Test of isotropy for rough textures of trended images.

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Supplementary Material

This supplementary material contains proofs for Theorem 3.4 and some propositions stated in the main paper (Sections S1 and S2). It also includes a description of a method used to estimate the covariance of quadratic variations (Section S3).

S1 Convergence study

S1.1 A multivariate Breuer Major Theorem

The original Breuer-Major theorem was shown for stationary processes in Breuer and Major (1983), and extended to multivariate fields by Arcones (1994). Another formulation of the Breuer-Major Theorem is demonstrated by Biermé et al. (2011, Theorem 3.2) using the Malliavin calculus. We state a specific version of this theorem which is sufficient for the proof of Theorem 3.4.

Theorem 1 (Breuer-Major theorem)**.** *Let $d, l \in \mathbb{N}^*$, and $X_N = (X_N[k])_{k\in\mathbb{Z}^d}$ be centered Gaussian stationary fields with values in $\mathbb{R}^l$. Assume that there exist functions $g_{a,b}^N$ in $L^2([0,2\pi]^d)$ (spectral densities) such that, for all $a, b \in [\![1, l]\!]$ and $k \in \mathbb{Z}^d$,*

$$\mathrm{Cov}(X_a^N[k], X_b^N[0]) = \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} e^{i\langle w,k\rangle} g_{a,b}^N(w)dw.$$

Further assume that, for all $a, b \in [\![1, l]\!]$, $g_{a,b}^N$ converges in $L^2([0,2\pi]^d)$ to a function $g_{a,b}$ as N tends to $+\infty$. Define

$$\forall\, k \in \mathbb{Z}^2,\ r_{a,b}[k] = \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} e^{i\langle w,k\rangle} g_{a,a}(w)dw,$$

and assume that $r_{a,b}[0] = 1$. Then,

$$\frac{1}{N^{d/2}}\sum_{k\in[\![1,N]\!]^d}((X^N[k])^2 - \mathbf{1}) \xrightarrow[N\to+\infty]{d} \mathcal{N}(0,\Sigma),$$

where $\mathbf{1}$ is the unit vector of size l and Γ is a $l\times l$-matrix having terms

$$\Gamma_{a,b} = 2\sum_{k\in\mathbb{Z}^d}(r_{a,b}[k])^2 = \frac{2}{(2\pi)^d}\int_{[0,2\pi]^d}|g_{a,b}(w)|^2dw.$$

S1.2 Proof of Theorem 3.4

In a first part, we prove the asymptotic normality of the random vector

$$U^N = \left(\frac{W_a^N}{\mathbb{E}(W_a^N)} \right)_{a \in \mathcal{F}} \tag{S1.1}$$

defined with quadratic variations W_a^N of Equation (23) (main paper). Then, we deduce the asymptotic normality (26) (main paper) of $Y^N = (Y_a^N)_{a\in\mathcal{F}}$. In a second part, we further specify terms of this convergence.

Part 1. For establishing the asymptotic normality of U^N, we use a multivariate version of the Breuer-Major theorem recalled above. For that, let us first notice that

$$N^{\frac{d}{2}}(U_a^N - 1) = \frac{N^{\frac{d}{2}}}{N_e} \sum_{m \in \mathcal{E}_N} \Big((X_a^N[k])^2 - 1 \Big) \underset{N\to+\infty}{\sim} \frac{1}{N^{\frac{d}{2}}} \sum_{k \in [\![1,N]\!]^d} \Big((X_a^N[k])^2 - 1 \Big),$$

with $X_a^N[m] = V_a^N[m]/\sqrt{\mathbb{E}((V_a^N[m])^2)}$. So, if the Breuer-Major theorem could be applied to the vector-valued random field $X^N = ((X_a^N[m])_{a\in\mathcal{F}}, m \in \mathbb{Z}^d)$, it would follow that

$$N^{\frac{d}{2}}(U^N - \mathbf{1}) \underset{N\to+\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(0, \Sigma), \tag{S1.2}$$

where $\mathbf{1}$ is the unit vector of the same size as U^N, and Σ is a covariance matrix. But, using Proposition 3.3 (main paper), the spectral density of X^N can be specified as

$$g_{a,b}^N(w) = \frac{f_{a,b}^N(w)}{\sqrt{\mathbb{E}((V_a^N[m])^2)}\sqrt{\mathbb{E}((V_b^N[m])^2)}},$$

where $\mathbb{E}((V_a^N[m])^2) = \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} f_{a,a}^N(w)dw$. Thus, it suffices to show the convergence in $L^2([0,2\pi]^d)$ of $g_{a,b}^N$. This convergence results from the next lemma whose proof is postponed at the end of the section.

Lemma 1. *Take the same conditions as in Theorem 3.4 (main paper). Consider the multivariate spectral density $f_{a,b}^N$ of V^N given by Equation (22) of Proposition 3.3 (main paper). Then, for any $a, b \in \mathcal{F}$, as N tends to $+\infty$, $N^{2H} f_{a,b}^N$ converges in $L^2([0,2\pi]^d)$ to the function $f_{a,b}$ defined by Equation (28) (main paper).*

Due to Lemma 1, $N^{2H} f_{a,b}^N$ tends to $f_{a,b}$ in $L^2([0,2\pi]^d)$ and, *a fortiori* in $L^1([0,2\pi]^d)$. Hence, for $a \in \mathcal{F}$, $N^{2H}\mathbb{E}((V_a^N[m])^2)$ converges to

$$C_a = \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} f_{a,a}(w)dw. \tag{S1.3}$$

Therefore, $g_{a,b}^N$ tends in $L^2([0,2\pi]^d)$ to $g_{a,b} = f_{a,b}/\sqrt{C_a C_b}$.

Consequently, the Breuer-Major theorem yields the asymptotic normality (26) (main paper) for a covariance matrix Σ whose terms are defined by Equation (27) (main paper).

Now, let G be the differentiable function mapping $(\mathbb{R}_*^+)^{n_f}$ into $\mathbb{R}^{n_f}$ defined by $G(y)_a = \log(y_a)$ for $a \in \mathcal{F}$. Since $N^{d/2}(U^N - \mathbf{1}) \xrightarrow[N\to+\infty]{d} \mathcal{N}(0,\Sigma)$, we have

$$N^{d/2}(G(U^N) - G(\mathbf{1})) \xrightarrow[N\to+\infty]{d} \nabla G(\mathbf{1})' \mathcal{N}(0,\Sigma).$$

using the multivariate Δ-method. But, for $a \in \mathcal{F}$,

$$G(U^N)_a - G(\mathbf{1})_a = Y_a^N + \log(N^{2H}) - \log(C_a) + R_a^N,$$

where $R_a^N = \log(C_a) - \log(N^{2H}\mathbb{E}(W_a^N))$ and C_a is defined by Equation (S1.3). Moreover, due to Lemma 1, $\lim\limits_{N\to+\infty} R_a^N = 0$. Hence, the asymptotic normality (26) (main paper) follows for $\zeta^N = (\log(C_a) - \log(N^{2H}))_{a\in\mathcal{F}}$ and Σ defined by Equation (27) (main paper).

Part 2. Let us notice that

$$C_a = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \int_{\mathbb{R}^d} |\hat{v}(T_a'(w+z))|^2 \delta\left(\frac{(w+z)}{|w+z|}\right) |w+z|^{-2H-d} d\Delta(z) dw,$$

where $d\Delta(z) = \sum_{k\in\mathbb{Z}^d} \delta_{2k\pi}(z)$ is a counting measure on $\mathbb{R}^d$. But, $C_a < +\infty$. Hence, by application of the Lebesgue-Fubini theorem, we obtain

$$C_a = \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{Z}^d} \int_{[0,2\pi]^d} |\hat{v}(T_a'(w+2k\pi))|^2 \delta\left(\frac{w+2k\pi}{|w+2k\pi|}\right) |w+2k\pi|^{-2H-d} dw.$$

After a variable change $\zeta = w + 2k\pi$ in each integral, we further get

$$C_a = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{v}(T_a'\zeta)|^2 \delta\left(\frac{\zeta}{|\zeta|}\right) |\zeta|^{-2H-d} d\zeta.$$

Next, using the variable change $w = |u_a|\zeta$, we have $C_a = |u_a|^{2H} C_H(\arg(u_a), v)$, where $\arg(u_a)$ is the angle of the rotation $\frac{T_a'}{|u_a|}$ and $C_H(\arg(u_a), v)$ is defined by Equation (31) (main paper). Then, the expression of ζ given by Equations (29) and (30) of the main paper follows.

Furthermore, let us notice that, when the texture of Z is isotropic, the function $\delta \equiv \tau_0 \in \mathbb{R}_*^+$. Hence, in this case, we obtain

$$C_H(\arg(u_a), v) = \frac{\tau_0}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{v}(w)|^2 |w|^{-2H-d} d\zeta.$$

by applying the variable change $w = \frac{T_a'}{|u_a|}\zeta$. Therefore, in this case, $C_H(\arg(u_a), v)$ only depends on v.

Proof of Lemma 1. Let us define

$$G^N(w) = N^{2H+d} f(Nw) - \delta\left(\frac{w}{|w|}\right) |w|^{-2H-d},$$

and, for $L \in \mathbb{N}^*$, consider $S_L^N(w) = \sum_{0<|k|\leq L} G^N(w + 2k\pi)$. Since f satisfies Condition (3) (main paper), we have

$$f(N(w+2k\pi)) \leq C_N |k|^{-2H-d},$$

for $k \neq 0$, $w \in [0, 2\pi]^d$, and N sufficiently large. Hence, $\sum_{0<|k|} f(N(w + 2k\pi))$ is normally convergent, and, as L tends to $+\infty$, S_L^N tends uniformly to

$$S^N = N^{2H+d} \sum_{0<|k|} f(N(w + 2k\pi)) - \sum_{0<|k|} \delta\left(\frac{w + 2k\pi}{|w + 2k\pi|}\right) |w + 2k\pi|^{-2H-d}.$$

Let us now consider

$$U^N = \int_{[0,2\pi]^d} |\hat{v}(T_a' w)\hat{v}(T_b' w)|^2 \left(S^N(w) + G^N(w)\right)^2 dw,$$

and show that U^N tends to 0 as N tends to $+\infty$.

First, let us quote that, for all $w \in [0, 2\pi]^d$, $S^N(w)$ is bounded by

$$I^N = \int_{\mathbb{R}^d \setminus B(0,A)} |G^N(w)| dw = \int_{S^{d-1}} \int_A^{+\infty} |G^N(\rho s)| \rho^{d-1} d\rho ds,$$

where $B(0, A)$ denotes a ball of $\mathbb{R}^d$ centered at 0 of radius $0 < A \leq 2\pi$. Further notice that

$$|G^N(\rho s)| \leq G_1^N(\rho s) + G_2^N(\rho s), \tag{S1.4}$$

$$\text{where } G_1^N(\rho s) = N^{2H+d} \left| f(N\rho s) - \tau(s)(N\rho)^{-2\beta(s)-d} \right|, \tag{S1.5}$$

$$\text{and } G_2^N(\rho s) = \left| N^{2(H-\beta(s))} \tau(s) \rho^{-2\beta(s)-d} - \delta(s) \rho^{-2H-d} \right|. \tag{S1.6}$$

Hence, $I^N \leq I_1^N + I_2^N$ with $I_j^N = \int_{S^{d-1}} \int_A^{+\infty} G_j^N(\rho s) \rho^{d-1} d\rho ds$.
Since f satisfies Condition (3) (main paper) and τ is bounded, we have

$$I_1^N \leq c_1 N^{-\gamma} A^{-2H-\gamma},$$

for some $c_1 > 0$, and large N. So, $\lim_{N \to +\infty} I_1^N = 0$.

Besides, for $\eta > 0$, let us define sets

$$E_\eta = \{s \in S^{d-1}, H < \beta(s) < H + \eta\} \text{ and } F_\eta = \{s \in S^{d-1}, \beta(s) \geq H + \eta\}. \tag{S1.7}$$

When $s \in E_\eta \cup F_\eta$, $\delta(s) = 0$, so that $G_2^N(\rho s) \leq c N^{2(H-\beta(s))} \rho^{-2\beta(s)-d}$. When $s \in E_0$, $\tau(s) = \delta(s)$ and $\beta(s) = H$, so that $G_2^N(\rho s) = 0$. Hence,

$$I_2^N \leq c_2 \int_{E_\eta \cup F\eta} N^{2(H-\beta(s))} A^{-2\beta(s)} ds.$$

Thus, $I_2^N \leq \tilde{c}_2(\mu(E_\eta) + N^{-2\eta})$, where $\mu(E_\eta)$ is the Lebesgue measure of E_η over the sphere S^{d-1}. Let us show that $\lim_{\eta \to 0^+} \mu(E_\eta) = 0$. Assume it is not the case. Then, there exists $c_0 > 0$ and a decreasing sequence $(\eta_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to +\infty} \eta_n = 0$ and $\mu(E_{\eta_n}) > c_0$. Since $E_\eta \subset E'_\eta$ when $\eta < \eta'$, the sequence $\mu(E_\eta)$ decreases, and admits a positive limit as η decreases to 0. This implies that $\mu(\underset{\eta < \eta_0}{\cap} E_\eta) > 0$. So, take $s \in \underset{\eta < \eta_0}{\cap} E_\eta$. It satisfies $H < \beta(s) < H + \eta$ for all

$\eta > 0$. This yields $\beta(s) = H$, which is contradictory.
Now, for $0 < \alpha < 1$, let us set $\eta_N = \log(N)^{-\alpha}/2$. We then obtain

$$I_2^N \leq \tilde{c}_2(\mu(E_{\log(N)^{-\alpha}}) + e^{-\log(N)^{1-\alpha}}), \tag{S1.8}$$

and $\lim_{N\to+\infty} I_2^N = 0$. Therefore, $\lim_{N\to+\infty} I^N = 0$. Thus, on $[0, 2\pi]^d$, S^N converges uniformly to 0, as N tends to 0.

Consequently, the integral U^N is bounded by $c_1(\sup_{w\in[0,2\pi]^d} |S^N(w)|)^2 + c_2 J^N$, where

$$J^N = \int_{[0,2\pi]^d} |\hat{v}(T'_a w)\hat{v}(T'_b w)|^2((G^N(w))^2 + G^N(w))dw.$$

Let us then decompose J^N into a sum of two integrals $J_1^N = \int_{B(0,A)} \cdots dw$ and $J_2^N = \int_{[0,2\pi]^d\setminus B(0,A)} \cdots dw$, and study separately these integrals.

Notice that $\hat{v}(y) = Q_v(e^{iy_1}, \cdots, e^{iy_d})$ where Q_v is the characteristic polynomial of v. Hence, using Proposition 3.2 and Taylor expansions of Q_v in the neighborhood of 0, we obtain $|\hat{v}(y)|^2 \leq C|y|^{2K+2}$ for some $C > 0$. Therefore, J_1^N is bounded by

$$\begin{aligned} &c_3 N^{4(H-K-1)+d} \textstyle\int_{B(0,\frac{A}{N})} |w|^{4(K+1)}(f^2(w) + f(w)|w|^{-2H-d} + |w|^{-4H-2d})dw \\ &+c_4 N^{2(H-2K-2)} \textstyle\int_{B(0,\frac{A}{N})} |w|^{4(K+1)}(f(w) + |w|^{-2H-d})dw, \end{aligned}$$

for some $c_3, c_4 > 0$. In this upper bound, integrals of the form $\int_{B(0,\epsilon)} |w|^u f(w)dw$ are finite for both $u = 4(K+1) - 2H - d$ and $u = 4(K+1)$, since Z is a M-IRF and $K \geq M/2 + d/4$. Moreover, $\sup_{w\in B(0,\epsilon)} |w|^{2K+3} f(w) \leq c < +\infty$, since Z is a M-IRF and $K_a \geq M$. Therefore,

$$\int_{B(0,\epsilon)} |w|^{4(K+1)} f^2(w)dw \leq c \int_{B(0,\epsilon)} |w|^{2K+1} f(w)dw < +\infty,$$

since $K \geq M+1$. Besides, integrals of the form $\int_{B(0,\epsilon)} |w|^u dw$ are finite for $u = 4(K+1)-2H-d$ and $u = 4(K+1) - 4H - 2d$, since $K \geq d/4$. Consequently,

$$J_1^N \leq \tilde{c}_3 N^{4(H-K-1)+d} + \tilde{c}_4 N^{2(H-2K-2)},$$

and $\lim_{N\to+\infty} J_1^N = 0$, since $K \geq d/4$.
Besides, using the bound (S1.4), we obtain

$$J_2^N \leq \int_{S^{d-1}} \int_A^{+\infty} (G_1^N(\rho s) + G_2^N(\rho s))^2 \rho^{d-1} d\rho ds.$$

Then, using previous bounds on G_1^N and G_2^N, we get

$$J_2^N \leq c_5 \left(N^{-2\gamma} + N^{-2\eta} + \mu(E_\eta)\right) \Gamma_H,$$

with $c_5 > 0$ and $\Gamma_H = \int_A^{+\infty} \rho^{-4H-d-1} d\rho < +\infty$. Setting again $\eta_N = \log(N)^{-\alpha}/2$ with $0 < \alpha < 1$, we obtain $\lim_{N\to+\infty} J_2^N = 0$. Therefore, $\lim_{N\to+\infty} J^N = 0$.

Consequently, $\lim_{N\to+\infty} U^N = 0$. This implies the convergence of $N^{2H} f_{a,b}^N$ to $f_{a,b}$ in $L^2([0, 2\pi]^d)$.

S2 Proofs of Propositions

Proof of Proposition 3.1. By definition, the kernel v leads to K-order increments of Z if and only if $\sum_{k\in[\![0,L]\!]^d} v[k](m-k)^l = 0, \forall\, m \in \mathbb{Z}^d, \forall l \in [\![0,K]\!]^d, |l| \leq K$, which is also equivalent to

$$\sum_{k\in L} v[k]k^l = 0, \forall\, l \in [\![0,K]\!]^d, |l| \leq K.$$

Besides, we have

$$\frac{\partial^{|l|}\mathcal{Q}_v}{\partial z^l}(z) = \sum_{k_1=l_1}^{L_1}\cdots\sum_{k_d=l_d}^{L_d} v[k]\prod_{j_1=0}^{l_1-1}(k_1-j_1)\cdots\prod_{j_d=0}^{l_d-1}(k_d-j_d)z^{k-l},$$

using the convention that $\prod_{j_i=0}^{l_i-1}(k_i-j_i) = 1$ if $l_i = 0$. From that, we deduce the recurrence equations

$$\frac{\partial^{|l|}\mathcal{Q}_v}{\partial z^l}(z) = \sum_{k\in[\![0,L]\!]^d} v[k]k^l z^{k-l} - \sum_{j\in[\![1,d]\!], l_j\geq 1}\frac{l_j-1}{z_j}\frac{\partial^{|l|-1}\mathcal{Q}_v}{\partial z^{l-e_j}}(z),$$

where e_j is the jth vector of the canonical basis of $\mathbb{R}^d$. In particular,

$$\forall j, l,\ \frac{\partial^{|l|}\mathcal{Q}_v}{\partial z^l}(1,\cdots,1) + \sum_{j\in[\![1,d]\!], l_j\geq 1}(l_j-1)\frac{\partial^{|l|-1}\mathcal{Q}_v}{\partial z^{l-e_j}}(1,\cdots,1) = \sum_{k\in[\![0,L]\!]^d} v[k]k^l.$$

We conclude the proof by recurrence on the order of the partial derivatives of Q. □

Proof of Proposition 3.2. Let P be a polynomial of degree $l \leq K$. We notice

$$\sum_k v[k]P\left(\frac{m-T_u k}{N}\right) = \sum_k v[k]P\circ T_u\left(\frac{m'-k}{N}\right), \text{with } m' = T_u^{-1}m.$$

But, any rescaling or rotation $P\circ T_u$ of a polynomial P remains a polynomial of the same degree. Hence, $\sum_k v[k]P\left(\frac{m-T_u k}{N}\right) = 0$ for all P of degree $l \leq K$ if and only if $\sum_k v[k]P\left(\frac{m'-k}{N}\right) = 0$ for all P of degree $l \leq K$. According to Proposition 3.1, this only holds if an and only if Condition (17) (main paper) is satistied. □

Proof of Proposition 3.3. Since, for $a \in \mathcal{F}$ and $m \in \mathbb{Z}^d$, $V_a^N[m]$ is an increment of Z of order $\geq M$, it has zero mean. Moreover, for any $a, b \in \mathcal{F}$ and $m, n \in \mathbb{Z}^d$, Theorem 2.3 yields

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \sum_{k,l\in\mathbb{Z}^d} v[k]v_b[l]K_Z\left(\frac{m-T_a k}{N} - \frac{n-T_b l}{N}\right) < +\infty,$$

where K_Z is a generalized covariance of the form (8). But, for any even polynomial P of degree $2M$, we can write $P(x-y) = \sum_{|l|=0}^{M} q_l(y)x^l + \sum_{|l|=0}^{M} q_l(x)y^l$ where q_l are polynomials of degree up to $2M$. Hence, since $V_a^N[m]$ and $V_b^N[n]$ are increments of order $K \geq M$,

$$\sum_{k,l\in\mathbb{Z}^d} v[k]v[l]P\left(\frac{m-T_a k}{N} - \frac{n-T_b l}{N}\right) = 0,$$

for any even polynomial P of degree $2M$, including P_M and Q of Equation (8). Therefore,

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\sum_{k,l\in\mathbb{Z}^d} v[k]v[l]\cos\left(\left\langle \frac{m-T_ak}{N} - \frac{n-T_bl}{N}, w\right\rangle\right) f(w)dw,$$

and, since f is even,

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\sum_{k,l\in\mathbb{Z}^d} v[k]v[l]e^{i\langle \frac{m-n}{N} - \frac{T_ak}{N} + \frac{T_bl}{N}, w\rangle} f(w)dw.$$

From this, we deduce

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{v}\left(\frac{T_a'w}{N}\right)\bar{\hat{v}}_b\left(\frac{T_b'w}{N}\right)e^{i\langle \frac{m-n}{N}, w\rangle} f(w)dw.$$

Since this expression exclusively depends on $m-n$, and not on m and n, V^N is stationary. Using a variable change $\zeta = w/N$, and a decomposition of the integral domain, we further obtain

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}\int_{[0,2\pi]^d+2k\pi}\hat{v}(T_a'\zeta)\bar{\hat{v}}(T_b'\zeta)e^{i2\pi\langle m-n,\zeta\rangle} f(N\zeta)N^d d\zeta,$$

After a variable change $\zeta = w + 2k\pi$ in each integral, we then get

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{[0,2\pi]^d} g_{a,b}^N(w,z)dwd\Delta(z),$$

where $d\Delta(u) = \sum_{k\in\mathbb{Z}^d}\delta_{2k\pi}(u)$ is a counting measure on $\mathbb{R}^d$ and $g_{a,b}^N(w,z) = N^d\hat{v}(T_a'w)\bar{\hat{v}}(T_b'w)e^{i\langle m-n,w\rangle} f(N(w+z))$. Since $\mathbb{E}(V_a^N[m]V_b^N[n]) < +\infty$, the Lebesgue Fubini theorem implies that $f_{a,b}^N(w) = \int_{\mathbb{R}^d} g_{a,b}^N(w,z)d\Delta(z)$ is almost everywhere defined, and that $\int_{[0,2\pi]^d} f_{a,b}^N(w)dw < +\infty$. □

S3 Covariance estimation

In this section, we construct an estimate of the covariance matrix Σ^N of log-variations Y^N involved in the linear model (29) (main paper). According to the proof in Section S1, the random vector Y^N has the same asymptotical covariance Σ as the random vector U^N defined by Equation (S1.1). Hence, we approximate Σ^N by an estimate of the covariance matrix of U^N. We have

$$\mathbb{E}(U_a^N U_b^N) = \frac{1}{N_e^2\mathbb{E}(W_a^N)\mathbb{E}(W_b^N)}\sum_{p,q\in\mathcal{E}_N}\mathbb{E}((V_a^N[p])^2(V_a^N[q])^2).$$

But $(V_a^N[p], V_a^N[q])$ are centered Gaussian vectors. Thus, $E((V_a^N[p])^2(V_a^N[q])^2) = 2(E(V_a^N[p]V_a^N[q]))^2$. Moreover,

$$E(V_a^N[p]V_a^N[q]) = \sum_{k,l} v[k]v[l]K_Z\left(\frac{p-T_ak-q+T_bl}{N}\right),$$

where K_Z is the generalized covariance of the IRF Z. Let $\tilde{H}$ be an estimate of the Hölder irregularity of Z (*e.g.* an OLS estimate of H in the linear model (29) (main paper). Approximating the generalized covariance K_Z by the one of a fractional Brownian field of order $\tilde{H}$, it follows that

$$E(V_a^N[p]V_a^N[q]) \simeq C_{\tilde{H}} \sum_{k,l} v[k]v[l]|p - T_a k - q + T_b l|^{2\tilde{H}}.$$

Using the same approximation, we also have

$$\mathbb{E}(W_a^N) = \mathbb{E}((V_a^N[0])^2) \simeq \sum_{k,l} v[k]v[l]|T_a(l-k)|^{2\tilde{H}}.$$

Hence, we get

$$\Sigma_{a,b}^N \simeq \sum_{\delta \in \Delta\mathcal{E}_N} \frac{N_\delta \left(\sum_{k,l} v[k]v[l]|\delta - T_a k + T_b l|^{2\tilde{H}}\right)^2}{N_e^2 \left(\sum_{k,l} v[k]v[l]|T_a(l-k)|^{2\tilde{H}}\right)\left(\sum_{k,l} v[k]v[l]|T_b(l-k)|^{2\tilde{H}}\right)},$$

where $\Delta\mathcal{E}_N = \{\delta = p - q, p, q \in \mathcal{E}_N\}$ and N_δ is the number of couples $(p,q) \in \mathcal{E}_N^2$ for which $\delta = p - q$.

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