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SEMIPARAMETRIC REGRESSION WITH TIME-DEPENDENT COEFFICIENTS FOR FAILURE TIME DATA ANALYSIS

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Supplementary Material

This note contains the proof of Theorem 1, Theorem 3, and some asymptotic expansions.

S1. Proof of Theorem 1

Let $H = diag(I_p, hI_p)$, where I_p is a $p \times p$ identity matrix. Let $U_{ij}(u,t) = H^{-1}\widetilde{X}_{ij}(u,t)$, and reparametrize $\alpha = H(b - b^0)$, where b^0 is the true value of the corresponding parameters, i.e, $\beta(t)$ and $\beta'(t)$. We also introduce the following notation:

$$
\begin{aligned}
S_{nr}(\alpha, u) &= n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{J} Y_{ij}(u)U_{ij}(u)^{\otimes r} e^{\widetilde{X}_{ij}(u,t)^T b^0 + U_{ij}(u)^T\alpha},\\
S_r(\alpha, u) &= E\{\sum_{j=1}^{J} P_j(u|X_{ij}(u))U_{ij}(u)^{\otimes r} e^{\widetilde{X}_{ij}(u,t)^T b^0 + U_{ij}(u)^T\alpha}\},
\end{aligned}
$$

where $r = 0, 1, 2$. Application of Lemma 1 of Cai and Sun (2003) and by conditions A, we can show $S^*_{nr}(u) \to_p S^*_r(u)$ and $S_{nr}(\alpha, u) \to_p S_r(\alpha, u)$ uniformly in a small neighborhood of t. One can first show the consistency using the Lenglart Inequality and Lemma A.1 of Spiekerman and Lin (1998), then prove the asymptotic normality using quadratic approximation formula on page 210 of Fan and Gijbels (1996). The Cramer-Wold device (Durrett 1995, p.170) and Linderberg-Feller Central Limit Theorem (Durrett, 1995, p.116) is used for the asymptotic normality. We omit the details to save space.

S2. Proof of Theorem 3

S2.1 Notation and Regularity Conditions B

We make the following notation: $S_{pn0}(u,\beta,\gamma) = n^{-1}\sum_{ij} Y_{ij}(u)e^{X_{ij}\beta(u,\gamma)+Z_{ij}^T\gamma}$
$S_{pn1}(u,\beta,\gamma) = n^{-1}\sum_{ij} Y_{ij}(u)\{X_{ij}\beta_\gamma(u,\gamma)+Z\}e^{X_{ij}\beta(u,\gamma)+Z_{ij}^T\gamma}$
$S_{pn2}(u,\beta,\gamma) = n^{-1}\sum_{ij} Y_{ij}(u)\{X_{ij}\beta_{\gamma\gamma}(u,\gamma)+(X_{ij}\beta(u,\gamma)+Z_{ij})^{\otimes 2}\}e^{X_{ij}\beta(u,\gamma)+Z_{ij}^T\gamma}$.
Conditions B *(1) The kernel function $K(s)$ is a bounded symmetric function with a bounded support, and $h \to 0$, $nh \to \infty$, as $n \to \infty$, and $nh^5 = O(1)$; (2) $P(Y_{ij}(t) = 1, for \quad all \quad t \in [0,\tau]) > 0$ for each i,j; (3) Covariates X_j, Z_j are bounded, time independent for $j = 1,\cdots,J$; (4) $\Sigma_s(t,\gamma)$ is positive definite for all $t \in [0,\tau]$ in a small neighborhood of γ_0; (5) $\beta_\gamma(t,\gamma)$, $\beta(t,\gamma)$ are bounded and have second continuous derivative as a function of t for γ in a small neighborhood of γ_0. They are also of bounded variation in $[0,\gamma]$; (6) $S_{pnr}(t,\beta,\gamma)$ converges to its asymptotic limit uniformly over $t \in [0,\gamma]$ and a small neighborhood of γ_0 for $r = 0,1,2$; (7) $f_{T_{ij},\Delta_{ij},X_{ij},Z_{ij}}(t,\delta,x,z)$ has the same marginal density for all $j = 1,\cdots,J$.*

The uniform convergence of S_{pnr} can be satisfied by bounded conditions on covariates, $\beta(u,\gamma)$, $\beta_\gamma(u,\gamma)$, and Theorem III.1 of Andersen and Gill (1982).

S2.2 The Nonparametric component

Denote by $\beta_0(t)$ and γ_0 as the true values of $\beta(t)$ and γ. We first show that the asymptotic distribution of $\widehat{\beta}(t,\widehat{\gamma})$ is the same as that of $\widehat{\beta}(t,\gamma)$, where $\beta(t,\gamma)$ is the solution of the asymptotic limit of $U_1(b,t)$, i.e., $E\{U_1(b,t;\gamma)\} = 0$, in (2.5) . One can easily see that if $\beta(t,\gamma_0) = \beta_0(t)$. By Taylor's expansion,

$$
\begin{aligned}
&\sqrt{nh}\{\widehat{\beta}(t,\widehat{\gamma}) - \beta(t,\gamma_0)\} \\
&\quad = \sqrt{nh}\{\widehat{\beta}(t,\widehat{\gamma}) - \widehat{\beta}(t,\gamma_0)\} + \sqrt{nh}\{\widehat{\beta}(t,\gamma_0) - \beta(t,\gamma_0)\} \\
&\quad = \sqrt{h}\widehat{\beta}_\gamma(t,\gamma_0)^T\{\sqrt{n}(\widehat{\gamma}-\gamma_0)\} + \sqrt{nh}\{\widehat{\beta}(t,\gamma_0) - \beta(t,\gamma_0)\},
\end{aligned}
$$

where $\widehat{\beta}(t,\widehat{\gamma})$ is the nonparametric estimator $\beta(t)$ given $\gamma = \hat{\gamma}$. The first term in the last equation is $o_p(1)$ provided that $\widehat{\gamma}$ is $\sqrt{n}$-consistent. The leading term is $\sqrt{nh}\{\widehat{\beta}(t,\gamma_0) - \beta(t,\gamma_0)\}$. Hence the asymptotic distributions of $\widehat{\beta}(t,\widehat{\gamma})$ and $\widehat{\beta}(t,\gamma)$ are the same. The results follow from the proof in Appendix 1.

S2.3 The Parametric component

We focus our proof on the correlated data. The results for independent data are a special case. We present three steps to prove asymptotic normality, similar to Andersen and Gill (1982).

Consistency

Since $\widehat{\beta}(u,\gamma) - \beta(u,\gamma) = o_p(1)$ for each $u \in [0,\tau]$ and by assuming it is a totally bounded functions set over $[0,\tau]$, the uniform convergence follows (see Lemma 11.16 and Corollary 11.19 of Carothers 2000). One can then replace $\widehat{\beta}(u,\gamma)$ by $\beta(u,\gamma)$ and work on $p\ell_2(\gamma,\beta(u,\gamma))$ for the proof of consistency of γ.

$$
\begin{aligned}
p\ell_2(\gamma,\beta(u,\gamma)) \quad - \quad & p\ell_2(\gamma_0,\beta(u,\gamma_0)) = n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{J}\int_0^\tau \Big[X_{ij}\{\beta(u,\gamma)-\beta(u,\gamma_0)\} \\
& +Z_{ij}^T(\gamma-\gamma_0) - log\frac{S_{pn0}(u,\beta,\gamma)}{S_{pn0}(u,\beta,\gamma_0)}\Big]dN_{ij}(u) = A_{pn}(\tau) + X_{pn}(\tau),
\end{aligned}
$$

where $A_{pn}(\tau) = n^{-1}\sum_{i=1}^n\sum_{j=1}^J\Big[X_{ij}\{\beta(u,\gamma)-\beta(u,\gamma_0)\}+Z_{ij}^T(\gamma-\gamma_0)-log\frac{S_{pn0}(u,\beta,\gamma)}{S_{pn0}(u,\beta,\gamma_0)}\Big]$ $\times\, Y_{ij}(u)e^{X_{ij}\beta(u)+Z_{ij}\gamma_0}\lambda_0(u)du$ and $X_{pn}(\tau) = n^{-1}\sum_{i=1}^n\sum_{j=1}^J\int_0^\tau\Big[X_{ij}\{\beta(u,\gamma) - \beta(u,\gamma_0)\} + Z_{ij}^T(\gamma-\gamma_0) - log\frac{S_{pn0}(u,\beta,\gamma)}{S_{pn0}(u,\beta,\gamma_0)}\Big]dM_{ij}(u)$.

It is easy to show $X_{pn}(\tau) = o_p(1)$ by Lemma A.1 of Spiekerman and Lin (1998) and noting X_{ij}, Z_{ij} are time independent. Now we consider $A_{pn}(\tau)$. By Conditions B,

$$
\begin{aligned}
A_{pn}(\tau) \quad = \quad & \sum_{j=1}^{J}\int_0^\tau\Big[\{\beta(u,\gamma)-\beta(u,\gamma_0)\}E\{Y_j(u)X_je^{X_j\beta(u)+Z_j^T\gamma_0}\} \\
& +(\gamma-\gamma_0)E\{Y_j(u)Z_je^{X_j\beta(u)+Z_j^T\gamma_0}\} \\
& -log\frac{S_{p0}(u,\beta,\gamma)}{S_{p0}(u,\beta,\gamma_0)}E\{Y_j(u)e^{X_j\beta(u)+Z_j^T\gamma_0}\}\Big]\lambda_0(u)du + o_p(1),
\end{aligned}
$$

where $S_{p0}(u,\beta,\gamma) = \sum_{j=1}^J E\{Y_j(u)e^{X_j\beta(u,\gamma)+Z_j^T\gamma}\}$. The uniform convergence of $S_{np0}(u,\beta,\gamma) \to_p S_{p0}(u,\beta,\gamma)$ follows from the bounded variation condition of $\beta(u,\gamma), \beta_\gamma(u,\gamma)$ and Theorem III.1 of Andersen and Gill(1982). Taking a derivative of A_{pn} with respect to γ, we have $\partial A_{pn}(\tau)/\partial\gamma|_{\gamma=\gamma_0} = 0$ by noting $\beta(u,\gamma_0) = \beta(u)$. One can also verify that $\partial^2 A_{pn}(\tau)/\partial\gamma\gamma^T$ is negative definite at $\gamma = \gamma_0$. So γ_0 is the maximizer of A_{pn} asymptotically. Then by the concave lemma in Andersen and Gill (1982), we have the maximizer $\widehat{\gamma}$ of $pl_2(\cdot)$ converges to γ_0 in probability.

Asymptotic Normality of $\widehat{\gamma}$

We need to show that the profile estimating equation $\sqrt{n}U_2$ evaluated at true value γ_0 converges to a normal random vector in distribution. By a Taylor expansion of $\widehat{\beta}(T_{ij},\gamma_0)$ and $\widehat{\beta}_\gamma(T_{ij},\gamma_0)$ around $\beta(T_{ij},\gamma_0)$ and $\beta_\gamma(T_{ij},\gamma_0)$ respectively, we have

$$\sqrt{n}U_2\{\gamma_0, \widehat{\beta}(T_{ij},\gamma_0), \widehat{\beta}_\gamma(T_{ij},\gamma_0)\} \approx \frac{1}{\sqrt{n}}\sum_{ij}\Delta_{ij}[Z_{ij}+X_{ij}\beta_\gamma(T_{ij},\gamma_0)$$
$$-\frac{\sum_{rl}Y_{rl}(T_{ij})\{X_{rl}\beta_\gamma(T_{ij},\gamma_0)+Z_{rl}\}e^{X_{rl}\beta(T_{ij},\gamma_0)+Z_{rl}^T\gamma_0}}{\sum_{rl}Y_{rl}(T_{ij})e^{X_{rs}\beta(T_{ij},\gamma_0)+Z_{rs}^T\gamma_0}}]$$
$$+\frac{1}{\sqrt{n}}\sum_{ij}\Delta_{ij}\{\widehat{\beta}(T_{ij},\gamma_0)-\beta(T_{ij},\gamma_0)\}Q(T_{ij})$$
$$+\frac{1}{\sqrt{n}}\sum_{ij}\Delta_{ij}\{\widehat{\beta}_\gamma(T_{ij},\gamma_0)-\beta_\gamma(T_{ij},\gamma_0)\}\Big\{X_{ij}-R(T_{ij})\Big\} = A+B+C$$

where

$$\begin{aligned} Q(T_{ij}) &= -\frac{\sum_{rl}Y_{rl}(T_{ij})\{X_{rl}\beta_\gamma(T_{ij},\gamma_0)+Z_{rl}\}X_{rl}e^{X_{rl}\beta(T_{ij},\gamma_0)+Z_{rl}^T\gamma_0}}{\sum_{rl}Y_{rl}(T_{ij})e^{X_{rl}\beta(T_{ij},\gamma_0)+Z_{rl}^T\gamma_0}} \\ &\quad+\frac{\sum_{rl}Y_{rl}(T_{ij})X_{rl}e^{X_{rl}\beta(T_{ij},\gamma_0)+Z_{rl}^T\gamma_0}}{\{\sum_{rl}Y_{rl}(T_{ij})e^{X_{rl}\beta(T_{ij},\gamma_0)+Z_{rl}^T\gamma_0}\}^2} \\ &\quad\times\sum_{rl}Y_{rl}(T_{ij})\{X_{rl}\beta_\gamma(T_{ij},\gamma_0)+Z_{rl}\}e^{X_{rl}\beta(T_{ij},\gamma_0)+Z_{rl}^T\gamma_0} \\ R(T_{ij}) &= \frac{\sum_{rl}Y_{rl}(T_{ij})X_{rl}e^{X_{rl}\beta(T_{ij},\gamma_0)+Z_{rl}^T\gamma_0}}{\sum_{rl}Y_{rl}(T_{ij})e^{X_{rl}\beta(T_{ij},\gamma_0)+Z_{rl}^T\gamma_0}}, \end{aligned}$$

where $\sum_{ij}=\sum_{i=1}^n\sum_{j=1}^j$ and $\sum_{rl}=\sum_{r=1}^n\sum_{l=1}^l$. We have

$$A=\frac{1}{\sqrt{n}}\sum_{ij}\int_0^\tau Z_{ij}+X_{ij}\beta_\gamma(u,\gamma_0)-\frac{\sum_{rl}Y_{rl}(u)\{X_{rl}\beta_\gamma(u,\gamma_0)+Z_{rl}\}e^{X_{rl}\beta(u,\gamma_0)+Z_{rl}^T\gamma_0}}{\sum_{rl}Y_{rl}(u)e^{X_{rl}\beta(u,\gamma_0)+Z_{rl}^T\gamma_0}}dM_{ij}(u),$$

where we have used the fact that $\beta(u,\gamma_0)=\beta_0(u)$. Plugging the expansion $\widehat{\beta}(t,\gamma_0)-\beta(t,\gamma_0)$ with $t=T_{ij}$ from Appendix 3 into expression B, and exchange the summation, we have

$$\begin{aligned} B &= \frac{1}{\sqrt{n}}\sum_{rs}\Delta_{rs}\{X_{rs}-R(T_{rs})\}\frac{1}{n}\sum_{ij}\sigma_s(T_{ij},\gamma_0)^{-1}\Delta_{ij}Q(T_{ij})K_h(T_{ij}-T_{rs}) \\ &\quad+\frac{h^2}{2\sqrt{n}}\sum_{ij}\Delta_{ij}Q(T_{ij})\beta^{(2)}(T_{ij})d(T_{ij},\gamma_0), \end{aligned}$$

where $d(t,\gamma_0)=\Sigma_s(t,\gamma_0)^{-1}\sum_{j=1}^J E[\Delta_j|T_j=t]g_1\{\beta(t)\}f_T(t)$ and g_1 is a scalar function defined in Appendix 3. Since $Q(T_{ij})=o_p(1)$ as proved in Appendix 3 i.e., $Q(t)f_T(t)=o_p(1)$. Hence, the second term is $o_p(1)$. Similarly, the inner sum of the first term is $o_p(1)$. Hence, $B = o_p(1)$. Substituting $\widehat{\beta}_\gamma(t,\gamma_0) - \beta_\gamma(t,\gamma_0)$ from Appendix 3.

into expression C and exchanging the summation, we have

$$\begin{aligned} C &= \frac{h^2}{2\sqrt{n}} \sum_{i=1}^{n}\sum_{j=1}^{J} \Delta_{ij}\beta^{(2)}(T_{ij}) b_1(T_{ij},\gamma_0)\Big\{X_{ij} - R(T_{ij})\Big\} \\ &+ \frac{1}{\sqrt{n}}\sum_{rl} \Delta_{rl}\{X_{rl} - R(T_{rl})\}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{J} \Sigma_s^2(T_{ij},\gamma_0)\Sigma_{s\gamma}(T_{ij},\gamma_0)\Delta_{ij} \\ &\times\Big\{X_{ij} - R(T_{ij})\Big\}K_h(T_{rl} - T_{ij}), \end{aligned}$$

where $b_1(T_{ij},\gamma_0)$ is a vector defined in Appendix 3, $\Sigma_{s\gamma}(T_{ij},\gamma_0)$ is the derivative of $\Sigma_s(T_{ij},\gamma_0)$ with respect to γ. The first term is easy to handle. Noting $R(T_{ij}) = E[X_{ij}|T = T_{ij}, \Delta_{ij} = 1] + o_p(1)$, then by adding and subtracting $E[X_{ij}|T = T_{ij}, \Delta_{ij} = 1]$, one can see the first term is $o_p(1)$. Similarly, the inner part of the second term is $o_p(1)$ by calculation and noting the asymptotic form of $R(T_{ij})$. Hence the second term is $o_p(1)$ and $C = o_p(1)$. Therefore,

$$\begin{aligned} &\sqrt{n}U(\gamma_0, \widehat{\beta}(T_{ij},\gamma_0), \widehat{\beta}_\gamma(T_{ij},\gamma_0)) = \frac{1}{\sqrt{n}}\sum_{ij}\int_0^\tau Z_{ij} + X_{ij}\beta_\gamma(u,\gamma_0) \\ &-\sum_{j=1}^{J} E[\{X_{i1}\beta_\gamma(u,\gamma_0) + Z_{i1}\}|T_{i1} = u, \Delta_{i1} = 1]dM_{ij}(u) + o_p(1) \qquad \text{(S1)} \end{aligned}$$

The derivation of this equation also uses Lemma A. 1. of Lin and Spiekerman (1998) and Lemma 1 in Appendix S3.3. The asymptotic normality of $\widehat{\gamma}$ follows easily from here.

S3. Asymptotic expansions of $\widehat{\beta}(t,\gamma)$ and $\widehat{\beta}_\gamma(t,\gamma)$

The asymptotic derivation of the profile estimator $\widehat{\gamma}$ requires the asymptotic properties of $\widehat{\beta}(t,\gamma)$ and $\widehat{\beta}_\gamma(t,\gamma)$. We study them here.

S3.1 Asymptotic Expansion of $\hat{\beta}(t,\gamma_0) - \beta(t)$

Using the kernel estimating equations (2.5) , some calculations give

$$\begin{aligned} &\widehat{\beta}(t,\gamma) - \beta(t) = \Sigma_s(t,\gamma)^{-1}n^{-1}\sum_{ij}\Delta_{ij}K_h(T_{ij} - t)\Big\{X_{ij} - R(T_{ij})\Big\} \\ &+\frac{h^2}{2}\beta^{(2)}(t)\sum_{j=1}^{J} E\{\Delta_{ij}|T_{ij} = t\}\Sigma_s(t,\gamma)^{-1}g_1(\beta(t))f_T(t) + o_p(h^2) + o_p(1) \end{aligned}$$

where

$$R(T_{ij}) = \frac{\sum_{lr} Y_{lr}(T_{ij}) X_{lr} e^{X_{lr}\beta(T_{ij})+Z_{lr}\gamma}}{\sum_{lr} Y_{lr}(T_{ij}) e^{X_{lr}\beta(T_{ij})+Z_{lr}\gamma}},$$

g_1 is the derivative of $R(\cdot)$ as a function of $\beta(T_{ij})$, $f_T(t)$ is the marginal distribution of observed time.

S3.2 Asymptotic expansion of $\widehat{\beta}_\gamma(t,\gamma)$

Differentiating (2.5) with respect to γ gives the estimating equation of $\widehat{\beta}_\gamma(t,\gamma)$. Denote its asymptotic limit by $\beta_\gamma(t,\gamma)$. Taking a linear expansion about $\beta_\gamma(t,\gamma)$, we have

$$\begin{aligned}\widehat{\beta}_\gamma(t,\gamma) &= \Sigma_s(\gamma,t)n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{J}\int_0^\tau K_h(u-t)\frac{\partial}{\partial\gamma}\{-\frac{\widetilde{S}_{n1}(0,\gamma,u)}{S_{n0}(0,\gamma,u)}\}dN_{ij}(u)\\ &- \Sigma_s(t,\gamma)^{-2}\Sigma_{s\gamma}(t,\gamma)n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{J}\int_0^\tau K_h(u-t)\{X_{ij}-\frac{\widetilde{S}_{n1}(0,\gamma,u)}{S_{n0}(0,\gamma,u)}\}dN_{ij}(u),\end{aligned}$$

where $\Sigma_{s\gamma}(t,\gamma)$ is the derivative of $\Sigma_s(t,\gamma)$ with respect to γ. The first term is $O_p(1)$, and the second term is $o_p(1)$ at $\gamma=\gamma_0$. Some calculations show that the asymptotic limit $\beta_\gamma(t,\gamma_0)$ satisfies

$$\beta_\gamma(t,\gamma_0) = -\Sigma_s(t,\gamma_0)^{-1}\{Q_1(t,\gamma_0,z) - Q_1(t,\gamma_0)Q_0(t,\gamma_0,z)/Q_0(t,\gamma_0)\},$$

where $Q_r(t,\gamma,z) = \sum_{j=1}^J Y_j(t)X_j^r Z_j exp\{X_j\beta(t)+Z_j^T\gamma\}$ for $j=0,1$ and $Q_0(t,\gamma) = \sum_{j=1}^J Y_j(t)exp\{X_j\beta(t)+Z_j^T\gamma\}$. By condition B.7 and lemma 1 in S3.3, we have

$$\beta_\gamma(t,\gamma_0) = -\frac{E\{X_1Z_{i1}|T_1=t,\Delta_1=1\} - E\{X_1|T_1=t,\Delta_1=1\}E\{Z|T_1=t,\Delta_1=1\}}{E\{X^2|T_1=1,\Delta_1=1\} - E\{X_1|T_1=t,\Delta_1=1\}^2}.$$

Therefore, denoting $d_1(t,\gamma_0) = \Sigma_s(t,\gamma_0)^{-2}\Sigma_{s\gamma}(t,\gamma_0)\sum_j E\{\Delta_j|T_j=t\}g_1\{\beta(t)\}f_T(t)$, we have

$$\begin{aligned}\widehat{\beta}_\gamma(t,\gamma_0) - \beta_\gamma(t,\gamma_0) &= -\Sigma_s(t,\gamma_0)^{-2}n^{-1}\Sigma_{s\gamma}(t,\gamma_0)\sum_{ij}\Delta_{ij}K_h(T_{ij}-t)\Big\{X_{ij}-R(T_{ij})\Big\}\\ &\quad -h^2/2\beta^{(2)}(t)d_1(t,\gamma_0) + o_p(1).\end{aligned}$$

S3.3 Proof of $Q(t) = o_p(1)$

Some calculations show that $Q(t)$ can be written as

$$\begin{aligned} Q(t) &= -\sum_{j}^{J} E\{X_j^2 \beta_\gamma(t,\gamma_0) + Z_j X_j | T_j = t, \Delta_j = 1\} \\ &\quad + \sum_{j}^{J} E\{X_j | T_j = t, \Delta_j = 1\} \sum_{j}^{J} E\{X_j \beta_\gamma(t,\gamma_0) + Z_j | T_j = t, \Delta_j = 1\} \end{aligned}$$

One can then easily see $Q(t) = o_p(1)$ from Lemma 1 and by condition B.7.

Lemma 1 Let

$$\begin{aligned} S_V(t) &= E\{\sum_{j=1}^{J} Y_j(t) V(t, X_j, Z_j) e^{X_j \beta(t,\gamma_0) + Z_j^T \gamma_0}\} \\ S_0(t) &= E\{\sum_{j=1}^{J} Y_j(t) e^{X_j \beta(t,\gamma_0) + Z_j^T \gamma_0}\}. \end{aligned}$$

Using Conditions B, we have $\frac{S_V(t)}{S_0(t)} = \frac{1}{J} \sum_{j=1}^{J} E\{V(t, X_j, Z_j) | T_j = t, \Delta_j = 1\}$. This can be easily verified using an argument similar to Lemma 2 of Sasieni(1992a). Note that we here omit subscript i.