolute value of the 68 observations at our disposal, and the analysis focused on the magnitude of the extreme swings of the results curve, irrespective of whether such a swing corresponds to a win or a loss. It should be pointed out that a statistical analysis did not reveal a significant difference between the tail indices of winning and losing swings at the 5% error rate.

Since we work on time series data, there are particular concerns about independence and stationarity. These hypotheses were checked using the turning point test (see Kendall and Stuart (1968)) contained in the R package `randtests`; the p-value of this test was 0.278 and thus we did not reject the i.i.d. assumption based on this procedure. Since such a test is known to be poor against trends, we also ran the KPSS test for trend stationarity (Kwiatkowski et al. (1992)) contained in the R package `tseries`, whose p-value was greater than 0.1 for an estimated trend parameter of $\widehat{m} = -15.236$ (estimated via a linear regression) and a lag parameter of 1 in the Newey-West variance estimator. The stationarity assumption could then be assumed to be reasonable on the detrended time series $X_t - \widehat{m}t$, which is the sample of data we applied our procedures on in what follows; this was confirmed by the KPSS test for level stationarity, also part of the `tseries` package, whose p-value was greater than 0.1. Finally, let us note that the plot of the sample autocorrelation function (see Figure 3 in the Supplementary Material) did not indicate significant correlation in the data.

Our next aim was to estimate the extreme value index γ of the detrended sample. Since the sample size was fairly small, we used the Hill estimator together with a bias-reduced version inspired by the work of Peng (1998):

$$\widehat{\gamma}_\beta^{RB}(\tau) = \frac{1}{\widehat{\rho}_{\beta_1}(\tau)}\widehat{\gamma}_\beta + \left(1 - \frac{1}{\widehat{\rho}_{\beta_1}(\tau)}\right)\frac{\widehat{\gamma}_\beta^S}{2\widehat{\gamma}_\beta},$$

with

$$\widehat{\gamma}_\beta^S = \frac{1}{\lceil n(1-\beta)\rceil}\sum_{i=1}^{\lceil n(1-\beta)\rceil}\left(\log X_{n-i+1,n} - \log X_{n-\lceil n(1-\beta)\rceil,n}\right)^2$$

and $\widehat{\rho}_{\beta_1}(\tau)$ is the consistent estimator of ρ presented at (2.18) of Fraga Alves, Gomes and de Haan (2003), which depends on a different sample fraction $1 - \beta_1$

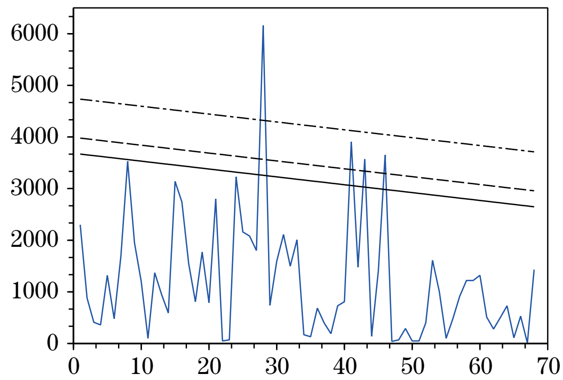

Figure 1. Poker data set (measurement unit: thousands of USD). Full line: 95% quantile line, dashed line: 97% quantile line, dashed-dotted line: 99% quantile line.

Table 4. Data sets: estimates of γ. Left: poker data set, right: Secura Belgian Re data set.

Estimator $\widehat{\gamma}$	β^*	Estimate
Hill	0.75	0.351
Bias-reduced, $\tau = 1$	0.794	0.260
Bias-reduced, $\tau = 3/4$	0.912	0.167
Bias-reduced, $\tau = 1/2$	0.912	0.158
Bias-reduced, $\tau = 1/4$	0.912	0.146
Bias-reduced, $\tau = 0$	0.853	0.118

Estimator $\widehat{\gamma}$	β^*	Estimate
Hill	0.854	0.292
Bias-reduced, $\tau = 1$	0.782	0.263
Bias-reduced, $\tau = 3/4$	0.792	0.262
Bias-reduced, $\tau = 1/2$	0.792	0.261
Bias-reduced, $\tau = 1/4$	0.792	0.260
Bias-reduced, $\tau = 0$	0.792	0.258

and a tuning parameter $\tau \le 0$. By Theorem 2.1 in Peng (1998),

$$\sqrt{n(1-\beta_n)}(\widehat{\gamma}^{RB}_{\beta_n}(\tau) - \gamma) \xrightarrow{d} \mathcal{N}\left(0, \gamma^2 \frac{1 - 2\rho + 2\rho^2}{\rho^2}\right) \tag{5.1}$$

provided (β_n) is an intermediate sequence. The generalized jackknife estimator $\widehat{\gamma}^{RB}_{\beta}(\tau)$ is thus essentially a suitably weighted combination of the Hill estimator and a similar estimator, the coefficients being estimates of those which make the asymptotic biases cancel out. We took $\beta_1 = 1 - \lceil n^{0.975} \rceil / n \approx 0.0882$, as recommended by Caeiro, Gomes and Rodrigues (2009).

Some estimates of γ are given in Table 4 and Hill plots are represented in Figure 4 of the Supplementary Material. The Hill estimator seems to drift away fairly quickly due to the finite-sample bias, and we decided to drop it for our analysis. We then estimated γ by the median of the bias-reduced estimates obtained for $\tau \in \{0, 1/4, 1/2, 3/4, 1\}$: in each case, the estimate was obtained by a straightforward adaptation of the selection procedure detailed in Section 4. We got $\widehat{\gamma} = 0.158$ for $\beta^* = 0.912$ and $\tau = 1/2$; especially, ρ was estimated

Table 5. Poker data set, detrended data: estimating some risk measures (measurement unit: thousands of USD). Between square brackets: asymptotic 95% confidence intervals.

δ	Estimator	$\widehat{\text{VaR}}$	$\widehat{\text{CTE}}$	$\widehat{\text{DP}(1/2)}$	$\widehat{\text{DP}(1/3)}$
0.95	AE	3684 [3121, 4247]	4373 [3705, 5041]	4747 [4022, 5472]	5010 [4245, 5775]
	PL	3684 [3121, 4247]	4911 [4161, 5661]	5450 [4618, 6282]	5805 [4918, 6692]
0.97	AE	3993 [2832, 5154]	4740 [3362, 6118]	5145 [3649, 6641]	5430 [3851, 7009]
	PL	3993 [2832, 5154]	5323 [3775, 6871]	5907 [4190, 7624]	6291 [4462, 8120]
0.99	AE	4748 [1958, 7538]	5636 [2325, 8947]	6118 [2524, 9712]	6457 [2663, 10251]
	PL	4748 [1958, 7538]	6329 [2611, 10047]	7023 [2897, 11149]	7480 [3085, 11875]

by $\widehat{\rho} = -1.130$. Finally, Table 5 gives estimates of some risk measures for the detrended data set and Figure 1 represents the estimates of some extreme quantile lines for the time series X_t, obtained by re-adding the trend component $\widehat{m}t$ to our estimates of the VaR. From these results, it appears in particular that the maximal value in this data set, corresponding to a losing streak costing more than 6.1 million USD, exceeds our estimate of the 99% quantile. It is also of the same order of magnitude as our estimates of the CTE and DP(1/2) (resp. DP(1/3)) risk measure in the 1% highest cases, which corresponds to the average value of the maximum of two (resp. three) consecutive extreme results. In our opinion, this losing streak can thus be regarded as an extreme period of loss.

5.2. The Secura Belgian Re actuarial data set

We consider here the Secura Belgian Re data set on automobile claims from 1998 until 2001, introduced in Beirlant et al. (2004) and further analyzed in Vandewalle and Beirlant (2006) from the extreme-value perspective. The data set consists of $n = 371$ claims which were at least as large as 1.2 million Euros and were corrected for inflation. Our aim was to revisit this data set and show how we can recover results similar to those of Vandewalle and Beirlant (2006) although they worked in a different context.

We started as in Section 5.1 by estimating the extreme value index γ. We again used the Hill estimator and some of its bias-reduced versions: Hill plots are represented in Figure 5 of the Supplementary Material, on which we can see that all our selected estimators give very close estimates. Results using our

Table 6. Insurance data set: estimating some risk measures (measurement unit: thousands of Euros). Between square brackets: asymptotic 95% confidence intervals.

δ	Estimator	$\widehat{\text{VaR}}$	$\widehat{\text{CTE}}$	$\widehat{\text{SP}}$
0.98	AE	4989 [3505, 6473]	6750 [4742, 8758]	35.220 [24.744, 45.696]
	PL	4989 [3505, 6473]	6864 [4822, 8906]	37.500 [26.346, 48.654]
0.99	AE	5978 [3673, 8283]	8087 [4969, 11205]	21.092 [12.960, 29.224]
	PL	5978 [3673, 8283]	8224 [5053, 11395]	22.459 [13.800, 31.118]
0.995	AE	7163 [3770, 10556]	9690 [5100, 14280]	12.636 [6.6506, 18.621]
	PL	7163 [3770, 10556]	9854 [5186, 14522]	13.455 [7.0817, 19.828]
0.999	AE	10899 [3506, 18291]	14744 [4743, 24745]	3.8452 [1.2371, 6.4533]
	PL	10899 [3506, 18292]	14993 [4823, 25163]	4.0944 [1.3172, 6.8716]

selection procedure are given in Table 4. Retaining the median estimate of γ yields $\widehat{\gamma} = 0.261$ for $\beta^* = 0.792$ and $\tau = 1/2$, with $\widehat{\rho} = -1.064$. Table 6 gives estimates of some risk measures for this data set.

The main example of excess-of-loss reinsurance policy that Vandewalle and Beirlant (2006) considered, namely the net premium principle, can actually be recovered from these estimates. Indeed, according to Vandewalle and Beirlant (2006), the net premium $\text{NP}(R)$ for a reinsurance policy in excess of a high retention level R is

$$\text{NP}(R) = \int_R^\infty [1 - F(x)]dx.$$

Rearranging equation (3.1) and setting $g(x) = x$ gives the identity

$$\text{NP}(q(\beta)) = (1 - \beta)(R_{g,\beta}(X) - \text{VaR}(\beta))$$

and, in particular, the right-hand side is actually $\text{SP}(\beta)$. When R is equal to 5 million Euros, as considered in Vandewalle and Beirlant (2006), it can be seen that the exceedance probability $\mathbb{P}(X > R)$ is estimated to be approximately 0.02, or in other words that R is essentially the estimated VaR at the 98% level. Estimates of our risk measures at this level are provided in Table 6; in particular, the net premium is estimated to be approximately 36,000 Euros, which is in line with the 41,798 Euros that Vandewalle and Beirlant (2006) obtained, with our estimate being slightly lower partly because a bias-reduced estimate of γ was used in the present work, whereas Vandewalle and Beirlant (2006) computed a simple Hill estimate.

6. Discussion

In the application of statistics to insurance and finance, the study of extreme risk is of prime importance. We believe that a major part of the value of our work lies in the flexibility and generality of the proposed class of extreme Wang distortion risk measures (DRMs) we introduce here. We also provide estimators for our concept of extreme Wang DRM when the underlying distribution is heavy-tailed. Our work makes it theoretically possible to give a detailed picture of extreme risk; the finite-sample procedure we introduce, which is completely data-driven and has decent performance when the tail index is moderate, is a step towards achieving this goal in practice.

Because the proposed class of extreme Wang DRMs allows for almost total freedom in choosing how to weight quantiles above a high level, it should be highlighted that it allows for yet many other interesting problems to be tackled. One may look for instance at extreme versions of Dual Power (DP) distortion risk measures; in certain situations, the DP risk measure is actually the expectation of the maximum $M_r = \max(X_1, \ldots, X_r)$ of independent copies of the random variable of interest above a high threshold. This is of course interesting in financial contexts, as our data application to the results curve of high-stakes poker player Tom Dwan shows.

As far as actuarial applications are concerned, a possible situation is the following: when insurance firms have to cover against flood risk, then assuming that r floods occur in a given year, a catastrophic event occurs when the maximum M_r of water levels during these flood episodes exceeds a given extreme level. If flood heights can reasonably be thought to be independent then such a problem can be examined as a simple application of the devices developed in this paper. Another appealing perspective lies in the fact that our class of extreme Wang DRMs can produce certain reinsurance objects such as the Stop-loss Premium and therefore, as in our application to the Secura Belgian Re data set, our framework may also be applied to certain reinsurance calculations.

Supplementary Materials

The Supplementary Material contains two tables listing risk indicators obtained by combining Wang DRMs in an appropriate way, five figures relevant to the data analyses and the proofs of our main results.

Acknowledgment

The authors acknowledge both anonymous referees and the Editor for their helpful and stimulating comments which led to improved presentation of this article.

References

Adrian, T. and Brunnermeier, M. K. (2008). CoVaR. *Federal Reserve of New York Staff Reports* **348**.

Artzner, P., Delbaen, F., Eber, J. M. and Heath, D. (1999). Coherent measures of risk. *Math. Finance* **9**, 203–228.

Basel Committee on Banking Supervision (2006). International convergence of capital measurement and capital standards: a revised framework (comprehensive version).

Basel Committee on Banking Supervision (2011). *Revisions to the Basel II market risk framework.*

Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J. L. (2004). *Statistics of Extremes – Theory and Applications*. Wiley, New York.

Belles-Sampera, J., Guillén, M. and Santolino, M. (2014). Beyond Value-at-Risk: GlueVaR distortion risk measures. *Risk Analysis* **34**, 121–134.

Caeiro, F., Gomes, M. I. and Rodrigues, L. H. (2009). Reduced-bias tail index estimators under a third-order framework. *Comm. Statist. Theory Methods* **38**, 1019–1040.

Cont, R., Deguest, R. and Scandolo, G. (2010). Robustness and sensitivity analysis of risk measurement procedures. *Quantitative Finance* **10**, 593–606.

Cotter, J. and Dowd, K. (2006). Extreme spectral risk measures: An application to futures clearinghouse margin requirements. *Journal of Banking and Finance* **30**, 3469–3485.

Daníelsson, J., Jorgensen, B. N., Samorodnitsky, G., Sarma, M. and de Vries, C. G. (2013). Fat tails, VaR and subadditivity. *J. Econometrics* **172**, 283–291.

Davis, M. H. A. (2013). Consistency of risk measure estimates. Preprint, available at `http://ssrn.com/abstract=2342279`.

de Haan, L. and Ferreira, A. (2006). *Extreme Value Theory: An Introduction*. Springer, New York.

de Sousa, B. and Michailidis, G. (2004). A diagnostic plot for estimating the tail index of a distribution. *J. Comput. Graph. Statist.* **13**, 974–995.

Drees, H. (2003). Extreme quantile estimation for dependent data, with applications to finance. *Bernoulli* **9**, 617–657.

Drees, H., de Haan, L. and Resnick, S. (2000). How to make a Hill plot. *Ann. Statist.* **28**, 254–274.

El Methni, J., Gardes, L. and Girard, S. (2014). Nonparametric estimation of extreme risks from conditional heavy-tailed distributions. *Scand. J. Stat.* **41**, 988–1012.

Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*. Springer-Verlag, Berlin.

Embrechts, P., Wang, B. and Wang, R. (2015). Aggregation-robustness and model uncertainty of regulatory risk measures. *Finance and Stochastics* **19**, 763–790.

Fraga Alves, M. I., Gomes, M. I. and de Haan, L. (2003). A new class of semi-parametric estimators of the second order parameter. *Port. Math.* **60**, 193–213.

Frahm, G., Junker, M. and Schmidt, R. (2005). Estimating the tail-dependence coefficient: properties and pitfalls. *Insurance Math. Econom.* **37**, 80–100.

Furman, E. and Landsman, Z. (2006). On some risk-adjusted tail-based premium calculation principles. *Journal of Actuarial Practice* **13**, 175–190.

Gardes, L. and Stupfler, G. (2014). Estimation of the conditional tail index using a smoothed local Hill estimator. *Extremes* **17**, 45–75.

Gneiting, T. (2011). Making and evaluating point forecasts. *J. Amer. Statist. Assoc.* **106**, 746–762.

Gordy, M. B. (2003). A risk-factor model foundation for ratings-based bank capital rules. *Journal of Financial Intermediation* **12**, 199–232.

Hill, B. M. (1975). A simple general approach to inference about the tail of a distribution. *Ann. Statist.* **3**, 1163–1174.

Kendall, M. G. and Stuart, A. (1968). *The Advanced Theory of Statistics, Volume 3: Design and Analysis, and Time-Series* (second edition). Griffin, London.

Keppo, J., Kofman, L. and Meng, X. (2010). Unintended consequences of the market risk requirement in banking regulation. *Journal of Economic Dynamics and Control* **34**, 2192–2214.

Kou, S. G., Peng, X. and Heyde, C. C. (2013). External risk measures and Basel accords. *Math. Oper. Res.* **38**, 393–417.

Kou, S. G. and Peng, X. (2014). On the measurement of economic tail risk. Preprint, available at `http://arxiv.org/pdf/1401.4787v2.pdf`.

Kuan, C.-M., Yeh, J.-H. and Hsu, Y.-C. (2009). Assessing value at risk with CARE, the Conditional Autoregressive Expectile models. *J. Econometrics* **2**, 261–270.

Kwiatkowski, D., Phillips, P.C.B., Schmidt, P. and Shin, Y. (1992). Testing the null hypothesis of stationarity against the alternative of a unit root. *J. Econometrics* **54**, 159–178.

Linton, O. and Xiao, Z. (2013). Estimation of and inference about the expected shortfall for time series with infinite variance. *Econometric Theory* **29**, 771–807.

McNeil, A. J., Frey, R. and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques, and Tools*. Princeton University Press.

Newey, W. K. and Powell, J. L. (1987). Asymmetric least squares estimation and testing. *Econometrica* **55**, 819–847.

Peng, L. (1998). Asymptotically unbiased estimators for the extreme-value index. *Statist. Probab. Lett.* **38**, 107–115.

Resnick, S. and Stărică, C. (1997). Smoothing the Hill estimator. *Adv. in Appl. Probab.* **29**, 271–293.

Rootzén, H. and Tajvidi, N. (1997). Extreme value statistics and wind storm losses: A case study. *Scand. Actuar. J.* **1**, 70–94.

Stupfler, G. (2013). A moment estimator for the conditional extreme-value index. *Electron. J. Stat.* **7**, 2298–2343.

Stupfler, G. (2016). Estimating the conditional extreme-value index under random right-censoring. *J. Multivariate Anal.* **144**, 1–24.

Vandewalle, B. and Beirlant, J. (2006). On univariate extreme value statistics and the estimation of reinsurance premiums. *Insurance Math. Econom.* **38**, 441–459.

Wang, S. S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bull.* **26**, 71–92.

Wang, S. S. (2000). A class of distortion operators for pricing financial and insurance risks. *Journal of Risk and Insurance* **67**, 15–36.

Wang, R. and Ziegel, J. F. (2015). Elicitable distortion risk measures: A concise proof. *Statist. Probab. Lett.* **100**, 172–175.

Weissman, I. (1978). Estimation of parameters and large quantiles based on the k largest observations. *J. Amer. Statist. Assoc.* **73**, 812–815.

Wirch, J. L. and Hardy, M. R. (2002). Distortion risk measures: Coherence and stochastic dominance. In *International Congress on Insurance: Mathematics and Economics*, Lisbon.

Ziegel, J. F. (2015). Coherence and elicitability. *Math. Finance*, to appear, available at `http://arxiv.org/pdf/1303.1690v3.pdf`.

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(Received December 2015 ; accepted May 2016)