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CONSTRUCTION OF NESTED (NEARLY) ORTHOGONAL DESIGNS FOR COMPUTER EXPERIMENTS

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Abstract: We propose several methods for constructing nested (nearly) orthogonal designs intended for multi-fidelity computer experiments. Such designs are two (nearly) orthogonal designs with one nested within the other. Our methods exploit nesting in such discrete structures as fractional factorial designs, Hadamard matrices, and rotation matrices. Examples are given illustrating the proposed methods.

Key words and phrases: Computer experiment, design of experiments, Hadamard matrix, Latin hypercube design, nested Hadamard matrix, nested Latin hypercube design, orthogonal array, space-filling design.

1. Introduction

Construction of (nearly) orthogonal Latin hypercube designs and related (nearly) orthogonal designs has recently drawn a surge of interest in computer experiments (Santner, Williams, and Notz (2003); Fang, Li, and Sudjianto (2005)). Such designs can be obtained by using permutation matrices (Ye (1998)), building on rotation matrices and factorial designs (Steinberg and Lin (2006)), or coupling small (nearly) orthogonal designs with orthogonal matrices (Bingham, Sitter, and Tang (2009)). Other work in this direction includes Owen (1994) Tang (1998), Lin, Mukerjee, and Tang (2009), and Pang, Liu, and Lin (2009).

The purpose of this article is to construct nested (nearly) orthogonal designs intended for running a pair of low-accuracy and high-accuracy computer experiments (Kennedy and O'Hagan (2000); Qian et al. (2006)). This work is based on the first author's Ph.D. thesis (Li (2010)). We define nested (nearly) orthogonal designs to be two (nearly) orthogonal designs with one nested within the other. Because taking an arbitrary subset of a (nearly) orthogonal design is not guaranteed to give a smaller (nearly) orthogonal design, systematic methods are needed for construction.

We introduce some notation and definitions. Take the correlation of vectors $\mathbf{a} = (a_1, \ldots, a_n)'$ and $\mathbf{b} = (b_1, \ldots, b_n)'$ to be

$$\rho = \frac{\sum_{i=1}^{n}(a_i - \bar{a})(b_i - \bar{b})}{(n-1)s_a s_b},$$

where $\bar{a} = \sum_{i=1}^n a_i$, $\bar{b} = \sum_{i=1}^n b_i$, and s_a and s_b are the sample standard deviations of $\mathbf{a}$ and $\mathbf{b}$, respectively. We call $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if $\rho = 0$. The *average correlation* of an $n \times m$ matrix $\boldsymbol{A} = (a_{ij})$ is given by

$$\rho(\boldsymbol{A}) = \sqrt{\frac{\sum_{i<j} \rho_{ij}^2(\boldsymbol{A})}{m(m-1)/2}}, \tag{1.1}$$

where ρ_{ij} is the correlation between columns i and j of $\boldsymbol{A}$. If $\rho(\boldsymbol{A}) = 0$, $\boldsymbol{A}$ is orthogonal. Throughout, a design is said to be orthogonal if its columns are orthogonal to each other.

Let $\mathrm{LH}(n, m)$ denote a Latin hypercube of n equally spaced levels in m factors (McKay, Beckman, and Conover (1979)). Let $\mathrm{OLH}(n, m)$ denote an $\mathrm{LH}(n, m)$ with orthogonal columns. Our definition of *nested Latin hypercubes* is motivated by the concept of nested orthogonal arrays in Mukerjee, Qian, and Wu (2008), who define a nested orthogonal array to be an orthogonal array containing a subarray that is a smaller orthogonal array itself. For integers $n_1 > n_2$, a nested Latin hypercube $\mathrm{NLH}(n_1, n_2, m)$ is a Latin hypercube of n_1 levels in m factors containing a subarray of n_2 runs that is an $\mathrm{LH}(n_2, m)$ itself. Let $\mathbf{B} \subset \mathbf{A}$ be an $\mathrm{NLH}(n_1, n_2, m)$ and suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal. Then $\boldsymbol{A}$, or $\mathbf{B} \subset \mathbf{A}$, is called a *nested orthogonal Latin hypercube*, denoted by $\mathrm{NOLH}(n_1, n_2, m)$. In Section 3, we construct nested Hadamard matrices that are a pair of Hadamard matrices of different orders with one nested within the other. Nested Hadamard matrices serve as an important building block for constructing nested (nearly) orthogonal designs via the Kronecker product in Section 4.

2. Obtaining Nested Orthogonal Latin Hypercubes Using Nested Rotation Matrices and Nested Factorial Designs

This section proposes an approach for constructing nested orthogonal Latin hypercubes by exploiting *nesting* in rotation matrices and factorial designs. As in Steinberg and Lin (2006), for an integer d and a prime p, let $\boldsymbol{D}_1$ be a $p^d \times d$ full factorial design with columns $1, 2, \ldots, d$. Each element $[a_0 + a_1 x + a_2 x^2 + \cdots + a_{d-1} x^{d-1}]$ of $GF(p^d)$ is denoted by a vector $[a_0, \ldots, a_{d-1}]$, corresponding to a generalized interaction $1^{a_0} 2^{a_1} \ldots d^{a_{d-1}}$ for $\boldsymbol{D}_1$, e.g., $[1 + x]$ corresponds to the interaction of factors 1 and 2. Every non-zero element of $GF(p^d)$ can be expressed as $[x^i]$ for some integer i. From Theorem 1 of Steinberg and Lin (2006) for a nonnegative integer k,

$$\big|[x^k], [x^{k+1}], \ldots, [x^{k+d-1}]\big| \tag{2.1}$$

constitutes a $p^d \times d$ full factorial design, where $|\cdot|$ denotes column juxtaposition. Two different designs generated by (2.1) are called *disjoint* if their columns do not overlap when expressed in terms of $[x]$ powers. Two such disjoint designs are mutually orthogonal in that any column of one design is orthogonal to any column of the other. Juxtaposing, column by column, a set of mutually orthogonal full factorial designs obtained from (2.1) yields an orthogonal fractional factorial design that has at most

$$b = \left\lfloor \frac{p^d - 1}{d(p-1)} \right\rfloor \tag{2.2}$$

mutually orthogonal full factorial design components (Pang, Liu, and Lin (2009)). Note that $(p^d - 1)/[d(p-1)]$ is an integer for $p \geq 3$.

An $n \times n$ matrix $\boldsymbol{R}$ is a rotation matrix if $\boldsymbol{R}'\boldsymbol{R}$ is proportional to the $n \times n$ identity matrix, denoted by $\boldsymbol{I}_n$. Here is a recursive method for constructing such matrices proposed in Beattie and Lin (1997) and Pang, Liu, and Lin (2009). For a prime p, let $\boldsymbol{R}_0 = 1$ for $c = 0$, and let

$$\boldsymbol{R}_c = \left[\begin{array}{c|c} \boldsymbol{R}_{c-1} & -p^{2^{c-1}}\boldsymbol{R}_{c-1} \\ \hline p^{2^{c-1}}\boldsymbol{R}_{c-1} & \boldsymbol{R}_{c-1} \end{array} \right]_{2^c \times 2^c}, \text{ for } c \geq 1,$$

where $\boldsymbol{R}_c'\boldsymbol{R}_c = a_c\boldsymbol{I}_{2^c}$ with $a_c = (p^{2^{c+1}} - 1)/(p^2 - 1)$. Let $d = 2^c$. Take $\boldsymbol{D} = (d_{ij})$ to be a full factorial design in d factors of p^d runs with levels $1, \ldots, p$. Centering $\boldsymbol{D}$ by replacing d_{ij} with

$$d_{ij} - \frac{p+1}{2} \tag{2.3}$$

gives

$$\boldsymbol{D}'\boldsymbol{D} = \lambda_1 \boldsymbol{I}_d, \quad (\boldsymbol{D}\boldsymbol{R}_c)'(\boldsymbol{D}\boldsymbol{R}_c) = \lambda_2 \boldsymbol{I}_d,$$

where $\lambda_1 = p^d(p^2 - 1)/12$ and $\lambda_2 = p^d(p^{2d} - 1)/12$. Here, $\boldsymbol{D}\boldsymbol{R}_c$ is an OLH(p^d, d), with each column a permutation on $\{-(p^d-1)/2, -(p^d-1)/2+1, \ldots, (p^d-1)/2\}$.

A key of the proposed approach is to rotate a *nested factorial design.* We take a nested factorial design to be a factorial design containing a small factorial design as a subset. Such a design can be constructed by juxtaposing two identical small fractional factorial designs in $\boldsymbol{D}_1$ in (2.1). For $d = 2^c$, the proposed method has three steps.

Step 1. Construct a $p^{2d} \times 2d$ partitioned full factorial design $\boldsymbol{D}_1 = |\boldsymbol{D}_1^{(1)}, \boldsymbol{D}_1^{(2)}|$ with levels $1, \ldots, p$ that contains $|\boldsymbol{D}_1^0, \boldsymbol{D}_1^0|$ as the first p^d rows, where $\boldsymbol{D}_1^0$ is a $p^d \times d$ full factorial design. Center $\boldsymbol{D}_1$ as in (2.3).

Step 2. For $i = 2, \ldots, b$, with b defined in (2.2), use (2.1) to generate a $p^{2d} \times 2d$ full factorial design $\boldsymbol{D}_i = |\boldsymbol{D}_i^{(1)}, \boldsymbol{D}_i^{(2)}|$ from $\boldsymbol{D}_1$, where $\boldsymbol{D}_i^{(1)}$ is from $\{[x^{(i-1)d}], [x^{(i-1)d+1}], \ldots, [x^{(i-1)d+d-1}]\}$ associated with $\boldsymbol{D}_1^{(1)}$, and $\boldsymbol{D}_i^{(2)}$ is from $\{[x^{(i-1)d}], [x^{(i-1)d+1}], \ldots, [x^{(i-1)d+d-1}]\}$ associated with $\boldsymbol{D}_1^{(2)}$.

Step 3. For $i = 1, \ldots, b$, obtain $\boldsymbol{A}_i$ by taking the first d columns of $\boldsymbol{D}_i\boldsymbol{R}_{c+1}$. Put $\boldsymbol{A} = |\boldsymbol{A}_1, \ldots, \boldsymbol{A}_b|$. Obtain a matrix $\boldsymbol{B}$ by taking the first p^d rows of $\boldsymbol{A}$.

Theorem 1. *For $\boldsymbol{A}$ and $\boldsymbol{B}$ constructed above, a prime p and an integer $c \geq 0$,* (i) *$\boldsymbol{A}$ is an* OLH(n_1, m)*;* (ii) *$\boldsymbol{B} \subset \boldsymbol{A}$ and $\boldsymbol{B}$ is an* OLH(n_2, m)*, where $d = 2^c$, $b = \lfloor (p^d - 1)/[d(p-1)] \rfloor$, $n_1 = p^{2d}$, $n_2 = p^d$, and $m = bd$.*

Proof. For $i = 1, \ldots, b$, $\boldsymbol{A}_i$ is an OLH(p^{2d}, d) with each column being a permutation on $\{-(p^{2d}-1)/2, -(p^{2d}-1)/2+1, \ldots, (p^{2d}-1)/2\}$. Because $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_b$ are mutually orthogonal, $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_b$ are mutually orthogonal as well, which proves part (i).

Let $\boldsymbol{B}_i$ denote the submatrix of $\boldsymbol{A}_i$ consisting of its first p^d rows. Since the first p^d rows of $\boldsymbol{D}_1\boldsymbol{R}_{c+1}$ are $|(p^d+1)\boldsymbol{D}_1^0\boldsymbol{R}_c, (1-p^d)\boldsymbol{D}_1^0\boldsymbol{R}_c|$,$\boldsymbol{B}_1 = (p^d+1)\boldsymbol{D}_1^0\boldsymbol{R}_c$ is an OLH. By the construction of $\boldsymbol{D}_2, \ldots, \boldsymbol{D}_b$, for $i = 1, \ldots, b$, $\boldsymbol{B}_i$ is an OLH(p^d, d) with each column being a permutation on $\{-(p^{2d}-1)/2, -(p^{2d}-1)/2 + (p^d + 1), \ldots, (p^{2d}-1)/2\}$, and $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_b$ are mutually orthogonal. Thus, $\boldsymbol{B}$ is an OLH(p^d, bd). Note that $\boldsymbol{B}$ is a Latin hypercube without level collapsing. Clearly, $\boldsymbol{B} \subset \boldsymbol{A}$ and the levels of $\boldsymbol{A}$ and those of $\boldsymbol{B}$ are equally spaced on $[-(p^{2d} - 1)/2, (p^{2d}-1)/2]$. This completes the proof.

Example 1. Let $p = 3$ and $c = 1$ with $d = 2$, $n_1 = 81$, $n_2 = 9$, and $b = 2$. Use the primitive polynomial $f(x) = x^2 + 2x + 2$ for $GF(9)$. Let $\boldsymbol{D}_1 = |\boldsymbol{D}_1^{(1)},\boldsymbol{D}_1^{(2)}|$ be an 81×4 full factorial design with columns $1, 2, 3, 4$, where the first nine rows of $\boldsymbol{D}_1$ are $|\boldsymbol{D}_1^0, \boldsymbol{D}_1^0|$ and $\boldsymbol{D}_1^0$ is a 9×2 full factorial design. Then $\boldsymbol{D}_1^{(1)} = |1, 2|$ and $\boldsymbol{D}_1^{(2)} = |3, 4|$, respectively. For $i = 2$, by taking the polynomial elements $\{[x^2] = [1 + x], [x^3] = [1 + 2x]\}$ in $GF(9)$, obtain a full factorial design $\boldsymbol{D}_2 = |12, 12^2, 34, 34^2|$ from $\boldsymbol{D}_1$. For $i = 1$, 2, let $\boldsymbol{A}_i$ be the first two columns of $\boldsymbol{D}_i\boldsymbol{R}_2$, and let $\boldsymbol{A} = |\boldsymbol{A}_1, \boldsymbol{A}_2|$. Let $\boldsymbol{B}$ be the first nine rows of $\boldsymbol{A}$. From Theorem 1, $\boldsymbol{B} \subset \boldsymbol{A}$ is an NOLH$(81, 9, 4)$; it is given in Table 1.

Example 2. Let $p = 2$ and $c = 1$ with $d = 2$, $n_1 = 16$, $n_2 = 4$, and $b = 1$. The pair of designs $\boldsymbol{B} \subset \boldsymbol{A}$ from Theorem 1 is an NOLH$(16, 4, 2)$; it is given in Table 2. Qian (2009) uses random nested permutations to generate nested Latin hypercube designs that do not have guaranteed (nearly) orthogonal properties. To illustrate this difference, we used the method in Qian (2009) to generate a pair of nested Latin hypercube designs of the same size as $\boldsymbol{B} \subset \boldsymbol{A}$ 1,000 times. The mean of the average correlations of these 1,000 pairs of nested designs is 0.4892 for the small design and 0.2115 for the large design, respectively; these are significantly different from zero.

Table 1. An NOLH$(81, 9, 4)$ in Example 1, where the subarray above the dash line is an OLH$(9, 4)$ after every entry is divided by ten.

Run #	x_1	x_2	x_3	x_4	Run #	x_1	x_2	x_3	x_4	Run #	x_1	x_2	x_3	x_4
1	0	0	-20	-40	28	-6	28	29	13	55	15	35	9	-27
2	10	-30	-10	30	29	-5	25	24	8	56	16	32	13	-29
3	-10	30	30	10	30	-7	31	28	6	57	14	38	5	-25
4	30	10	20	40	31	-12	26	27	9	58	-27	-9	-2	-4
5	40	-20	-30	-10	32	-11	23	31	7	59	-26	-12	-1	3
6	20	40	10	-30	33	-13	29	23	11	60	-28	-6	3	1
7	-30	-10	0	0	34	27	9	16	32	61	-24	-8	2	4
8	-20	-40	40	-20	35	28	6	17	39	62	-23	-11	-3	-1
9	-40	20	-40	20	36	26	12	21	37	63	-25	-5	1	-3
10	1	-3	-19	-33	37	31	7	15	35	64	-29	-13	4	-2
11	-1	3	-15	-35	38	29	13	19	33	65	-31	-7	-4	2
12	3	1	-16	-32	39	24	8	18	36	66	-18	-36	34	-22
13	4	-2	-21	-37	40	25	5	22	34	67	-17	-39	35	-15
14	2	4	-17	-39	41	23	11	14	38	68	-19	-33	39	-17
15	-3	-1	-18	-36	42	36	-18	-29	-13	69	-15	-35	38	-14
16	-2	-4	-14	-38	43	37	-21	-28	-6	70	-14	-38	33	-19
17	-4	2	-22	-34	44	35	-15	-24	-8	71	-16	-32	37	-21
18	9	-27	-11	23	45	39	-17	-25	-5	72	-21	-37	36	-18
19	8	-24	-6	28	46	38	-14	-26	-12	73	-22	-34	32	-16
20	12	-26	-7	31	47	33	-19	-27	-9	74	-36	18	-38	14
21	13	-29	-12	26	48	34	-22	-23	-11	75	-35	15	-37	21
22	11	-23	-8	24	49	32	-16	-31	-7	76	-37	21	-33	19
23	6	-28	-9	27	50	18	36	7	-31	77	-33	19	-34	22
24	7	-31	-5	25	51	19	33	8	-24	78	-32	16	-39	17
25	5	-25	-13	29	52	17	39	12	-26	79	-34	22	-35	15
26	-9	27	25	5	53	21	37	11	-23	80	-39	17	-36	18
27	-8	24	26	12	54	22	34	6	-28	81	-38	14	-32	16

3. Construction of Nested Hadamard Matrices

This section presents methods for constructing nested Hadamard matrices, serving as a stepping-stone for generating new nested (nearly) orthogonal designs in Section 4. A Hadamard matrix $\boldsymbol{H}_n$ is an $n \times n$ orthogonal matrix with entries ± 1 (Hedayat, Sloane, and Stufken (1999)). Suppose that $\boldsymbol{A}$ is an $\boldsymbol{H}_n$ and its subarray consisting of the first m rows and first m columns, denoted by $\boldsymbol{B}$, is an $\boldsymbol{H}_m$. Then $\boldsymbol{A}$, or more precisely $\boldsymbol{B} \subset \boldsymbol{A}$, is called a *nested Hadamard matrix*, denoted by NHM(n, m). Since $\boldsymbol{A}$ has more columns than $\boldsymbol{B}$, this pair of matrices can generate nested (nearly) orthogonal designs in which the larger design accommodates more factors than the smaller design, see Section 4. For illustration, Tables 3 and 4 present an NHM$(12, 4)$ and an NHM$(20, 4)$, respectively. The definition of nested Hadamard matrices does not involve any level-collapsing and is

Table 2. An NOLH$(16, 4, 2)$ in Example 2, where the subarray above the dash line is an OLH$(4, 2)$ after every entry is divided by five

15	-5
-5	-15
5	15
-15	5
-1	-13
7	11
-9	3
11	-7
3	9
-13	1
13	-1
-3	-9
-11	7
9	-3
-7	-11
1	13

Table 3. An NHM$(12, 4)$, where the whole array is an $\boldsymbol{H}_{12}$ and the subarray in the top left block is an $\boldsymbol{H}_4$

1	1	1	1	1	1	1	1	1	1	1	1
1	-1	1	-1	1	-1	1	1	-1	-1	-1	1
1	-1	-1	1	-1	1	1	1	1	-1	-1	-1
1	1	-1	-1	1	-1	1	-1	1	1	-1	-1
1	1	1	-1	-1	1	-1	-1	1	-1	-1	1
1	1	1	1	-1	-1	-1	1	-1	1	-1	-1
1	-1	1	1	1	-1	-1	-1	1	-1	1	-1
1	-1	-1	1	1	1	-1	-1	-1	1	-1	1
1	1	-1	-1	1	1	-1	1	-1	-1	1	-1
1	-1	-1	-1	-1	-1	-1	1	1	1	1	1
1	-1	1	-1	-1	1	1	-1	-1	1	1	-1
1	1	-1	1	-1	-1	1	-1	-1	-1	1	1

in the spirit of the concept of nested Latin hypercubes in Section 2. In a pair of nested Latin hypercubes, a small Latin hypercube is nested within a large Latin hypercube with more runs but the same number of columns. Because Hadamard matrices are square matrices by definition, in a pair of nested Hadamard matrices, the small and the large matrices have different numbers of rows and different numbers of columns as well.

Table 4. An NHM$(20, 4)$, where the subarray in the top left block is an $\boldsymbol{H}_4$

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	-1	1	-1	1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	-1
1	-1	-1	1	1	1	1	1	-1	-1	-1	1	-1	1	1	-1	-1	-1	-1	1
1	1	-1	-1	1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	-1	-1
1	-1	-1	1	-1	1	-1	1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1
1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	1	1	-1	1	1	-1	-1
1	-1	1	-1	1	-1	-1	1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1
1	1	1	1	-1	-1	1	-1	1	-1	-1	-1	-1	1	-1	1	-1	-1	1	1
1	1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	-1	-1	-1	1	1	-1	1
1	1	1	1	-1	1	-1	1	-1	-1	-1	-1	1	-1	1	-1	-1	1	1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1
1	-1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	-1	1	1	-1	1	-1	1
1	-1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	-1	-1	1	1	1	1	-1
1	1	-1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	1	-1	1	-1	1	1
1	-1	-1	1	-1	1	-1	-1	1	1	1	1	-1	1	-1	-1	-1	1	1	-1
1	1	-1	-1	-1	-1	1	1	-1	1	1	1	1	-1	-1	-1	-1	-1	1	1
1	-1	1	-1	1	-1	-1	1	1	-1	1	-1	1	1	-1	-1	-1	1	-1	1
1	-1	1	1	-1	-1	1	-1	1	-1	1	1	1	-1	1	-1	1	-1	-1	-1
1	1	-1	1	1	-1	-1	-1	-1	1	1	-1	1	1	1	1	-1	-1	-1	-1
1	1	1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	-1	1	1	-1	-1	-1

3.1. A linear subspace approach

For an odd prime p, $s_1 = p^{u_1}$ and $s_2 = p^{u_2}$ with $u_1 = 4$ and $u_2 = 2$, we propose a linear subspace approach to constructing an NHM$(2(s_1+1), 2(s_2+1))$. Let $\mathcal{F}$ denote $GF(s_1)$ with a primitive polynomial $f(x)$ and $s_1 = p^4$, and let α denote the primitive element $[x]$ of $\mathcal{F}$. The elements of $\mathcal{F}$ are denoted by $\alpha_0 = 0$ and $\alpha_i = \alpha^i$, for $i = 1, \ldots, s_1 - 1$. This approach exploits *nesting* in a family of Hadamard matrices as constructed in Paley (1933). The basic idea here to find a linear subspace of $\mathcal{F}_0 \subset \mathcal{F}$ such that $\sum_{\gamma \in \mathcal{F}_0} \chi(\gamma(\gamma + 1)) = -1$. Define the indication function of the quadratic residues of $\mathcal{F}$ Hedayat, Sloane, and Stufken (1999) as

$$\chi(\xi) = \begin{cases} 1, & \text{if } \xi \text{ is a quadratic residue of } \mathcal{F}, \\ 0, & \text{if } \xi = 0, \\ -1, & \text{if } \xi \text{ is not a quadratic residue of } \mathcal{F}. \end{cases} \tag{3.1}$$

The proposed method has four steps.

Step 1. Obtain an $s_1 \times s_1$ matrix $\boldsymbol{Q}_1 = (q_{ij})$ with

$$q_{ij} = \chi(\alpha_i - \alpha_j), \; i, j = 0, 1, \ldots, s_1 - 1. \tag{3.2}$$

Put

$$\boldsymbol{C}_1 = \begin{cases} \boldsymbol{I}_{s_1} + \left[\begin{array}{c|c} 0 & -\boldsymbol{1}' \\ \hline \boldsymbol{1} & \boldsymbol{Q}_1 \end{array}\right], \text{if } s_1 \equiv 3 \pmod 4, \\ \left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & -1 \end{array}\right] \otimes \boldsymbol{I}_{s_1} + \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & -1 \end{array}\right] \otimes \left[\begin{array}{c|c} 0 & \boldsymbol{1}' \\ \hline \boldsymbol{1} & \boldsymbol{Q}_1 \end{array}\right], \text{if } s_1 \equiv 1 \pmod 4. \end{cases} \quad (3.3)$$

Step 2. Let η be a quadratic nonresidue of $\mathcal{F}$ for which $\{\eta + a | a \in GF(p)\backslash\{0\}\}$ has exactly $(p-1)/2$ quadratic residues and exactly $(p-1)/2$ quadratic nonresidues of $\mathcal{F}$. Define

$$\mathcal{F}_0 = \{a\eta + b | a, b \in GF(p)\}. \quad (3.4)$$

Step 3. Let $\boldsymbol{Q}_2$ be the $s_2 \times s_2$ submatrix of $\boldsymbol{Q}_1$ consisting of all q_{ij} entries in (3.2) with $\alpha_i, \alpha_j \in \mathcal{F}_0$. Put

$$\boldsymbol{C}_2 = \begin{cases} \boldsymbol{I}_{s_2} + \left[\begin{array}{c|c} 0 & -\boldsymbol{1}' \\ \hline \boldsymbol{1} & \boldsymbol{Q}_2 \end{array}\right], \text{if } s_2 \equiv 3 \pmod 4, \\ \left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & -1 \end{array}\right] \otimes \boldsymbol{I}_{s_2} + \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & -1 \end{array}\right] \otimes \left[\begin{array}{c|c} 0 & \boldsymbol{1}' \\ \hline \boldsymbol{1} & \boldsymbol{Q}_2 \end{array}\right], \text{if } s_2 \equiv 1 \pmod 4. \end{cases} \quad (3.5)$$

Let $\boldsymbol{J}_n$ be the $n \times n$ matrix of ones. By Paley (1933), $\boldsymbol{Q}_1$ in (3.2) has three properties.

Property 1. *The matrix* $\boldsymbol{Q}_1$ *is symmetric if* $s_1 \equiv 1 \ (mod\ 4)$*, and is skew-symmetric if* $s_1 \equiv 3 \ (mod\ 4)$*.*

Property 2. *The relationship* $\boldsymbol{Q}_1 \boldsymbol{J}_{s_1} = \boldsymbol{J}_{s_1} \boldsymbol{Q}_1 = \boldsymbol{0}$ *holds.*

Property 3. *The relationship* $\boldsymbol{Q}_1 \boldsymbol{Q}_1' = s_1 \boldsymbol{I}_{s_1} - \boldsymbol{J}_{s_1}$ *holds.*

Lemma 1 is critical to verifying that these properties also hold for $\boldsymbol{Q}_2$.

Lemma 1. *Let* $\mathcal{B} = GF(p)$ *and* $\mathcal{B}_1 = GF(p)\backslash\{0\}$*. For an odd prime* p*, suppose that* η *is a quadratic nonresidue of* $\mathcal{F}$ *such that* $\{\eta + a | a \in \mathcal{B}_1\}$ *has exactly* $(p-1)/2$ *quadratic residues and exactly* $(p-1)/2$ *quadratic nonresidues of* $\mathcal{F}$ *defined in* (3.1)*. Let* $p_0 = (p-1)/2$ *and* $\mathcal{B}_2 = \{c_1, \ldots, c_{p_0}\}$ *be a subset of* $\mathcal{B}_1$ *for which* $\eta + c_i$ *is a quadratic residue of* $\mathcal{F}$*, for* $i = 1, \ldots, p_0$*. Then we have*

$$\sum_{\gamma \in \mathcal{F}_0} \chi(\gamma(\gamma + 1)) = -1 \quad (3.6)$$

for $\mathcal{F}_0$ *defined in* (3.4)*.*

Proof. Note that every element in $\mathcal{B}_1$ is a quadratic residue of $\mathcal{F}$. Because $\sum_{b \in \mathcal{B}} \chi(b(b+1)) = p - 2$,

$$\sum_{\gamma \in \mathcal{F}_0} \chi(\gamma(\gamma + 1)) = \sum_{a \in \mathcal{B}_1} \sum_{b \in \mathcal{B}} \chi((a\eta + b)(a\eta + b + 1)) + (p-2). \quad (3.7)$$

To simplify (3.7), let $\Lambda = \{(a\eta+b, a\eta+b+1)|a \in \mathcal{B}_1,\ b \in \mathcal{B}\}$. For $a \in \mathcal{B}_1$, let δ_a be the number of pairs in Λ satisfying the condition $\chi((a\eta+b)(a\eta+b+1)) = -1$. Since $\sum_{b\in\mathcal{B}} \chi((a\eta+b)(a\eta+b+1)) = p - 2\delta_a$, (3.7) becomes

$$(p^2-2) - 2\sum_{a\in\mathcal{B}_1} \delta_a. \tag{3.8}$$

For $a \in \mathcal{B}_1$, define $V(a) = \{a\eta + b|b \in \mathcal{B}_1\}$, so $\{a(\eta + a^{-1}b)|b \in \mathcal{B}_1\} = aV(1)$. Because $V(1)$ has exactly p_0 quadratic residues, so does $V(a)$, for $a = 2,\ldots,p-1 \in \mathcal{B}_1$. Recall that any $a \in \mathcal{B}_1$ is a quadratic residue and η is a nonresidue of $\mathcal{F}$.

Let r_a be the number of pairs in Λ for which both $a\eta+b$ and $a\eta+b+1$ are quadratic residues. For $a \in \mathcal{B}_1$, link r_a and δ_a as follows. If $r_a = (p-1)/2-1$ so that all the quadratic residues in $V(a)$ are consecutive, then $\delta_a = 2$. More generally, $\delta_a = 2+2\left((p-1)/2-1-r_a\right) = (p-1)-2r_a$, which simplifies (3.8) to

$$(p^2-2) - 2\Big[(p-1)^2 - 2\sum_{a\in\mathcal{B}_1} r_a\Big]. \tag{3.9}$$

We now calculate $\sum_{a\in\mathcal{B}_1} r_a$ in (3.9). For $a \in \mathcal{B}_1$ and $b \in \mathcal{B}$, finding a pair of a, b such that both $a\eta+b$ and $a\eta+b+1$ are quadratic residues of $\mathcal{F}$ is equivalent to solving a linear system

$$\begin{cases} a\eta + b \quad = a(\eta + c_i), \\ a\eta + b + 1 = a(\eta + c_j), \text{ for } c_i, c_j \in \mathcal{B}_2, \end{cases}$$

as

$$\begin{cases} a(c_j - c_i) = 1, \\ \qquad\quad b = ac_i. \end{cases}$$

For c_i, $c_j \in \mathcal{B}_2$, $i \neq j$, precisely one $a \in \mathcal{B}_1$ satisfies the condition $a(c_j - c_i) = 1$. As $p_0(p_0-1)$ different (c_i, c_j)'s, $i \neq j$, take values in $\mathcal{B}_2$, $\sum_{a\in\mathcal{B}_1} r_a = p_0(p_0-1)$, which simplifies (3.9) to $(p^2-2)-2(p-1)\left[(p-1)-((p-1)/2-1)\right] = -1$. The proof is now complete.

Table 5 provides a list of choices for η for Lemma 1 with $p \leq 13$.

As $s_1 = p^4 \equiv 1 \pmod 4$ and $s_2 = p^2 \equiv 1 \pmod 4$, the condition $\boldsymbol{C}_2 \subset \boldsymbol{C}_1$ holds. Theorem 2 is the main result of this construction.

Theorem 2. *Under the conditions of Lemma* 1, (i) $\boldsymbol{C}_1$ *is an* $\boldsymbol{H}_{n_1}$ *with* $n_1 = 2(p^4+1)$*;* (ii) $\boldsymbol{Q}_2$ *satisfies Properties* 1–3 *with* s_1 *replaced by* s_2*;* (iii) $\boldsymbol{C}_2 \subset \boldsymbol{C}_1$ *and* $\boldsymbol{C}_2$ *is an* $\boldsymbol{H}_{n_2}$ *with* $n_2 = 2(p^2+1)$.

Proof. Part (i) is clear from Properties 1–3 for $\boldsymbol{Q}_1$. Since $\chi(-1) = 1$, $\boldsymbol{Q}_2$ is symmetric. This result, combined with the fact that $s_2 = p^2 \equiv 1 \equiv s_1 \pmod 4$, verifies Property 1. Property 2 follows by noting that $\mathcal{F}_0$ has

Table 5. A list of primitive polynomials of $\mathcal{F}$ and their corresponding choices of η for Lemma 1 with $p \le 13$

p	$f(x)$	η
3	x^4+x+2	$[x]$
5	x^4+x^3+2x+3	$[x]$
7	$x^4+6x^3+x^2+3$	$[x]$
11	x^4+x^2+7x+7	$[10x^3+4x^2+4x]$
13	x^4+x^2+x+2	$[12x^3+2x^2+3x+2]$

$(s_2-1)/2$ quadratic residues and $(s_2-1)/2$ nonresidues of $\mathcal{F}$. The (i,j) element of $\boldsymbol{Q}_2\boldsymbol{Q}_2{}'$, $\sum_{k=0}^{s_2-1}\chi(\beta_i-\beta_k)\chi(\beta_j-\beta_k)$, is

$$\sum_{k\neq i}\chi^2(\beta_i-\beta_k)\chi((\beta_i-\beta_k)^{-1}[(\beta_j-\beta_i)+(\beta_i-\beta_k)]) = \sum_{k\neq i}\chi((\beta_i-\beta_k)^{-1}(\beta_j-\beta_i)+1). \tag{3.10}$$

Now simplify (3.10) case by case for $i=j$ and $i\neq j$. For $i=j$, (3.10) is $\sum_{k\neq i}\chi(1)=s_2-1$. For $i\neq j$, let $\zeta=\zeta_{ij}=(\beta_j-\beta_i)^{-1}$, for β_i, $\beta_j\in\mathcal{F}_0$, and take $\mathcal{F}_0^*=\{\gamma\zeta|\gamma\in\mathcal{F}_0\}$. Note that $\mathcal{F}_0^*\supset GF(p)$ and $\mathcal{F}_0^*$ has exactly $(s_2-1)/2$ quadratic residues and exactly $(s_2-1)/2$ nonresidues. Thus, Lemma 1 holds for $\mathcal{F}_0^*$ and hence (3.10) equals $\sum_{\gamma\in\mathcal{F}_0,\gamma\neq 0}\chi((\gamma\zeta)^{-1}+1)$, which, letting $\tilde\gamma=\gamma\zeta$, equals

$$\sum_{\tilde\gamma\in\mathcal{F}_0^*,\tilde\gamma\neq 0}\chi(\tilde\gamma^{-1}+1)=\sum_{\tilde\gamma\in\mathcal{F}_0^*,\tilde\gamma\neq 0}\chi^2(\tilde\gamma^2)\chi(\tilde\gamma^{-1}+1)=\sum_{\tilde\gamma\in\mathcal{F}_0^*,\tilde\gamma\neq 0}\chi(\tilde\gamma(\tilde\gamma+1))=-1.$$

The last equality follows from Lemma 1. Thus, $\boldsymbol{Q}_2\boldsymbol{Q}_2{}'=s_2\boldsymbol{I}_{s_2}-\boldsymbol{J}_{s_2}$, which verifies Property 3. Parts (ii) and (iii) are direct consequences of part (i).

Example 3. For $p=3$, use the primitive polynomial $f(x)=x^4+x+2$ for $GF(p^4)$ and let $\eta=[x]$, where $\mathcal{F}_0$ has nine elements. Theorem 2 produces an NHM$(164,20)$, with the embedded small Hadamard matrix given in Table 6.

Example 4. For $p=5$, use the primitive polynomial $f(x)=x^4+x^3+2x+3$ for $GF(p^4)$ and let $\eta=[x]$, where $\mathcal{F}_0$ has 25 elements. Theorem 2 gives an NHM$(1252,52)$.

3.2. A subfield approach

To complement the linear subspace approach in Section 3.1, we propose a subfield approach to constructing NHM's by replacing the subspace in Step 2 of the subspace approach with a subfield $\mathcal{G}$ of order s_2. Take $\delta=u_1/u_2>1$. For

Table 6. An $\boldsymbol{H}_{20}$ is nested within an $\boldsymbol{H}_{164}$ in Example 3.

1	1	1	1	1	1	1	1	1	1	-1	1	1	1	1	1	1	1	1	1
1	1	1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	1	-1
1	1	1	1	1	-1	-1	-1	-1	1	1	1	-1	1	1	-1	-1	-1	-1	1
1	1	1	1	-1	1	-1	1	-1	-1	1	1	1	-1	-1	1	-1	1	-1	-1
1	-1	1	-1	1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1
1	-1	-1	1	1	1	1	1	-1	-1	1	-1	-1	1	1	-1	1	1	-1	-1
1	1	-1	-1	1	1	1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	1	-1
1	-1	-1	1	-1	1	-1	1	1	1	1	-1	-1	1	-1	1	-1	-1	1	1
1	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	-1	1
1	-1	1	-1	1	-1	-1	1	1	1	1	-1	1	-1	1	-1	-1	1	1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1
1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1
1	1	-1	1	1	-1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	1	1	1	-1
1	1	1	-1	-1	1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1	-1	1	1
1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	-1	1	-1	-1	-1	1	1	-1
1	-1	-1	1	1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	-1	-1	1	1
1	1	-1	-1	1	1	-1	-1	1	-1	-1	-1	1	1	-1	-1	-1	1	-1	1
1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	1	-1	1	-1	1	-1	-1	-1
1	1	-1	-1	-1	-1	1	1	-1	1	-1	-1	1	1	1	1	-1	-1	-1	-1
1	-1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	-1	1	1	-1	-1	-1

integers u_1 and u_2 with an odd δ, note that $\lambda = (p^{u_1} - 1)/(p^{u_2} - 1)$ is odd. The approach here proceeds as follows. As in Section 3.1, Step 1 obtains $\boldsymbol{Q}_1$ from $\mathcal{F}$. Step 2 takes $\mathcal{G}$ to be a subfield of $\mathcal{F}$ given by $\{0, \beta, \ldots, \beta^{s_2-1}\}$, where $\beta = \alpha^{\lambda}$. Step 3 takes $\boldsymbol{Q}_2$ to be the submatrix of $\boldsymbol{Q}_1$ with entries $\chi(\beta_i - \beta_j)$, for β_i, $\beta_j \in \mathcal{G}$, and then uses $\boldsymbol{Q}_2$ to construct $\boldsymbol{C}_2$ in (3.5). Because δ is odd, we have either $s_1 \equiv s_2 \equiv 1 \pmod 4$ or $s_1 \equiv s_2 \equiv 3 \pmod 4$ in (3.3) and (3.5). Thus, $\boldsymbol{C}_2$ is guaranteed to be a subset of $\boldsymbol{C}_1$.

Proposition 1. For p, u_1, u_2, and δ defined above, (i) $\boldsymbol{C}_1$ is an $\boldsymbol{H}_{n_1}$ with $n_1 = s_1 + 1$ for $s_1 \equiv 3 \pmod 4$ and $n_1 = 2(s_1 + 1)$ for $s_1 \equiv 1 \pmod 4$; (ii) $\boldsymbol{Q}_2$ satisfies Properties 1–3 with s_1 replaced by s_2; (iii) $\boldsymbol{C}_2 \subset \boldsymbol{C}_1$ and $\boldsymbol{C}_2$ is an $\boldsymbol{H}_{n_2}$ with $n_2 = s_2 + 1$ for $s_2 \equiv 3 \pmod 4$ and $n_2 = 2(s_2 + 1)$ for $s_2 \equiv 1 \pmod 4$.

This proposition follows by noting that under the assumed conditions, an element in $\mathcal{G}$ is a quadratic residue of $\mathcal{G}$ if and only if it is a quadratic residue of $\mathcal{F}$.

Example 5. Let $p = 3$, $u_1 = 3$ and $u_2 = 1$ with $s_1 = 27$ and $s_2 = 3$. The pair of nested arrays $\boldsymbol{C}_2 \subset \boldsymbol{C}_1$ from Proposition 1 is an NHM$(28, 4)$; it is given in Table 7.

Since the Kronecker product of two Hadamard matrices yields a larger Hadamard matrix (Hedayat, Sloane, and Stufken (1999)), this product can be

Table 7. An NHM(28, 4) in Example 5, where the subarray in the top left block is an $\boldsymbol{H}_4$

1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	-1	1	1	1	1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1
1	1	1	-1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	-1	1	-1	-1	-1	1	1	1	1	-1	1	1	-1
1	-1	1	1	1	1	1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	1
1	-1	-1	-1	1	-1	1	1	1	1	-1	-1	1	-1	1	-1	-1	-1	1	1	-1	1	1	1	-1	1	-1	1
1	-1	-1	-1	1	1	-1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1
1	-1	-1	-1	-1	1	1	1	1	1	-1	1	-1	1	-1	-1	-1	1	-1	-1	1	1	1	-1	1	-1	1	1
1	1	1	1	-1	-1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	1	1	-1
1	1	1	1	-1	-1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	1	1	1	-1	1	1	-1	-1	1	1
1	1	1	1	-1	-1	-1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	-1	-1	1	1	-1	1	1	1	-1	1
1	1	1	-1	1	-1	1	1	-1	1	1	-1	1	1	1	1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	1
1	-1	1	1	1	1	-1	1	1	-1	1	1	-1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	-1	1	-1	-1
1	1	-1	1	-1	1	1	-1	1	1	-1	1	1	1	1	1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	1	-1
1	1	-1	1	1	1	-1	1	-1	1	-1	-1	-1	1	-1	1	1	1	1	-1	-1	1	-1	1	-1	-1	-1	1
1	1	1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	-1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1
1	-1	1	1	1	-1	1	-1	1	1	-1	-1	-1	-1	1	1	1	1	1	-1	1	-1	1	-1	-1	-1	1	-1
1	1	-1	1	1	-1	1	1	1	-1	1	1	1	-1	-1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1
1	1	1	-1	1	1	-1	-1	1	1	1	1	1	-1	-1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	1
1	-1	1	1	-1	1	1	1	-1	1	1	1	1	-1	-1	-1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	-1
1	-1	1	-1	-1	-1	1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1	1	-1	1	1	1	1	-1	-1	-1
1	-1	-1	1	1	-1	-1	1	-1	-1	-1	1	1	1	1	-1	1	1	-1	1	1	-1	1	1	1	-1	-1	-1
1	1	-1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	1	-1	1	1	1	1	1	-1	-1	-1
1	-1	-1	1	-1	1	-1	-1	-1	1	1	-1	1	1	1	-1	1	-1	1	-1	-1	-1	1	-1	1	1	1	1
1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	-1	1	1	1
1	-1	1	-1	1	-1	-1	-1	1	-1	-1	1	1	1	-1	1	-1	1	1	-1	-1	-1	-1	1	1	1	1	1
1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	1	1	-1	1	1	1	-1	-1	-1	1	-1	1
1	1	-1	-1	1	-1	-1	-1	-1	1	1	1	-1	1	1	-1	-1	1	1	1	1	1	-1	-1	-1	1	1	-1
1	-1	1	-1	-1	1	-1	1	-1	-1	-1	1	1	-1	1	1	1	-1	1	1	1	1	-1	-1	-1	-1	1	1

used to generate new NHM's from existing ones. Let $\boldsymbol{F}$ be an $\boldsymbol{H}_n$ and let $\boldsymbol{G}$ be an $\boldsymbol{H}_m$. Put $\boldsymbol{K} = \boldsymbol{F} \otimes \boldsymbol{G}$. Then $\boldsymbol{G} \subset \boldsymbol{K}$ constitutes an NHM(nm, m). More generally, the Kronecker product of an NHM and a Hadamard matrix, or that of two NHM's, yields a larger NHM. This approach, however, cannot obtain the NHM's in Theorem 2, where n_1 is not a multiple of n_2.

4. Using the Kronecker Product to Obtain New Nested (Nearly) Orthogonal Designs

This section presents two approaches to constructing new nested (nearly) orthogonal designs by taking the Kronecker product of small (nearly) orthogonal designs and two-level orthogonal designs. First we give a lemma from Bingham, Sitter, and Tang (2009).

Lemma 2. *Let $\boldsymbol{F}$ be an $n_1 \times m_1$ orthogonal matrix with two levels, -1 and $+1$, where each of the two levels appears equally often in every column. Let $\boldsymbol{D}_0$ be an $n_2 \times m_2$ nearly orthogonal design. After centering $\boldsymbol{D}_0$ column by column, define*

$$\boldsymbol{D} = \boldsymbol{F} \otimes \boldsymbol{D}_0. \tag{4.1}$$

Then $\rho(\boldsymbol{D})$ is

$$\sqrt{\frac{m_2 - 1}{m_1 m_2 - 1}\rho^2(\boldsymbol{D}_0)},$$

with $\rho(\cdot)$ as in (1.1). Furthermore, if $\boldsymbol{D}_0$ is orthogonal, then $\boldsymbol{D}$ is orthogonal.

Remark 1. For $\rho(\boldsymbol{D})$ and ${m_1}^{-1}\rho^2(\boldsymbol{D}_0)$ in Lemma 2, $\rho^2(\boldsymbol{D}) \leq {m_1}^{-1}\rho^2(\boldsymbol{D}_0)$, where m_1^{-1} depends on $\boldsymbol{F}$ only. This result and Lemma 2 still hold if an additional column $\mathbf{1}_{n_1}$ is added to $\boldsymbol{F}$.

The two approaches impose a nested structure in $\boldsymbol{F}$ or $\boldsymbol{D}_0$ in (4.1). The first approach has three steps. First, let $\boldsymbol{F}$ be the first m columns of an NHM(n, m), from Section 3 or another source, and take $\boldsymbol{G}$ to be the subset of $\boldsymbol{F}$ consisting of its first m rows and m columns. Clearly, $\boldsymbol{G}$ is an $\boldsymbol{H}_m$. Second, let $\boldsymbol{D}_0$ be a $u \times v$ nearly orthogonal design and for each column, subtract the mean from all entries. Third, put

$$\boldsymbol{A} = \boldsymbol{F} \otimes \boldsymbol{D}_0,\ \boldsymbol{B} = \boldsymbol{G} \otimes \boldsymbol{D}_0. \tag{4.2}$$

Remark 2. For $\boldsymbol{A}$ and $\boldsymbol{B}$ constructed above, (i) $\rho^2(\boldsymbol{A}) \leq (1/m)\rho^2(\boldsymbol{D}_0)$; (ii) $\boldsymbol{B} \subset \boldsymbol{A}$ and $\rho^2(\boldsymbol{B}) \leq (1/m)\rho^2(\boldsymbol{D}_0)$. Furthermore, if $\boldsymbol{D}_0$ is orthogonal, then both $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal.

Remark 2 can be verified by using Lemma 2 and Remark 1.

Example 6. Let $\boldsymbol{D}_0$ be the 6×3 nearly orthogonal Latin hypercube

$$\begin{bmatrix} -5 & -1 & 1 & -3 & 3 & 5 \\ 1 & 5 & -5 & -3 & 3 & -1 \\ -3 & 3 & 5 & -1 & 1 & -5 \end{bmatrix}',$$

where the correlation between any two columns is -0.0286 and $\rho^2(\boldsymbol{D}_0)$ is 0.0008. Let $\boldsymbol{F}$ be the last four columns of an NHM$(8, 4)$ given by

$$\left[\begin{array}{cccc:cccc} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array}\right]'.$$

Remark 2 produces a pair of designs $\boldsymbol{B} \subset \boldsymbol{A}$ in 12 factors with 24 runs and 48 runs, respectively, where $\rho^2(\boldsymbol{A}) < 0.0002$ and $\rho^2(\boldsymbol{B}) < 0.0002$.

Taking $\boldsymbol{F}$ to be an $\boldsymbol{H}_n$ in (4.2) gives $\boldsymbol{B} \subset \boldsymbol{A}$ with $\rho^2(\boldsymbol{A}) \le (1/n)\rho^2(\boldsymbol{D}_0)$ and $\rho^2(\boldsymbol{B}) \le (1/m)\rho^2(\boldsymbol{D}_0)$, where $\boldsymbol{A}$ can accommodate more factors than $\boldsymbol{B}$.

The second proposed approach takes $\boldsymbol{D}_0 \subset \boldsymbol{D}_1$ to be a pair of nested (nearly) orthogonal designs, both with zero mean for each column. Let $\boldsymbol{F}$ be an $\boldsymbol{H}_m$. Put

$$\boldsymbol{A} = \boldsymbol{F} \otimes \boldsymbol{D}_1,\ \boldsymbol{B} = \boldsymbol{F} \otimes \boldsymbol{D}_0. \tag{4.3}$$

Remark 3. For $\boldsymbol{A}$ and $\boldsymbol{B}$ constructed above, (i) $\rho^2(\boldsymbol{A}) \le (1/m)\rho^2(\boldsymbol{D}_1)$; (ii) $\boldsymbol{B} \subset \boldsymbol{A}$ and $\rho^2(\boldsymbol{B}) \le (1/m)\rho^2(\boldsymbol{D}_0)$.

Example 7. Let $\boldsymbol{D}_0 \subset \boldsymbol{D}_1$ be the NOLH$(16, 4, 2)$ from Table 2. Let $\boldsymbol{F}$ be an $\boldsymbol{H}_m$. Then $\boldsymbol{B} \subset \boldsymbol{A}$ in (4.3) can accommodate $2m$ factors. From Remark 3, both $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal.

Note that Qian, Ai, and Wu (2009) and Qian,, Tang, and Wu (2009) define a nested orthogonal array to be an orthogonal array containing a subarray that becomes a smaller orthogonal array after some suitable level-mapping. Inspired by this definition, in a future project we will define a nested Latin hypercube as follows. For integers $n_1 > n_2$ with n_2 dividing n_1, NLH(n_1, n_2, m) denotes a Latin hypercube of n_1 levels in m factors containing a subarray of n_2 runs that becomes an LH(n_2, m) after collapsing the n_2 groups of n_1/n_2 consecutive levels to n_2 equally spaced levels. Methods for constructing nested Latin hypercube designs with (nearly) orthogonal columns according to this definition will be developed.

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