EXTREME VERSIONS OF WANG RISK MEASURES AND THEIR ESTIMATION FOR HEAVY-TAILED DISTRIBUTIONS

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Supplementary Material

S1 Proofs of the main results

Proof of Theorem 1. Write for any j:

$$\frac{\widehat{R}^{AE}_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\beta_n}(X^{a_j})} = \left[\frac{X_{\lceil n\beta_n\rceil,n}}{q(\beta_n)}\right]^a \times \frac{\int_0^1 s^{-a_j\widehat{\gamma}_n} dg_j(s)}{\int_0^1 s^{-a_j\gamma} dg_j(s)} \times \frac{[q(\beta_n)]^a \int_0^1 s^{-a_j\gamma} dg_j(s)}{R_{g_j,\beta_n}(X^{a_j})}.$$

We start by showing the consistency statement: from Lemma 3(i) and the continuity of the maps $t \mapsto \int_0^1 s^{-a_j t} dg_j(s)$, $1 \le j \le d$ at the point γ, we obtain

$$\frac{\widehat{R}^{AE}_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\beta_n}(X^{a_j})} = \left[\frac{X_{\lceil n\beta_n\rceil,n}}{q(\beta_n)}\right]^a (1 + \mathrm{o}_{\mathbb{P}}(1))$$

Write now $X_{\lceil n\beta_n\rceil,n} = U(Y_{\lceil n\beta_n\rceil,n})$ where Y has a standard Pareto distribution, and use Corollary 2.2.2 in de Haan and Ferreira (2006) together with the regular variation property of U to get

$$\frac{\widehat{R}^{AE}_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\beta_n}(X^{a_j})} \xrightarrow{\mathbb{P}} 1.$$

To show the asymptotic normality of the estimator, use first the hypothesis on $X_{\lceil n\beta_n \rceil,n}$ and Lemma 3(ii) together with a Taylor expansion to get

$$\frac{\widehat{R}^{AE}_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\beta_n}(X^{a_j})} = \frac{\int_0^1 s^{-a_j\widehat{\gamma}_n} dg_j(s)}{\int_0^1 s^{-a_j\gamma} dg_j(s)} \left[1 + \frac{a_j}{\sqrt{n(1-\beta_n)}}\left\{\Theta - \lambda\frac{\int_0^1 s^{-a_j\gamma}\frac{s^{-\rho}-1}{\rho} dg_j(s)}{\int_0^1 s^{-a_j\gamma} dg_j(s)} + \mathrm{o}_{\mathbb{P}}(1)\right\}\right]. \tag{S1.1}$$

Set then $\kappa(x) = e^x - 1 - x$ and notice that

$$\begin{aligned}\frac{\int_0^1 s^{-a_j\widehat{\gamma}_n} dg_j(s)}{\int_0^1 s^{-a_j\gamma} dg_j(s)} &= 1 + a_j(\widehat{\gamma}_n - \gamma)\frac{\int_0^1 s^{-a_j\gamma}\log(1/s) dg_j(s)}{\int_0^1 s^{-a_j\gamma} dg_j(s)} \\ &+ \frac{\int_0^1 s^{-a_j\gamma}\kappa(a_j(\widehat{\gamma}_n-\gamma)\log(1/s)) dg_j(s)}{\int_0^1 s^{-a_j\gamma} dg_j(s)}.\end{aligned}$$

A Taylor inequality for the exponential function at order 2 gives $|\kappa(x)| \le x^2 e^{|x|}/2$ and thus

$$\begin{aligned}&\left|\int_0^1 s^{-a_j\gamma}\kappa(a_j(\widehat{\gamma}_n-\gamma)\log(1/s)) dg_j(s)\right| \\ \le\ & \frac{a_j^2}{2}(\widehat{\gamma}_n-\gamma)^2 \int_0^1 s^{-a_j\gamma}\log^2(1/s) s^{-a_j|\widehat{\gamma}_n-\gamma|} dg_j(s).\end{aligned}$$

Since $\int_0^1 s^{-a_j\gamma-\eta} dg_j(s) < \infty$, it follows by the $\sqrt{n(1-\beta_n)}-$consistency of $\widehat{\gamma}_n$ that

$$\left|\int_0^1 s^{-a_j\gamma}\kappa(a_j(\widehat{\gamma}_n-\gamma)\log(1/s)) dg_j(s)\right| = \mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\beta_n)}}\right)$$

and thus

$$\frac{\int_0^1 s^{-a_j\widehat{\gamma}_n}dg_j(s)}{\int_0^1 s^{-a_j\gamma}dg_j(s)} = 1+\frac{a_j}{\sqrt{n(1-\beta_n)}}\frac{\int_0^1 s^{-a_j\gamma}\log(1/s)dg_j(s)}{\int_0^1 s^{-a_j\gamma}dg_j(s)}\Gamma + \mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\beta_n)}}\right). \tag{S1.2}$$

Combining (S1.1) and (S1.2) completes the proof. ■

Proof of Theorem 2. First, recall that for any $t\in\mathbb{R}$ we have $\lfloor t\rfloor+\lceil -t\rceil=0$, where $\lfloor\cdot\rfloor$ denotes the floor function. Whence the equality

$$\widehat{R}_{g_j,\beta_n}(X^{a_j})=\int_0^1 X^{a_j}_{n-\lfloor ls\rfloor,n}\,dg_j(s)$$

with $l=l(n)=n(1-\beta_n)\to\infty$. Clearly:

$$\forall s\in[0,1],\ X_{n-\lfloor(\lfloor l\rfloor+1)s\rfloor,n}\le X_{n-\lfloor ls\rfloor,n}\le X_{n-\lfloor\lfloor l\rfloor s\rfloor,n},$$

and thus it is enough to prove that, for any sequence of integers $k=k(n)$ such that $k(n)/l(n)\to 1$, we have:

$$\sqrt{k}\left(\frac{\int_0^1 X^{a_j}_{n-\lfloor ks\rfloor,n}\,dg_j(s)}{R_{g_j,\beta_n}(X^{a_j})}-1\right)_{1\le j\le d}\xrightarrow{d}\mathcal{N}(0,V).$$

For any $a>0$, let $U_a(x):=[U(x)]^a$ denote the left-continuous inverse of $1/(1-F_a)$, where F_a is the cdf of X^a. By Lemma 2:

$$\frac{R_{g_j,\beta_n}(X^{a_j})}{U_{a_j}(n/k)}=\int_0^1\frac{U_{a_j}(n/ks)}{U_{a_j}(n/k)}dg_j(s)\to\int_0^1 s^{-a_j\gamma}dg_j(s).$$

It is therefore enough to prove that:

$$\sqrt{k}\left(\frac{\int_0^1 X^{a_j}_{n-\lfloor ks\rfloor,n}\,dg_j(s) - R_{g_j,\beta_n}(X^{a_j})}{U_{a_j}(n/k)}\right)_{1\le j\le d} \xrightarrow{d} \mathcal{N}(0,M) \tag{S1.3}$$

where M is the $d\times d$ matrix with $(i,j)-$th entry

$$M_{i,j} = a_i a_j \gamma^2 \int_{[0,1]^2} \min(s,t) s^{-a_i\gamma-1} t^{-a_j\gamma-1} dg_i(s) dg_j(t).$$

Pick now $j\in\{1,\ldots,d\}$ and write

$$\int_0^1 X^{a_j}_{n-\lfloor ks\rfloor,n}\,dg_j(s) - R_{g_j,\beta_n}(X^{a_j}) = \zeta_{j,n} + \xi_{j,n} \tag{S1.4}$$

with

$$\begin{aligned}
\zeta_{j,n} &= \int_0^1 U_{a_j}(n/ks)\left(\frac{X^{a_j}_{n-\lfloor ks\rfloor,n}}{U_{a_j}(n/k)} - s^{-a_j\gamma}\right) s^{a_j\gamma} dg_j(s) \\
\text{and}\quad \xi_{j,n} &= \int_0^1 U_{a_j}(n/ks)\frac{X^{a_j}_{n-\lfloor ks\rfloor,n}}{U_{a_j}(n/k)}\left(\frac{U_{a_j}(n/k)}{U_{a_j}(n/ks)} - s^{a_j\gamma}\right) dg_j(s).
\end{aligned}$$

According to Lemma 4, we have:

$$\sqrt{k}\left(\frac{\zeta_{j,n}}{U_{a_j}(n/k)}\right)_{1\le j\le n} \xrightarrow{d} \mathcal{N}\left(\lambda C, M\right) \tag{S1.5}$$

where C is the column vector whose $j-$th entry is

$$C_j = a_j \int_0^1 \frac{s^{-\rho}-1}{\rho} s^{-a_j\gamma} dg_j(s).$$

To examine the convergence of $\xi_{j,n}$, we note that according to (S2.1), there exist Borel measurable functions $B_{a_1},\ldots,B_{a_d}$, respectively asymptotically equivalent to $a_1A_1, \ldots, a_dA_d$ and having constant sign, such that for any $\varepsilon > 0$:

$$\forall s \in (0,1], \ \left| \frac{1}{B_{a_j}(n/ks)} \left(\frac{U_{a_j}(n/k)}{U_{a_j}(n/ks)} - s^{a_j\gamma} \right) - s^{a_j\gamma} \frac{s^\rho - 1}{\rho} \right| \le \varepsilon s^{a_j\gamma + \rho - \varepsilon} \tag{S1.6}$$

for n sufficiently large. Consider then the following decomposition of $\xi_{j,n}$:

$$\xi_{j,n} = \xi_{j,n}^{(1)} + \xi_{j,n}^{(2)} \tag{S1.7}$$

with

$$\xi_{j,n}^{(1)} = \int_0^1 U_{a_j}(n/ks) B_{a_j}(n/ks) \frac{X_{n-\lfloor ks \rfloor,n}^{a_j}}{U_{a_j}(n/k)} s^{a_j\gamma} \frac{s^\rho - 1}{\rho} dg_j(s),$$
$$\xi_{j,n}^{(2)} = \int_0^1 U_{a_j}(n/ks) B_{a_j}(n/ks) \frac{X_{n-\lfloor ks \rfloor,n}^{a_j}}{U_{a_j}(n/k)} \left(\frac{1}{B_{a_j}(n/ks)} \left[\frac{U_{a_j}(n/k)}{U_{a_j}(n/ks)} - s^{a_j\gamma} \right] - s^{a_j\gamma} \frac{s^\rho - 1}{\rho} \right) dg_j(s).$$

Writing

$$\frac{X_{n-\lfloor ks \rfloor,n}^{a_j}}{U_{a_j}(n/k)} s^{a_j\gamma} = 1 + \left(\frac{X_{n-\lfloor ks \rfloor,n}^{a_j}}{U_{a_j}(n/k)} - s^{-a_j\gamma} \right) s^{a_j\gamma},$$

we get by Lemma 4:

$$\xi_{j,n}^{(1)} = \int_0^1 U_{a_j}(n/ks) B_{a_j}(n/ks) \frac{s^\rho - 1}{\rho} dg_j(s) + \mathrm{O}_{\mathbb{P}} \left(\frac{U_{a_j}(n/k) B_{a_j}(n/k)}{\sqrt{k}} \right).$$

Applying Lemma 2 to the regularly varying functions $t \mapsto U_{a_j}(t)|B_{a_j}(t)|$ and $t \mapsto t^{-\rho} U_{a_j}(t)|B_{a_j}(t)|$, which have respective regular variation indices

$a_j\gamma + \rho$ and $a_j\gamma$, we get

$$\begin{aligned} \sqrt{k}\frac{\xi_{j,n}^{(1)}}{U_{a_j}(n/k)} &= \sqrt{k}B_{a_j}(n/k)\int_0^1 s^{-a_j\gamma}\frac{1-s^{-\rho}}{\rho}dg_j(s) + \mathrm{o}_{\mathbb{P}}(1) \\ &= -a_j\lambda\int_0^1 s^{-a_j\gamma}\frac{s^{-\rho}-1}{\rho}dg_j(s) + \mathrm{o}_{\mathbb{P}}(1) \qquad \text{(S1.8)} \\ &= -\lambda C_j + \mathrm{o}_{\mathbb{P}}(1) \end{aligned}$$

since B_{a_j} is equivalent to a_jA. The quantity $\xi_{j,n}^{(2)}$ is controlled by applying inequality (S1.6): for any $\varepsilon \in (0,\eta)$, we have for sufficiently large n that:

$$|\xi_{j,n}^{(2)}| \leq \varepsilon\int_0^1 U_{a_j}(n/ks)|B_{a_j}(n/ks)|\frac{X_{n-\lfloor ks\rfloor,n}^{a_j}}{U_{a_j}(n/k)}s^{a_j\gamma+\rho-\varepsilon}dg_j(s).$$

The ideas used to control $\xi_{j,n}^{(1)}$ yield for n large enough:

$$\begin{aligned} \sqrt{k}\left|\frac{\xi_{j,n}^{(2)}}{U_{a_j}(n/k)}\right| &\leq \varepsilon a_j|\lambda|\int_0^1 s^{-a_j\gamma-\varepsilon}dg_j(s) + \mathrm{o}_{\mathbb{P}}(1) \\ &\leq \varepsilon a_j|\lambda|\int_0^1 s^{-a_j\gamma-\eta}dg_j(s) + \mathrm{o}_{\mathbb{P}}(1) \end{aligned}$$

which, since ε is arbitrary, entails

$$\sqrt{k}\left|\frac{\xi_{j,n}^{(2)}}{U_{a_j}(n/k)}\right| = \mathrm{o}_{\mathbb{P}}(1). \qquad \text{(S1.9)}$$

Combining (S1.7), (S1.8) and (S1.9) entails

$$\sqrt{k}\left(\frac{\xi_{j,n}}{U_{a_j}(n/k)}\right)_{1\leq j\leq d} \xrightarrow{\mathbb{P}} -\lambda C. \qquad \text{(S1.10)}$$

Combine finally (S1.4), (S1.5) and (S1.10) to obtain (S1.3): the proof is complete. ■

Proof of Theorem 3. We start by writing, for any j:

$$\frac{\widehat{R}^W_{g_j,\delta_n}(X^{a_j};\beta_n)}{R_{g_j,\delta_n}(X^{a_j})} = \left(\frac{1-\beta_n}{1-\delta_n}\right)^{a_j(\widehat{\gamma}_n-\gamma)} \frac{\widehat{R}_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\beta_n}(X^{a_j})} \times \frac{R_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\delta_n}(X^{a_j})} \left(\frac{1-\beta_n}{1-\delta_n}\right)^{a_j\gamma}.$$

Recall that for any $a>0$, U_a satisfies condition $\mathcal{C}_2(a\gamma,\rho,aA)$ by Lemma 1. Taking logarithms and applying Lemma 5 with $Y = X^{a_j}$, we get

$$\begin{aligned}\log\left(\frac{\widehat{R}^W_{g_j,\delta_n}(X^{a_j};\beta_n)}{R_{g_j,\delta_n}(X^{a_j})}\right) &= a_j(\widehat{\gamma}_n-\gamma)\log\left(\frac{1-\beta_n}{1-\delta_n}\right) + \log\left(\frac{\widehat{R}_{g_j,\beta_n}(X^{a_j})}{R_{g_j,\beta_n}(X^{a_j})}\right) \\ &+ \mathrm{O}\left(\frac{1}{\sqrt{n(1-\beta_n)}}\right).\end{aligned}$$

The $\sqrt{n(1-\beta_n)}$−relative consistency of $\widehat{R}_{g_j,\beta_n}(X^{a_j})$ entails

$$\log\left(\frac{\widehat{R}^W_{g_j,\delta_n}(X^{a_j};\beta_n)}{R_{g_j,\delta_n}(X^{a_j})}\right) = a_j(\widehat{\gamma}_n-\gamma)\log\left(\frac{1-\beta_n}{1-\delta_n}\right) + \mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\beta_n)}}\right).$$

Recall that $\log([1-\beta_n]/[1-\delta_n]) \to \infty$; a Taylor expansion and the hypothesis on $\widehat{\gamma}_n$ now make it clear that

$$\frac{\sqrt{n(1-\beta_n)}}{\log([1-\beta_n]/[1-\delta_n])}\left(\frac{\widehat{R}^W_{g_j,\delta_n}(X^{a_j};\beta_n)}{R_{g_j,\delta_n}(X^{a_j})} - 1\right) = a_j\xi(1+\mathrm{o}_{\mathbb{P}}(1))$$

which completes the proof. ■

S2 Preliminary results and their proofs

The first result is a very useful fact which we shall use several times in our proofs.

Lemma 1. *Assume that condition $\mathcal{C}_2(\gamma,\rho,A)$ is satisfied. Pick $a>0$ and define $U_a(x) := [U(x)]^a$. Then U_a satisfies condition $\mathcal{C}_2(a\gamma,\rho,aA)$.*

Proof of Lemma 1. Pick $x>0$. The function U satisfies condition $\mathcal{C}_2(\gamma,\rho,A)$ which is equivalent to:

$$U(tx) = U(t)\left(x^\gamma + A(t)\left[\frac{x^\gamma(x^\rho-1)}{\rho} + \mathrm{o}(1)\right]\right) \quad \text{as} \quad t\to\infty.$$

Thus

$$U_a(tx) = U_a(t)x^{a\gamma}\left(1 + A(t)\left[\frac{x^\rho-1}{\rho} + \mathrm{o}(1)\right]\right)^a \quad \text{as} \quad t\to\infty.$$

Using a Taylor expansion and rearranging terms, we get:

$$U_a(tx) = U_a(t)\left(x^{a\gamma} + aA(t)\left[\frac{x^{a\gamma}(x^\rho-1)}{\rho} + \mathrm{o}(1)\right]\right) \quad \text{as} \quad t\to\infty,$$

which is the result. ■

This result yields an important inequality which is actually contained in Theorem 2.3.9 in de Haan and Ferreira (2006): for any $a>0$, one may find a Borel measurable function B_a, asymptotically equivalent to aA and having constant sign, such that for any $\varepsilon>0$, there is $t_0>0$ such that for $t, tx \geq t_0$:

$$\left|\frac{1}{B_a(t)}\left(\frac{U_a(tx)}{U_a(t)} - x^{a\gamma}\right) - x^{a\gamma}\frac{x^\rho-1}{\rho}\right| \leq \varepsilon x^{a\gamma+\rho}\max(x^\varepsilon, x^{-\varepsilon}). \qquad \text{(S2.1)}$$

The second preliminary result is a technical lemma on some integrals, which we shall use frequently in our proofs.

Lemma 2. *Let g be a nondecreasing right-continuous function on $[0,1]$. Assume that f is a Borel measurable regularly varying function with index $b \in \mathbb{R}$. If for some $\eta > 0$:*

$$\int_0^1 s^{-b-\eta} dg(s) < \infty,$$

then for any $\delta \in \mathbb{R}$ such that $\delta < \eta$ and any continuous and bounded function φ on $(0,1]$ we have, provided (u_n) is a positive sequence tending to infinity:

$$\int_0^1 \frac{f(u_n/s)}{f(u_n)} s^{-\delta} \varphi(s) dg(s) \to \int_0^1 s^{-b-\delta} \varphi(s) dg(s).$$

Proof of Lemma 2. Pick $\delta < \eta$ and define $\varepsilon := (\eta - \delta)/2 > 0$, so that $\delta + \varepsilon < \eta$. We have

$$\begin{aligned} & \left| \int_0^1 \frac{f(u_n/s)}{f(u_n)} s^{-\delta} \varphi(s) dg(s) - \int_0^1 s^{-b-\delta} \varphi(s) dg(s) \right| \\ \leq & \int_0^1 s^{b+\varepsilon} \left| \frac{f(u_n/s)}{f(u_n)} - s^{-b} \right| s^{-b-\delta-\varepsilon} |\varphi(s)| dg(s). \end{aligned}$$

Notice that the function $f_1 : y \mapsto y^{-b-\varepsilon} f(y)$ is regularly varying with index $-\varepsilon < 0$. By a uniform convergence result for regularly varying functions (see *e.g.* Theorem 1.5.2 in Bingham *et al.*, 1987):

$$\sup_{0<s\leq 1} s^{b+\varepsilon} \left| \frac{f(u_n/s)}{f(u_n)} - s^{-b} \right| = \sup_{0<s\leq 1} \left| \frac{f_1(u_n/s)}{f_1(u_n)} - s^{\varepsilon} \right| = \sup_{t\geq 1} \left| \frac{f_1(u_n t)}{f_1(u_n)} - t^{-\varepsilon} \right| \to 0.$$

As a consequence

$$\begin{aligned}
& \left|\int_0^1 \frac{f(u_n/s)}{f(u_n)} s^{-\delta}\varphi(s)dg(s) - \int_0^1 s^{-b-\delta}\varphi(s)dg(s)\right| \\
= & \ \mathrm{O}\left(\sup_{0<s\leq 1} s^{b+\varepsilon}\left|\frac{f(u_n/s)}{f(u_n)} - s^{-b}\right|\right)
\end{aligned}$$

and the right-hand side converges to 0. The proof is complete. ∎

The third lemma gives an asymptotic expansion of a Wang DRM that is in particular the key to the construction of our first family of estimators.

Lemma 3. *Let g be a distortion function on $[0,1]$ and $a > 0$. Pick a sequence (β_n) such that $\beta_n \to 1$.*

(i) If U is regularly varying with index $\gamma > 0$ and there is $\eta > 0$ such that

$$\int_0^1 s^{-a\gamma-\eta} dg(s) < \infty$$

then we have that:

$$\frac{R_{g,\beta_n}(X^a)}{U_a([1-\beta_n]^{-1})} \to \int_0^1 s^{-a\gamma} dg(s) \quad as \quad n \to \infty.$$

(ii) If furthermore condition $\mathcal{C}_2(\gamma,\rho,A)$ is satisfied and $n(1-\beta_n) \to \infty$, $\sqrt{n(1-\beta_n)}A((1-\beta_n)^{-1}) \to \lambda \in \mathbb{R}$ then provided

$$\int_0^1 s^{-a\gamma-1/2-\eta} dg(s) < \infty$$

for some $\eta > 0$, we have that:

$$\begin{aligned}\frac{R_{g,\beta_n}(X^a)}{U_a([1-\beta_n]^{-1})} &= \int_0^1 s^{-a\gamma}dg(s) + \frac{a\lambda}{\sqrt{n(1-\beta_n)}}\int_0^1 \frac{s^{-\rho}-1}{\rho}s^{-a\gamma}dg(s)\\ &+ \mathrm{o}\left(\frac{1}{\sqrt{n(1-\beta_n)}}\right).\end{aligned}$$

Proof of Lemma 3. The first statement is proven by applying Lemma 2:

$$\frac{R_{g,\beta_n}(X^a)}{U_a([1-\beta_n]^{-1})} = \int_0^1 \frac{U_a([1-\beta_n]^{-1}/s)}{U_a([1-\beta_n]^{-1})}dg(s) = \int_0^1 s^{-a\gamma}dg(s)(1+\mathrm{o}(1)). \tag{S2.2}$$

To show the second statement, use (S2.1) to get:

$$\begin{aligned}\frac{R_{g,\beta_n}(X^a)}{U_a([1-\beta_n]^{-1})} &- \int_0^1 \left(1 + B_a([1-\beta_n]^{-1})\frac{s^{-\rho}-1}{\rho}\right) s^{-a\gamma}dg(s)\\ &= \mathrm{o}\left(B_a([1-\beta_n]^{-1})\int_0^1 s^{-a\gamma-\rho-\eta}dg(s)\right).\end{aligned}$$

Rearranging and using the convergence $\sqrt{n(1-\beta_n)}B_a((1-\beta_n)^{-1}) \to a\lambda \in \mathbb{R}$, we obtain

$$\begin{aligned}\frac{R_{g,\beta_n}(X^a)}{U_a([1-\beta_n]^{-1})} &= \int_0^1 s^{-a\gamma}dg(s) + \frac{a\lambda}{\sqrt{n(1-\beta_n)}}\int_0^1 \frac{s^{-\rho}-1}{\rho}s^{-a\gamma}dg(s)\\ &+ \mathrm{o}\left(\frac{1}{\sqrt{n(1-\beta_n)}}\right)\end{aligned} \tag{S2.3}$$

which completes the proof. ∎

The fourth lemma is the key to the proof of Theorem 2. It examines the asymptotic behavior of some weighted integrals of the empirical tail quantile process.

Lemma 4. *Assume that condition $\mathcal{C}_2(\gamma,\rho,A)$ is satisfied. Let $a_1,\ldots,a_d > 0$, $f_1,\ldots,f_d$ be Borel measurable regularly varying functions with respective indices $b_j \le a_j\gamma$ and $g_1,\ldots,g_d$ be distortion functions. Assume that $k = k(n)\to\infty$, $k/n\to 0$, $\sqrt{k}A(n/k)\to\lambda\in\mathbb{R}$ and for some $\eta>0$:*

$$\forall j\in\{1,\ldots,d\},\ \int_0^1 s^{-a_j\gamma-1/2-\eta}dg_j(s)<\infty.$$

Pick $\delta_1,\ldots,\delta_d\in\mathbb{R}$ such that $\delta_j<(a_j\gamma-b_j)+\eta$, and set

$$I_{j,n} := \frac{1}{f_j(n/k)}\int_0^1 f_j(n/ks)\sqrt{k}\left(\frac{X_{n-\lfloor ks\rfloor,n}^{a_j}}{U_{a_j}(n/k)} - s^{-a_j\gamma}\right)s^{a_j\gamma-\delta_j}dg_j(s).$$

Then we have:

$$(I_{1,n},\ldots,I_{d,n}) \xrightarrow{d} \mathcal{N}(\lambda C,\Sigma)$$

with C being the column vector with $j-$th entry

$$C_j = a_j\int_0^1 \frac{s^{-\rho}-1}{\rho}s^{-b_j-\delta_j}dg_j(s)$$

and Σ being the $d\times d$ matrix with $(i,j)-$th entry

$$\Sigma_{i,j} = a_ia_j\gamma^2\int_{[0,1]^2}\min(s,t)s^{-b_i-\delta_i-1}t^{-b_j-\delta_j-1}dg_i(s)dg_j(t).$$

Proof of Lemma 4. Define $\varepsilon := \min_{1\le j\le d}(\eta-\delta_j)/2>0$, so that $\delta_j+\varepsilon<\eta$ for all j, and let $\varepsilon'>0$ be so small that

$$\forall j\in\{1,\ldots,d\},\ a_j\gamma+\frac{1}{2}+\frac{\varepsilon}{2}-\frac{1-\varepsilon'}{1+2\varepsilon'}\left(a_j\gamma+\frac{1}{2}+\varepsilon\right)<0. \qquad \text{(S2.4)}$$

Set $s_n = k^{-(1-\varepsilon')/(1+2\varepsilon')}$. Pick $j \in \{1, \ldots, d\}$ and use the triangle inequality to get:

$$\left| \frac{1}{f_j(n/k)} \int_0^{s_n} f_j(n/ks)\sqrt{k} \left(\frac{X_{n-\lfloor ks \rfloor,n}^{a_j}}{U_{a_j}(n/k)} - s^{-a_j\gamma} \right) s^{a_j\gamma-\delta_j} dg_j(s) \right| \leq E_{j,n}^{(1)} + E_{j,n}^{(2)},$$

with

$$\begin{aligned} E_{j,n}^{(1)} &= \sqrt{k} \frac{X_{n,n}^{a_j}}{U_{a_j}(n/k)} \int_0^{s_n} \frac{f_j(n/ks)}{f_j(n/k)} s^{a_j\gamma-\delta_j} dg_j(s) \\ \text{and } E_{j,n}^{(2)} &= \sqrt{k} \int_0^{s_n} \frac{f_j(n/ks)}{f_j(n/k)} s^{-\delta_j} dg_j(s). \end{aligned}$$

Since the distribution of X is heavy-tailed it follows from Theorem 1.1.6, Theorem 1.2.1 and Lemma 1.2.9 in de Haan and Ferreira (2006) that $X_{n,n} = \mathrm{O}_{\mathbb{P}}(U(n))$. Thus

$$E_{j,n}^{(1)} = \mathrm{O}_{\mathbb{P}} \left(\sqrt{k} \frac{U_{a_j}(n)}{U_{a_j}(n/k)} \int_0^{s_n} \frac{f_j(n/ks)}{f_j(n/k)} s^{a_j\gamma-\delta_j} dg_j(s) \right)$$

Use now Potter bounds for U (see *e.g.* Theorem 1.5.6 in Bingham *et al.*, 1987) to get

$$\begin{aligned} E_{j,n}^{(1)} &= \mathrm{O}_{\mathbb{P}} \left(k^{a_j\gamma+1/2+\varepsilon/2} \int_0^{s_n} \frac{f_j(n/ks)}{f_j(n/k)} s^{a_j\gamma-\delta_j} dg_j(s) \right) \\ &= \mathrm{O}_{\mathbb{P}} \left(k^{a_j\gamma+1/2+\varepsilon/2} s_n^{a_j\gamma} \int_0^{s_n} \frac{f_j(n/ks)}{f_j(n/k)} s^{-\delta_j} dg_j(s) \right). \end{aligned}$$

Besides, note that

$$\int_0^{s_n} \frac{f_j(n/ks)}{f_j(n/k)} s^{-\delta_j} dg_j(s) \quad \leq \quad s_n^{1/2+\varepsilon} \int_0^{s_n} \frac{f_j(n/ks)}{f_j(n/k)} s^{-1/2-\delta_j-\varepsilon} dg_j(s) = \mathrm{o}\left(s_n^{1/2+\varepsilon}\right),$$

by Lemma 2. Thus

$$\begin{aligned} E_{j,n}^{(1)} &= \mathrm{o}_{\mathbb{P}}(k^{a_j\gamma+1/2+\varepsilon/2} s_n^{a_j\gamma+1/2+\varepsilon}) = \mathrm{o}_{\mathbb{P}}(1) \\ \text{and} \ \ E_{j,n}^{(2)} &= \mathrm{o}_{\mathbb{P}}(k^{1/2+\varepsilon/2} s_n^{1/2+\varepsilon}) = \mathrm{o}_{\mathbb{P}}(1) \end{aligned}$$

by (S2.4) and the fact that $s_n = k^{-(1-\varepsilon')/(1+2\varepsilon')}$. From this we deduce that for any $j \in \{1, \ldots, d\}$:

$$I_{j,n} = \frac{1}{f_j(n/k)} \int_{s_n}^1 f_j(n/ks)\sqrt{k} \left(\frac{X_{n-\lfloor ks \rfloor, n}^{a_j}}{U_{a_j}(n/k)} - s^{-a_j\gamma} \right) s^{a_j\gamma-\delta_j} dg_j(s) + \mathrm{o}_{\mathbb{P}}(1).$$

Now, by Theorem 2.4.8 in de Haan and Ferreira (2006), we may find a Borel measurable function A_0 which has constant sign and is asymptotically equivalent to A at infinity such that for any $\varepsilon' > 0$, we have

$$\sup_{0<s\leq 1} s^{\gamma+1/2+\varepsilon'} \left| \sqrt{k} \left(\frac{X_{n-\lfloor ks \rfloor, n}}{U(n/k)} - s^{-\gamma} \right) - \gamma s^{-\gamma-1} W_n(s) - \sqrt{k} A_0(n/k) s^{-\gamma} \frac{s^{-\rho}-1}{\rho} \right| \xrightarrow{\mathbb{P}} 0 \tag{S2.5}$$

where W_n is an appropriate sequence of standard Brownian motions. In other words:

$$\frac{X_{n-\lfloor ks \rfloor, n}}{U(n/k)} = s^{-\gamma} \left(1 + \frac{1}{\sqrt{k}} \gamma s^{-1} W_n(s) + A_0(n/k) \frac{s^{-\rho}-1}{\rho} + \frac{1}{\sqrt{k}} s^{-1/2-\varepsilon'} \, \mathrm{o}_{\mathbb{P}}(1) \right)$$

with the $\mathrm{o}_{\mathbb{P}}(1)$ being uniform in $s \in (0,1]$. Now for any n, $W_n \stackrel{d}{=} W$ where W is a standard Brownian motion, and the random process W has continuous sample paths and $s^{-1/2+\varepsilon'} W(s) \to 0$ almost surely as $s \to 0$. Moreover, for $s \in [s_n, 1]$, $s^{-1/2-\varepsilon'} \leq s_n^{-1/2-\varepsilon'} = \sqrt{k^{1-\varepsilon'}} = \mathrm{o}(\sqrt{k})$. Finally, $(s^{-\rho}-1)/\rho$ is bounded by a constant on $[s_n, 1]$ when $\rho < 0$, and is equal to $-\log(s)$ for $\rho = 0$ and thus dominated by $s^{-1/2-\varepsilon'}$ in a neighborhood of 0. A Taylor expansion therefore yields:

$$\begin{aligned}\frac{X^{a_j}_{n-\lfloor ks\rfloor,n}}{U_{a_j}(n/k)} &= s^{-a_j\gamma}\left(1+\frac{1}{\sqrt{k}}\gamma s^{-1}W_n(s)+A_0(n/k)\frac{s^{-\rho}-1}{\rho}+\frac{1}{\sqrt{k}}s^{-1/2-\varepsilon'}\,\mathrm{o}_{\mathbb{P}}(1)\right)^{a_j}\\ &= s^{-a_j\gamma}\left(1+\frac{1}{\sqrt{k}}a_j\gamma s^{-1}W_n(s)+a_jA_0(n/k)\frac{s^{-\rho}-1}{\rho}+\frac{1}{\sqrt{k}}s^{-1/2-\varepsilon'}\,\mathrm{o}_{\mathbb{P}}(1)\right)\end{aligned}$$

where the $\mathrm{o}_{\mathbb{P}}(1)$ is uniform in $s \in [s_n, 1]$. We deduce from this convergence that

$$\begin{aligned} I_{j,n} &= \zeta_{j,n}+\xi_{j,n}+\mathrm{o}_{\mathbb{P}}\left(\int_0^1\frac{f_j(n/ks)}{f_j(n/k)}s^{-1/2-\delta_j-\varepsilon'}dg_j(s)\right)+\mathrm{o}_{\mathbb{P}}(1)\\ \text{with}\quad \zeta_{j,n} &= a_j\gamma\int_0^1\frac{f_j(n/ks)}{f_j(n/k)}s^{-1-\delta_j}W_n(s)dg_j(s)\\ \text{and}\quad \xi_{j,n} &= a_j\sqrt{k}A_0(n/k)\int_0^1\frac{f_j(n/ks)}{f_j(n/k)}\frac{s^{-\rho}-1}{\rho}s^{-\delta_j}dg_j(s).\end{aligned}$$

By Lemma 2, we obtain

$$I_{j,n} = \zeta_{j,n}+\xi_{j,n}+\mathrm{o}_{\mathbb{P}}(1). \tag{S2.6}$$

The bias term $\xi_{j,n}$ is controlled by applying Lemma 2:

$$\xi_{j,n} = a_j\lambda\int_0^1\frac{s^{-\rho}-1}{\rho}s^{-b_j-\delta_j}dg_j(s)+\mathrm{o}(1)\to\lambda C_j. \tag{S2.7}$$

Notice now that

$$(\zeta_{1,n},\ldots,\zeta_{d,n}) \stackrel{d}{=} \left(a_j\gamma\int_0^1\frac{f_j(n/ks)}{f_j(n/k)}s^{-1-\delta_j}W(s)dg_j(s)\right)_{1\le j\le d}$$

where W is a standard Brownian motion. Since W has continuous sample paths and $s^{-1/2+\varepsilon'}W(s) \to 0$ almost surely as $s \to 0$, we get by Lemma 2 that

$$\begin{aligned}(\zeta_{1,n},\ldots,\zeta_{d,n}) &\stackrel{d}{=} \left(a_j\gamma\int_0^1 \frac{f_j(n/ks)}{f_j(n/k)} s^{-1/2-\delta_j-\varepsilon'}(s^{-1/2+\varepsilon'}W(s))dg_j(s)\right)_{1\le j\le d} \\ &\stackrel{d}{\longrightarrow} \left(a_j\gamma\int_0^1 s^{-1-b_j-\delta_j}W(s)dg_j(s)\right)_{1\le j\le d}.\end{aligned}$$

The entries of this random vector are almost surely finite. Let us recall that W is a centered Gaussian process with covariance function $\operatorname{Cov}(W(s),W(t)) = \min(s,t)$; consequently, for all $(u_1,\ldots,u_d)\in\mathbb{R}^d$, the random variable

$$\sum_{j=1}^d u_j a_j \gamma \int_0^1 s^{-1-b_j-\delta_j} W(s) dg_j(s)$$

is Gaussian centered and has variance

$$\gamma^2 \operatorname{Var}\left(\sum_{j=1}^d u_j a_j \int_0^1 s^{-1-b_j-\delta_j} W(s) dg_j(s)\right) = \sum_{i,j=1}^d u_i u_j \Sigma_{i,j} \qquad \text{(S2.8)}$$

by Fubini's theorem. It remains to combine Equations (S2.6), (S2.7) and (S2.8), and to use the Cramér-Wold theorem to complete the proof. ∎

The fifth and final lemma shall be useful to control the bias term in Theorem 3.

Lemma 5. *Assume that Y_i, $i \ge 1$ are independent random variables with common cdf F_Y, such that the left-continuous inverse U_Y of $1/(1-F_Y)$ satisfies condition $\mathcal{C}_2(\gamma_Y,\rho_Y,A_Y)$, with $\rho_Y<0$. Assume further that $\beta_n,\delta_n\to 1$,*

$n(1-\beta_n)\to\infty$, $(1-\delta_n)/(1-\beta_n)\to 0$ and $\sqrt{n(1-\beta_n)}A_Y((1-\beta_n)^{-1})\to \lambda\in\mathbb{R}$. Pick a distortion function g. If for some $\eta>0$,

$$\int_0^1 s^{-\gamma_Y-\eta}dg(s)<\infty,$$

then

$$\frac{R_{g,\delta_n}(Y)}{R_{g,\beta_n}(Y)}\left(\frac{1-\beta_n}{1-\delta_n}\right)^{-\gamma_Y}=1-\frac{\lambda/\rho_Y}{\sqrt{n(1-\beta_n)}}\frac{\int_0^1 s^{-\gamma_Y-\rho_Y}dg(s)}{\int_0^1 s^{-\gamma_Y}dg(s)}+\mathrm{o}\left(\frac{1}{\sqrt{n(1-\beta_n)}}\right).$$

Proof of Lemma 5. Set $k_1=k_1(n)=n(1-\beta_n)$, $r_n=(1-\beta_n)/(1-\delta_n)$, $k_2=k_2(n)=k_1/r_n$. Since for any $b\in(0,1)$,

$$R_{g,b}(Y)=\int_0^1 U_Y([(1-b)s]^{-1})dg(s),$$

we may write

$$R_{g,\delta_n}(Y)=r_n^{\gamma_Y}R_{g,\beta_n}(Y)+u_{1,n}+u_{2,n}\tag{S2.9}$$

where

$$\begin{aligned} u_{1,n}&=r_n^{\gamma_Y}\frac{r_n^{\rho_Y}-1}{\rho_Y}\int_0^1 U_Y(n/k_1s)A_0(n/k_1s)dg(s)\\ \text{and}\quad u_{2,n}&=\int_0^1 U_Y(n/k_1s)A_0(n/k_1s)\left(\frac{1}{A_0(n/k_1s)}\left[\frac{U_Y(n/k_2s)}{U_Y(n/k_1s)}-r_n^{\gamma_Y}\right]-r_n^{\gamma_Y}\frac{r_n^{\rho_Y}-1}{\rho_Y}\right)dg(s)\end{aligned}$$

with the notation of (S2.5). By Lemma 2 and the convergence $\sqrt{k_1}A_0(n/k_1)\to\lambda$,

$$\begin{aligned}\sqrt{k_1}\frac{u_{1,n}}{U_Y(n/k_1)}&=\lambda r_n^{\gamma_Y}\frac{r_n^{\rho_Y}-1}{\rho_Y}\int_0^1 s^{-\gamma_Y-\rho_Y}dg(s)+\mathrm{o}(r_n^{\gamma_Y})\\ &=-\frac{\lambda}{\rho_Y}r_n^{\gamma_Y}\int_0^1 s^{-\gamma_Y-\rho_Y}dg(s)+\mathrm{o}(r_n^{\gamma_Y})\end{aligned}\tag{S2.10}$$

because $r_n \to \infty$ and $\rho_Y < 0$. The sequence $u_{2,n}$ is controlled by using first inequality (S2.1) and Lemma 2: for any $\varepsilon \in (0, -\rho_Y)$, we have if n is large enough,

$$\begin{aligned}
\sqrt{k_1}\frac{|u_{2,n}|}{U_Y(n/k_1)} &\leq \varepsilon r_n^{\gamma_Y+\rho_Y+\varepsilon}|\sqrt{k_1}A_0(n/k_1)| \int_0^1 \frac{U_Y(n/k_1 s)|A_0(n/k_1 s)|}{U_Y(n/k_1)|A_0(n/k_1)|} dg(s) \\
&= \varepsilon|\lambda| r_n^{\gamma_Y+\rho_Y+\varepsilon} \int_0^1 s^{-\gamma_Y-\rho_Y} dg(s) + \mathrm{o}(r_n^{\gamma_Y+\rho_Y+\varepsilon}) \\
&= \mathrm{o}(r_n^{\gamma_Y}). \qquad \text{(S2.11)}
\end{aligned}$$

Combining (S2.10) and (S2.11) entails

$$\frac{\sqrt{k_1}}{U_Y(n/k_1)}(u_{1,n}+u_{2,n}) = -\frac{\lambda}{\rho_Y} r_n^{\gamma_Y} \int_0^1 s^{-\gamma_Y-\rho_Y} dg(s) + \mathrm{o}(r_n^{\gamma_Y}).$$

Use once more Lemma 2 to get

$$\frac{R_{g,\beta_n}(Y)}{U_Y(n/k_1)} = \int_0^1 \frac{U_Y(n/k_1 s)}{U_Y(n/k_1)} dg(s) \to \int_0^1 s^{-\gamma_Y} dg(s),$$

which yields

$$\frac{\sqrt{k_1}}{R_{g,\beta_n}(Y)}(u_{1,n}+u_{2,n}) = -\frac{\lambda}{\rho_Y} r_n^{\gamma_Y} \frac{\int_0^1 s^{-\gamma_Y-\rho_Y} dg(s)}{\int_0^1 s^{-\gamma_Y} dg(s)} + \mathrm{o}(r_n^{\gamma_Y}). \qquad \text{(S2.12)}$$

Combining (S2.9) and (S2.12) completes the proof. ■

S3 Tables and Figures

Risk measure $R_g(X)$	Distortion function g
VaR at level β	$g(x) = \mathbb{I}\{x \geq 1-\beta\}$ where $0 \leq \beta < 1$
TVaR above level β	$g(x) = \min\left\{\frac{x}{1-\beta}, 1\right\}$ where $0 \leq \beta < 1$
Proportional Hazard transform	$g(x) = x^\alpha$ where $0 < \alpha < 1$
Dual Power	$g(x) = 1-(1-x)^{1/\alpha}$ where $0 < \alpha < 1$
MAXMINVAR	$g(x) = (1-(1-x)^\alpha)^{1/\alpha}$ where $0 < \alpha < 1$
MINMAXVAR	$g(x) = 1-(1-x^{1/\alpha})^\alpha$ where $0 < \alpha < 1$
Gini's principle	$g(x) = (1+\alpha)x - \alpha x^2$ where $0 < \alpha \leq 1$
Denneberg's absolute deviation	$g(x) = \begin{cases} (1+\alpha)x & \text{if} \quad 0 \leq x \leq 1/2 \\ \alpha + (1-\alpha)x & \text{if} \quad 1/2 \leq x \leq 1 \end{cases}$ where $0 < \alpha \leq 1$
Exponential transform	$g(x) = \begin{cases} (1-\exp(-rx))/(1-\exp(-r)) & \text{if} \quad r > 0 \\ x & \text{if} \quad r = 0 \end{cases}$
Logarithmic transform	$g(x) = \begin{cases} (\log(1+rx))/(\log(1+r)) & \text{if} \quad r > 0 \\ x & \text{if} \quad r = 0 \end{cases}$
Square-root transform	$g(x) = \begin{cases} (\sqrt{1+rx}-1)/(\sqrt{1+r}-1) & \text{if} \quad r > 0 \\ x & \text{if} \quad r = 0 \end{cases}$
S-inverse shaped transform	$g(x) = a\left(\frac{x^3}{6} - \frac{\delta}{2}x^2 + \left(\frac{\delta^2}{2} + \beta\right)x\right)$ where $a = \left(\frac{1}{6} - \frac{\delta}{2} + \frac{\delta^2}{2} + \beta\right)^{-1}$ with $0 \leq \delta \leq 1$ and $\beta \in \mathbb{R}$
Wang's transform	$g(x) = \Phi(\Phi^{-1}(x) + \Phi^{-1}(\alpha))$ where Φ is the standard Gaussian cdf and $0 \leq \alpha \leq 1$
Beta's transform	$g(x) = \int_0^x \frac{1}{\beta(a,b)} t^{a-1}(1-t)^{b-1} dt$ where $\beta(a,b)$ is the Beta function with parameters a, $b > 0$

Table 1: Some risk measures and their distortion functions.

Risk measure	Expression as a combination of $\mathrm{CTM}_a(\beta)$ and $\mathrm{VaR}(\beta)$
$\mathrm{CTE}(\beta)$	$\mathrm{CTM}_1(\beta)$
$\mathrm{CVaR}_\lambda(\beta)$	$\lambda\mathrm{VaR}(\beta) + (1-\lambda)\mathrm{CTM}_1(\beta)$ where $\lambda \in [0,1]$
$\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}$	$\omega_1\mathrm{CTM}_1(\beta) + \omega_2\mathrm{CTM}_1(\alpha) + \omega_3\mathrm{VaR}(\alpha)$ where $\omega_1 = h_1 - \dfrac{(h_2-h_1)(1-\beta)}{\beta-\alpha}$, $\omega_2 = \dfrac{(h_2-h_1)(1-\alpha)}{\beta-\alpha}$ and $\omega_3 = 1-\omega_1-\omega_2 = 1-h_2$, with $h_1 \in [0,1]$, $h_2 \in [h_1,1]$ and $\alpha < \beta$
$\mathrm{SP}(\beta)$	$(1-\beta)(\mathrm{CTM}_1(\beta) - \mathrm{VaR}(\beta))$
$\mathrm{CTV}(\beta)$	$\mathrm{CTM}_2(\beta) - \mathrm{CTM}_1^2(\beta)$
$\mathrm{TSD}_\lambda(\beta)$	$\mathrm{CTM}_1(\beta) + \lambda\sqrt{\mathrm{CTM}_2(\beta) - \mathrm{CTM}_1^2(\beta)}$ where $\lambda \geq 0$
$\mathrm{CTS}(\beta)$	$\mathrm{CTM}_3(\beta)/(\mathrm{CTM}_2(\beta) - \mathrm{CTM}_1^2(\beta))^{3/2}$

Table 2: Link between the CTM and some risk measures when the cdf of X is continuous.

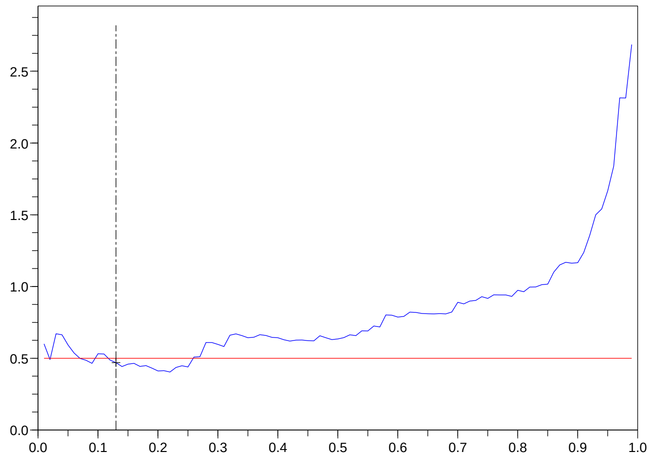

Figure 1: Choosing β on a random sample of $n = 100$ Burr observations with $\gamma = 1/2$ and $\rho = -1$; $x-$axis: $1-\beta$. The choice procedure is conducted with $\beta_0 = 0.5$ and $h = 0.1$. The blue line is the Hill estimator; we obtain $\beta^* = 0.86$ and $\widehat{\gamma} = 0.475$.

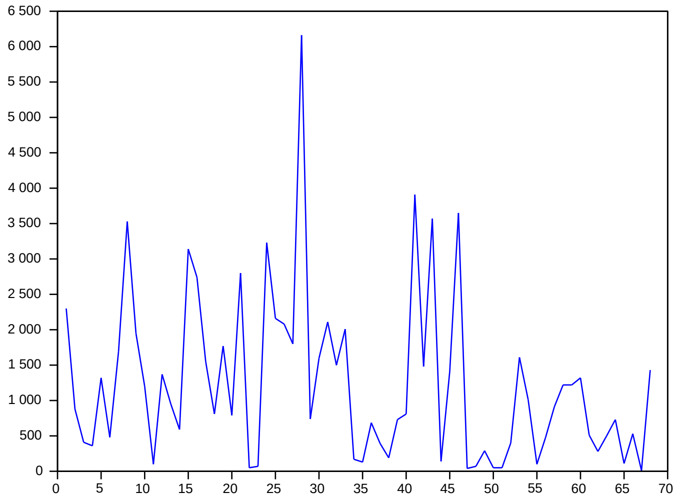

Figure 2: Poker data set: values of the consecutive swings of poker player Tom Dwan (absolute value of the aggregated results during alternative winning and losing streaks). Measurement unit: thousands of USD.

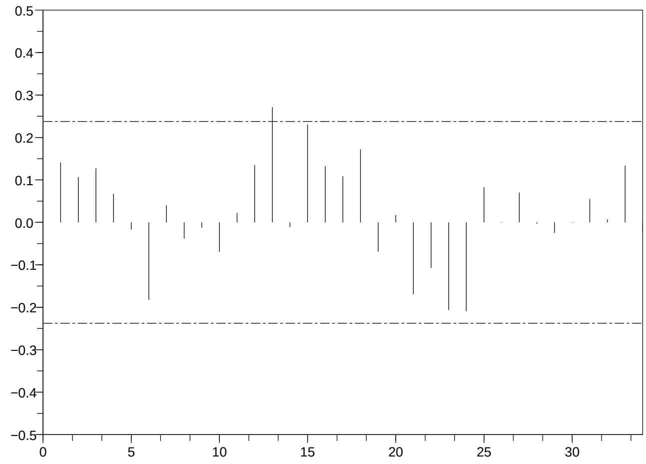

Figure 3: Poker data set: sample autocorrelation function until lag 34. Dashed line: 95% significance level.

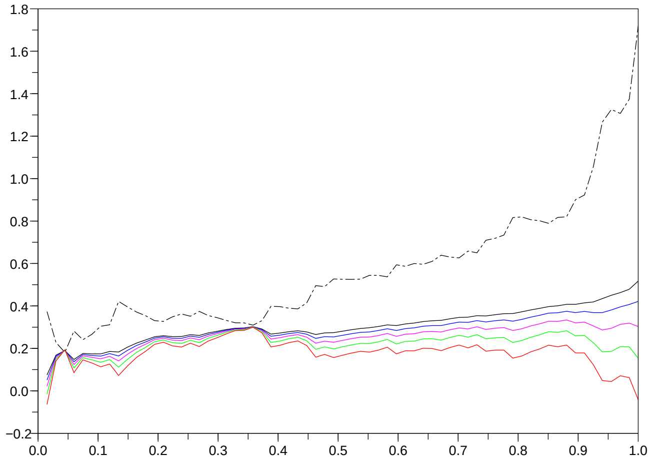

Figure 4: Poker data set, detrended data: Hill estimators; $x-$axis: $1-\beta$. Dashed line: standard Hill estimator, black line: estimator $\widehat{\gamma}_{\beta}^{RB}(1)$, blue line: estimator $\widehat{\gamma}_{\beta}^{RB}(3/4)$, purple line: estimator $\widehat{\gamma}_{\beta}^{RB}(1/2)$, green line: estimator $\widehat{\gamma}_{\beta}^{RB}(1/4)$, red line: estimator $\widehat{\gamma}_{\beta}^{RB}(0)$.

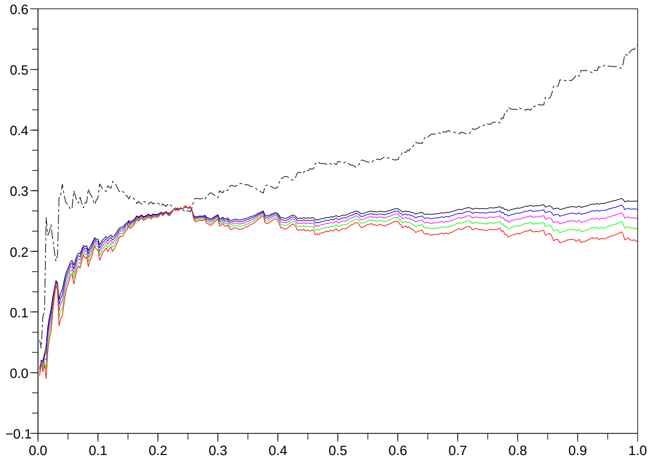

Figure 5: Secura Belgian Re data set: Hill estimators; x−axis: $1 - \beta$. Dashed line: standard Hill estimator, black line: estimator $\widehat{\gamma}_{\beta}^{RB}(1)$, blue line: estimator $\widehat{\gamma}_{\beta}^{RB}(3/4)$, purple line: estimator $\widehat{\gamma}_{\beta}^{RB}(1/2)$, green line: estimator $\widehat{\gamma}_{\beta}^{RB}(1/4)$, red line: estimator $\widehat{\gamma}_{\beta}^{RB}(0)$.

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