A COSET PATTERN IDENTITY BETWEEN A 2^{n-p} DESIGN AND ITS COMPLEMENT

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Supplementary Material

S1 Details for the Proof of Theorem 1

It can be verified that $Q_{n,k}(s,t) = 0$ for $\max(s,t) > n$. (Let $i = (t-s+k)/2$ and $j = (t+s-k)/2$. Then $Q_{n,k}(s,t) = 0$ when $i > k$ or $j > n-k$, which yields $s > n$ or $t > n$.)

(i) When $i = 0$, then $\tau(i) = 0$. Let $S_1 = D$, $S_2 = \{\bar{a}\}$, and $S_3 = \bar{D}\backslash\{\bar{a}\}$, where $\bar{a}$ is an arbitrary column in $\bar{D}$. Then $l_1 = n$ and $l_2 = 1$. From Lemma 1 we have

$$N_{j,0,0} = \frac{1}{m}\binom{n}{j} - \frac{1}{m}\sum_{j_1+j_2=j}(-1)^{j_2}\binom{n-m/2}{j_1}\binom{m/2}{j_2} + \sum_{t_1+t_2=j}\sum_{s_2+s_3=t_1}\sum_{u}(-1)^{[t_2/2]+t_2}\binom{n-m/2}{[t_2/2]}(-1)^{u+s_3}N_{0,u,s_3}Q_{1,u}(s_2,0).$$

Because $Q_{n,k}(s,0) = 1$ when $k = s$ and $= 0$ otherwise, $N_{j,0,0}$ can be written as

$$N_{j,0,0} = \frac{1}{m}\binom{n}{j} - \frac{1}{m}\sum_{j_1+j_2=j}(-1)^{j_2}\binom{n-m/2}{j_1}\binom{m/2}{j_2} + \sum_{t_1+t_2=j}\sum_{s_2+s_3=t_1}(-1)^{[t_2/2]+t_2}\binom{n-m/2}{[t_2/2]}(-1)^{t_1}N_{0,s_2,s_3}.$$

Therefore, Theorem 1 holds for $i = 0$ when we notice that $A_{0,j} = N_{j,0,0}$ and $\bar{A}_{0,t_1} = \sum_{s_2+s_3=t_1} N_{0,s_2,s_3} = N_{0,0,t_1} + N_{0,1,t_1-1}$.

(ii) When $1 \le i \le n$, then $2^{n-p} - n \le \tau(i) \le 2^{n-p} - 1$. Suppose that the coset leader of the corresponding coset for D is $a \in D$. Let $S_1 = D\backslash\{a\}$, $S_2 = \{a\}$, and $S_3 = \bar{D}$.

Then $l_1 = n-1$ and $l_2 = 1$. From Lemma 1 we have

$$N_{j,1,0} = \frac{1}{m}\binom{n-1}{j} - \frac{1}{m}\sum_{j_1+j_2=j}(-1)^{j_2}\binom{n-1-m/2}{j_1}\binom{m/2}{j_2} + \sum_{t_1+t_2=j}\sum_{s_2+s_3=t_1}\sum_{u}(-1)^{[t_2/2]+t_2}\binom{n-1-m/2}{[t_2/2]}(-1)^{u+s_3}N_{0,u,s_3}Q_{1,u}(s_2,1).$$

Because $N_{0,u,s_3}Q_{1,u}(s_2,1) = -N_{0,u,s_3}$ when $(u,s_2) = (0,1)$ or $(1,\ 0)$ and $= 0$ otherwise, we can write $N_{j,1,0}$ as

$$N_{j,1,0} = \frac{1}{m}\binom{n-1}{j} - \frac{1}{m}\sum_{j_1+j_2=j}(-1)^{j_2}\binom{n-1-m/2}{j_1}\binom{m/2}{j_2} + \sum_{t_1+t_2=j}(-1)^{[t_2/2]+t_2+t_1}\binom{n-1-m/2}{[t_2/2]}\{N_{0,0,t_1-1}+N_{0,1,t_1}\},$$

where the last two terms corresponding to $\{u=0, s_2=1, s_3=t_1-1\}$ and $\{u=1, s_2=0, s_3=t_1\}$, respectively. Following similar argument as in (i) and noticing that $l_1 = n-1$, we have

$$\begin{aligned} N_{j-1,0,0} &= \frac{1}{m}\binom{n-1}{j-1} - \frac{1}{m}\sum_{j_1+j_2=j-1}(-1)^{j_2}\binom{n-1-m/2}{j_1}\binom{m/2}{j_2} \\ &\quad + \sum_{t_1+t_2=j-1}(-1)^{[t_2/2]+t_2+t_1}\binom{n-1-m/2}{[t_2/2]}\{N_{0,0,t_1}+N_{0,1,t_1-1}\} \\ &= \frac{1}{m}\binom{n-1}{j-1} - \frac{1}{m}\sum_{j_1+j_2=j}(-1)^{j_2}\binom{n-1-m/2}{j_1-1}\binom{m/2}{j_2} \\ &\quad - \sum_{t_1+t_2=j}(-1)^{[t_2/2]+t_2+t_1}\binom{n-1-m/2}{[t_2/2]}N_{0,0,t_1-1} \\ &\quad + \sum_{t_1+t_2=j}(-1)^{[t_2/2]+t_2+t_1}\binom{n-1-m/2}{[t_2/2]-1}N_{0,1,t_1}. \end{aligned}$$

Therefore, Theorem 1 holds when we notice that $A_{i,j} = N_{j-1,0,0}+N_{j,1,0}$, $\bar{A}_{\tau(i),j} = N_{0,1,j}$, and $\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$.

(iii) When $n+1 \le i \le 2^{n-p}-1$, then $1 \le \tau(i) \le 2^{n-p}-1-n$. Suppose that the coset leader of the corresponding coset for $\bar{D}$ is $\bar{a}$. Let $S_1 = D$, $S_2 = \{\bar{a}\}$, and $S_3 = \bar{D}\backslash\{\bar{a}\}$. Then $l_1 = n$ and $l_2 = 1$. From Lemma 1 we have

$$N_{j,1,0} = \frac{1}{m}\binom{n}{j} - \frac{1}{m}\sum_{j_1+j_2=j}(-1)^{j_2}\binom{n-m/2}{j_1}\binom{m/2}{j_2} + \sum_{t_1+t_2=j}\sum_{s_2+s_3=t_1}\sum_{u}(-1)^{[t_2/2]+t_2}\binom{n-m/2}{[t_2/2]}(-1)^{u+s_3}N_{0,u,s_3}Q_{1,u}(s_2,1).$$

Because $N_{0,u,s_3}Q_{1,u}(s_2,1) = -N_{0,u,s_3}$ when $(u,s_2) = (0,1)$ or $(u,s_2) = (1,0)$ and $= 0$ otherwise, $N_{j,1,0}$ can be written as

$$N_{j,1,0} = \frac{1}{m}\binom{n}{j} - \frac{1}{m}\sum_{j_1+j_2=j}(-1)^{j_2}\binom{n-m/2}{j_1}\binom{m/2}{j_2} + \sum_{t_1+t_2=j}(-1)^{[t_2/2]+t_2+t_1}\binom{n-m/2}{[t_2/2]}\{N_{0,0,t_1-1}+N_{0,1,t_1}\},$$

where the last two terms corresponding to $\{u=0, s_2=1, s_3=t_1-1\}$ and $\{u=1, s_2=0, s_3=t_1\}$, respectively. Then Theorem 1 hold when we notice that $A_{i,j} = N_{j,1,0}$ and $\bar{A}_{\tau(i),j} = N_{0,0,j-1} + N_{0,1,j}$.

S2 Details for the First Pair of Designs

Note that $H_4(2)$ consists of 15 columns, which are denoted by

$$\{a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\}.$$

Choose 8 columns from $H_4(2)$ to form design D,

$$1 = a, 2 = b, 3 = c, 4 = d, 5 = ab, 6 = ac, 7 = ad, 8 = bcd.$$

Then D is a 2^{8-4} design with defining relations $I = 125 = 136 = 147 = 2348$. The defining contrast subgroup consists of

$$\begin{aligned}&125, 136, 147,\\ &2348, 2356, 2457, 2678, 3467, 3578, 4568,\\ &13458, 12468, 12378, 15678,\\ &1234567\end{aligned}$$

and the wordlength pattern is $(0, 0, 3, 7, 4, 0, 1, 0)$. The following table lists all the cosets.

rank	coset	factorial effects			$\tau^*(i_1\cdots i_l G)$
0	G			125, 136, 147	G
1	$1G$	1,	25, 36, 47		$\bar{1}\bar{4}\bar{G}$
2	$2G$	2,	15,	348, 356, 457, 678	$\bar{6}\bar{7}\bar{G}$
3	$3G$	3,	16,	248, 256, 467, 578	$\bar{5}\bar{7}\bar{G}$
4	$4G$	4,	17,	238, 257, 367, 568	$\bar{4}\bar{7}\bar{G}$
5	$5G$	5,	12,	236, 247, 378, 468	$\bar{3}\bar{7}\bar{G}$
6	$6G$	6,	13,	235, 278, 347, 458	$\bar{2}\bar{7}\bar{G}$
7	$7G$	7,	14,	245, 268, 346, 358	$\bar{1}\bar{7}\bar{G}$
8	$8G$	8,		234, 267, 357, 456	$\bar{1}\bar{4}\bar{7}\bar{G}$
9	$18G$		18,	237, 246, 258, 245, 368, 478, 567	$\bar{7}\bar{G}$
10	$23G$		23, 48, 56,	126, 135, 178	$\bar{1}\bar{G}$
11	$24G$		24, 38, 57,	127, 145, 168	$\bar{2}\bar{G}$
12	$26G$		26, 35, 78,	123, 148, 156	$\bar{4}\bar{G}$
13	$27G$		27, 45, 68,	124, 138, 157	$\bar{5}\bar{G}$
14	$28G$		28, 34, 67,	137, 146, 158	$\bar{3}\bar{G}$
15	$37G$		37, 46, 58,	128, 134, 167	$\bar{6}\bar{G}$

where all interactions involving more than three factors are omitted. Some cosets share exactly the same coset pattern.

coset of D	rows of A							
G	0	0	3	7	4	0	1	0
$1G$	1	3	0	4	7	1	0	0
$2G$, $3G$, $4G$, $5G$, $6G$, $7G$	1	1	4	4	3	3	0	0
$8G$	1	0	4	7	3	0	0	1
$18G$	0	1	7	4	0	3	1	0
$23G$, $24G$, $26G$, $27G$, $28G$, $37G$	0	3	3	4	4	1	1	0

The complementary design $\bar{D}$ consists of the remaining columns after deleting those corresponding to D,

$$\bar{1} = bc, \bar{2} = bd, \bar{3} = cd, \bar{4} = abc, \bar{5} = abd, \bar{6} = acd, \bar{7} = abcd.$$

Thus $\bar{D}$ is 2^{7-3} design with defining relations $I = \bar{1}\bar{2}\bar{3} = \bar{1}\bar{5}\bar{6} = \bar{2}\bar{4}\bar{6}$.

$$\bar{1}\bar{2}\bar{3}, \bar{1}\bar{5}\bar{6}, \bar{2}\bar{4}\bar{6}, \bar{3}\bar{4}\bar{5}, \quad \bar{1}\bar{2}\bar{4}\bar{5}, \bar{1}\bar{3}\bar{4}\bar{6}, \bar{2}\bar{3}\bar{5}\bar{6}$$

and the wordlength pattern is $(0, 0, 4, 3, 0, 0, 0)$. The following table lists all the cosets of $\bar{D}$.

rank	coset	factorial effects			$\tau^{*-1}(\bar{j}_1\cdots\bar{j}_m\bar{G})$
0	$\bar{G}$			$\bar{1}\bar{2}\bar{3}$, $\bar{1}\bar{5}\bar{6}$, $\bar{2}\bar{4}\bar{6}$, $\bar{3}\bar{4}\bar{5}$	G
1	$\bar{1}\bar{G}$	$\bar{1}$,	$\bar{2}\bar{3}$, $\bar{5}\bar{6}$,	$\bar{2}\bar{4}\bar{5}$, $\bar{3}\bar{4}\bar{6}$	$23G$
2	$\bar{2}\bar{G}$	$\bar{2}$,	$\bar{1}\bar{3}$, $\bar{4}\bar{6}$,	$\bar{1}\bar{4}\bar{5}$, $\bar{3}\bar{5}\bar{6}$	$24G$
3	$\bar{3}\bar{G}$	$\bar{3}$,	$\bar{1}\bar{2}$, $\bar{4}\bar{5}$,	$\bar{1}\bar{4}\bar{6}$, $\bar{2}\bar{5}\bar{6}$	$28G$
4	$\bar{4}\bar{G}$	$\bar{4}$,	$\bar{2}\bar{6}$, $\bar{3}\bar{5}$,	$\bar{1}\bar{2}\bar{5}$, $\bar{1}\bar{3}\bar{6}$	$26G$
5	$\bar{5}\bar{G}$	$\bar{5}$,	$\bar{1}\bar{6}$, $\bar{3}\bar{4}$,	$\bar{1}\bar{2}\bar{4}$, $\bar{2}\bar{3}\bar{6}$	$27G$
6	$\bar{6}\bar{G}$	$\bar{6}$,	$\bar{1}\bar{5}$, $\bar{2}\bar{4}$,	$\bar{1}\bar{3}\bar{4}$, $\bar{2}\bar{3}\bar{5}$	$37G$
7	$\bar{7}\bar{G}$	$\bar{7}$,			$18G$
8	$\bar{1}\bar{4}\bar{G}$		$\bar{1}\bar{4}$, $\bar{2}\bar{5}$, $\bar{3}\bar{6}$,	$\bar{1}\bar{2}\bar{6}$, $\bar{1}\bar{3}\bar{5}$, $\bar{2}\bar{3}\bar{4}$, $\bar{4}\bar{5}\bar{6}$	$1G$
9	$\bar{1}\bar{7}\bar{G}$		$\bar{1}\bar{7}$,	$\bar{2}\bar{3}\bar{7}$, $\bar{5}\bar{6}\bar{7}$	$7G$
10	$\bar{2}\bar{7}\bar{G}$		$\bar{2}\bar{7}$,	$\bar{1}\bar{3}\bar{7}$, $\bar{4}\bar{6}\bar{7}$	$6G$
11	$\bar{3}\bar{7}\bar{G}$		$\bar{3}\bar{7}$,	$\bar{1}\bar{2}\bar{7}$, $\bar{4}\bar{5}\bar{7}$	$5G$
12	$\bar{4}\bar{7}\bar{G}$		$\bar{4}\bar{7}$,	$\bar{2}\bar{6}\bar{7}$, $\bar{3}\bar{5}\bar{7}$	$4G$
13	$\bar{5}\bar{7}\bar{G}$		$\bar{5}\bar{7}$,	$\bar{1}\bar{6}\bar{7}$, $\bar{3}\bar{4}\bar{7}$	$3G$
14	$\bar{6}\bar{7}\bar{G}$		$\bar{6}\bar{7}$,	$\bar{1}\bar{5}\bar{7}$, $\bar{2}\bar{4}\bar{7}$	$2G$
15	$\bar{1}\bar{4}\bar{7}\bar{G}$			$\bar{1}\bar{4}\bar{7}$, $\bar{2}\bar{5}\bar{7}$, $\bar{3}\bar{6}\bar{7}$	$8G$

where all interactions involving more than three factors are omitted. Some cosets share exactly the same coset pattern.

Coset of $\bar{D}$	rows of $\bar{A}$						
$\bar{G}$	0	0	4	3	0	0	0
$\bar{1}\bar{G}$, $\bar{2}\bar{G}$, $\bar{3}\bar{G}$, $\bar{4}\bar{G}$, $\bar{5}\bar{G}$, $\bar{6}\bar{G}$	1	2	2	2	1	0	0
$\bar{7}\bar{G}$	1	0	0	4	3	0	0
$\bar{1}\bar{4}\bar{G}$	0	3	4	0	0	1	0
$\bar{1}\bar{7}\bar{G}$, $\bar{2}\bar{7}\bar{G}$, $\bar{3}\bar{7}\bar{G}$, $\bar{4}\bar{7}\bar{G}$, $\bar{5}\bar{7}\bar{G}$, $\bar{6}\bar{7}\bar{G}$	0	1	2	2	2	1	0
$\bar{1}\bar{4}\bar{7}\bar{G}$	0	0	3	4	0	0	1

S3 Details for the Second Pair of Designs

Note that $H_4(2)$ consists of 15 columns, which are denoted by

$$\{a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\}.$$

Choose 13 columns from $H_4(2)$ to form design D,

$$1 = a, 2 = b, 3 = c, 4 = d, 5 = ab, 6 = ac, 7 = bc, 8 = abc, 9 = ad,$$
$$t_0 = bd, t_1 = abd, t_2 = cd, t_3 = acd.$$

where t_0, ..., t_3 represent factors 10, ..., 13, respectively. Then D is a 2^{13-9} design with defining relations

$$\{125, 136, 237, 1238, 149, 24t_0, 124t_1, 34t_2, 134t_3\},$$

and wordlength patter $(0, 0, 22, 55, 72, 96, 116, 87, 40, 16, 6, 1, 0)$. The following table lists all the cosets of D, and all main effects and some 2fi's.

point	rank	coset	factorial effects		τ^*
0	0	G			$\bar{G}$
a	1	$1G$	1,	25,	$\bar{1}\bar{2}\bar{G}$
b	2	$2G$	2,	15,	
c	3	$3G$	3,	16, 27	
d	4	$4G$	4,	19, $2t_0$	
ab	5	$5G$	5,	12,	
ac	6	$6G$	6,	13, 28	
bc	7	$7G$	7,	23, 18	
abc	8	$8G$	8,	17, 26	
ad	9	$9G$	9,	14, $2t_1$	
bd	10	t_0G	t_0,	24,	
abd	11	t_1G	t_1,	$1t_0$, 29	
cd	12	t_2G	t_2,	$1t_1$, $1t_3$	
acd	13	t_3G	t_3,	$1t_2$,	
bcd	14	$2t_2G$		$2t_2$,	$\bar{1}\bar{G}$
$abcd$	15	$2t_3G$		$2t_3$,	$\bar{2}\bar{G}$

The distinct coset patterns are as follows.

coset of D	rows of A												
G	0	0	22	55	72	96	116	87	40	16	6	1	0
$1G$	1	6	16	40	87	116	96	72	55	22	0	0	1
all others	1	5	17	45	82	106	106	82	45	17	5	1	0
$2t_2G$, $2t_3G$	0	6	22	40	72	116	116	72	40	22	6	0	0

The complementary design $\bar{D}$ consists of the remaining columns after deleting those corresponding to D,

$$\bar{1} = bcd, \bar{2} = abcd,$$

which form a full 2^2 factorial design. In terms of $H_4(2)$, the remaining columns bcd and $abcd$ consist of 4 replicated 2^2 designs.

cosets	rows of $\bar{A}$	
$\bar{G}$	0	0
$\bar{1}\bar{G}$, $\bar{2}\bar{G}$	1	0
$\bar{1}\bar{2}\bar{G}$	0	1
$\varnothing$	0	0

The correspondence between coset patterns are as follows.

rows of A													rows of $\bar{A}$	
0	0	22	55	72	96	116	87	40	16	6	1	0	0	0
1	6	16	40	87	116	96	72	55	22	0	0	1	0	1
1	5	17	45	82	106	106	82	45	17	5	1	0	0	0
0	6	22	40	72	116	116	72	40	22	6	0	0	1	0

S4 Details for the Proof of Corollary 2

Denote $b = n - m/2 = n - 2^{n-p-1}$. When $\bar{h} \in \cup_{k\geq 2}\bar{\mathcal{R}}_k$, we have $\bar{A}_{\bar{h},0} = \bar{A}_{\bar{h},1} = 0$, while when $\bar{h} \in \bar{\mathcal{R}}_1$, we have $\bar{A}_{\bar{h},0} = 0$ and $\bar{A}_{\bar{h},1} = 1$. We also have

$$\sum_{\bar{h}\in\cup_{k\geq 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},i} = \binom{m-1-n}{i} - \bar{A}_{0,i} - \sum_{\bar{h}\in\bar{\mathcal{R}}_1} \bar{A}_{\bar{h},i} = \binom{m-1-n}{i} - \bar{A}_{0,i} - \bar{M}_{(1,i)_1}$$

$$\begin{aligned} M_{(1,2)_1} &= \sum_{h\in\mathcal{R}_1} A_{h,1}A_{h,2} = \sum_{\bar{h}\in\cup_{k\geq 2}\bar{\mathcal{R}}_k} (1-\bar{A}_{\bar{h},1}-\bar{A}_{\bar{h},0})(b+\bar{A}_{\bar{h},2}+\bar{A}_{\bar{h},1}-b\bar{A}_{\bar{h},0}) \\ &= \sum_{\bar{h}\in\cup_{k\geq 2}\bar{\mathcal{R}}_k} (b+\bar{A}_{\bar{h},2}) = bn + \binom{m-1-n}{2} - \bar{A}_{0,2} - \bar{M}_{(1,2)_1} \\ &= \text{const} - \bar{M}_{(1,2)_1}. \end{aligned}$$

$$\begin{aligned} M_{(2,2)_2} &= \sum_{h\in\mathcal{R}_2} A_{h,2}(A_{h,2}-1)/2 = \sum_{\bar{h}\in\bar{\mathcal{R}}_1} (b+\bar{A}_{\bar{h},2}+\bar{A}_{\bar{h},1})(b+\bar{A}_{\bar{h},2}+\bar{A}_{\bar{h},1}-1)/2 \\ &= \text{const} + (b+1)\sum_{\bar{h}\in\bar{\mathcal{R}}_1} \bar{A}_{\bar{h},2} + \sum_{\bar{h}\in\bar{\mathcal{R}}_1} \bar{A}_{\bar{h},2}(\bar{A}_{\bar{h},2}-1)/2 \\ &= \text{const} + (b+1)\bar{M}_{(1,2)_1} + \bar{M}_{(2,2)_1}. \end{aligned}$$

$$\begin{aligned} M_{(2,2)_1} &= \sum_{h\in\mathcal{R}_1} A_{h,2}(A_{h,2}-1)/2 = \sum_{\bar{h}\in\cup_{k\geq 2}\bar{\mathcal{R}}_k} (b+\bar{A}_{\bar{h},2})(b+\bar{A}_{\bar{h},2}-1)/2 \\ &= \text{const} + b\sum_{\bar{h}\in\cup_{k\geq 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},2} + \sum_{\bar{h}\in\cup_{k\geq 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},2}(\bar{A}_{\bar{h},2}-1)/2 \\ &= \text{const} - b\bar{M}_{(1,2)_1} + \bar{M}_{(2,2)_2} \end{aligned}$$

$$\begin{aligned} M_{(1,3)_1} &= \sum_{h\in\mathcal{R}_1} A_{h,1}A_{h,3} = \sum_{\bar{h}\in\cup_{k\geq 2}\bar{\mathcal{R}}_k} (1-\bar{A}_{\bar{h},1})(a-\bar{A}_{\bar{h},3}-\bar{A}_{\bar{h},2}) \\ &= \text{const} - \sum_{\bar{h}\in\cup_{k\geq 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},3} - \sum_{\bar{h}\in\cup_{k\geq 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},2} \\ &= \text{const} + \bar{A}_{0,3} + \bar{M}_{(1,3)_1} + \bar{M}_{(1,2)_1} \\ &= \text{const} + \bar{M}_{(1,3)_1} + (4/3)\bar{M}_{(1,2)_1} \end{aligned}$$

$$\begin{aligned}
M_{(2,3)_2} &= \sum_{h\in\mathcal{R}_2} A_{h,2}A_{h,3} = \sum_{\bar{h}\in\bar{\mathcal{R}}_1} (b+\bar{A}_{\bar{h},2}+\bar{A}_{\bar{h},1})(a-\bar{A}_{\bar{h},3}-\bar{A}_{\bar{h},2}+b\bar{A}_{\bar{h},1})\\
&= \text{const} - \sum_{\bar{h}\in\bar{\mathcal{R}}_1} \bar{A}_{\bar{h},2}\bar{A}_{\bar{h},3} - \sum_{\bar{h}\in\bar{\mathcal{R}}_1} \bar{A}^2_{\bar{h},2} - (b+1)\sum_{\bar{h}\in\bar{\mathcal{R}}_1} \bar{A}_{\bar{h},3} + (a-1)\sum_{\bar{h}\in\bar{\mathcal{R}}_1} \bar{A}_{\bar{h},2}\\
&= \text{const} - \bar{M}_{(2,3)_1} - 2\bar{M}_{(2,2)_1} - (b+1)\bar{M}_{(1,3)_1} + (a-2)\bar{M}_{(1,2)_1}
\end{aligned}$$

$$\begin{aligned}
M_{(2,3)_1} &= \sum_{h\in\mathcal{R}_1} A_{h,2}A_{h,3} = \sum_{\bar{h}\in\cup_{k\ge 2}\bar{\mathcal{R}}_k} (b+\bar{A}_{\bar{h},2})(a-\bar{A}_{\bar{h},3}-\bar{A}_{\bar{h},2})\\
&= \text{const} - \sum_{\bar{h}\in\cup_{k\ge 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},2}\bar{A}_{\bar{h},3} - \sum_{\bar{h}\in\cup_{k\ge 2}\bar{\mathcal{R}}_k} \bar{A}^2_{\bar{h},2} - b\sum_{\bar{h}\in\cup_{k\ge 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},3} + (a-b)\sum_{\bar{h}\in\cup_{k\ge 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},2}\\
&= \text{const} - \bar{M}_{(2,3)_2} - 2\bar{M}_{(2,2)_2} + b\bar{A}_{0,3} + b\bar{M}_{(1,3)_1} - (a-b-1)\bar{M}_{(1,2)_1}\\
&= \text{const} - \bar{M}_{(2,3)_2} - 2\bar{M}_{(2,2)_2} + b\bar{M}(1,3)_1 + (4b/3-a+1)\bar{M}_{(1,2)_1}
\end{aligned}$$

$$\begin{aligned}
M_{(1,4)_1} &= \sum_{h\in\mathcal{R}_1} A_{h,1}A_{h,4} = \sum_{\bar{h}\in\cup_{k\ge 2}\bar{\mathcal{R}}_k} (1-\bar{A}_{\bar{h},1})(\text{const}+\bar{A}_{\bar{h},4}+\bar{A}_{\bar{h},3}-b\bar{A}_{\bar{h},2})\\
&= \text{const} + \sum_{\bar{h}\in\cup_{k\ge 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},4} + \sum_{\bar{h}\in\cup_{k\ge 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},3} - b\sum_{\bar{h}\in\cup_{k\ge 2}\bar{\mathcal{R}}_k} \bar{A}_{\bar{h},2}\\
&= \text{const} - \bar{A}_{0,4} - \bar{M}_{(1,4)_1} - \bar{A}_{0,3} - \bar{M}_{(1,3)_1} + b\bar{M}_{(1,2)_1}\\
&= \text{const} - \bar{M}_{(1,4)_1} - (5/4)\bar{M}_{(1,3)_1} + (b-1/3)\bar{M}_{(1,2)_1}
\end{aligned}$$