DESIGNS OF VARIABLE RESOLUTION ROBUST TO NON-NEGLIGIBLE TWO-FACTOR INTERACTIONS

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Supplementary Materials

The supplementary materials contain an example of Construction 2 in the article, additional constructions due to Lin (2012) for robust designs of variable resolution, and the proofs of Propositions 7 and 8 in the article.

S1 An Example of Construction 2

Example S1 below uses Construction 2 to build robust designs of variable resolution.

Example S1. Let A be a $D(4, 3, 3)$ and B_i be a saturated design of 16 runs for $i = 1, 2, 3$. It is well known that B_i^*, a sub-design of resolution V from B_i, can have up to 5 columns (Wu and Hamada (2011)). Now take these 5 columns and the column of all 1's to be E_i^*. Construction 2 provides a $RD\{64, (6, 6, 6, 10, 10, 10), (6, 6, 6, 4, 4, 4); 3\}$.

S2 Additional Constructions for Robust Designs of Variable Resolution

The following constructions correspond to Constructions 1 and 2, and the first and second constructions in Section 3.3 in Lin (2012).

Construction S1. Let $A = (a_{ij})$ be an $n_1 \times m_1$ matrix with entries $a_{ij} = \pm 1$ and $d_0 = (D_{01}, \ldots, D_{0k})$ with D_{0i} a design of n_2 runs for p_i factors for $i = 1, \ldots, k$. Construction 1 in Lin (2012) gives

$$d = A \otimes d_0 = (A \otimes D_{01}, \ldots, A \otimes D_{0k}). \tag{S2.1}$$

Proposition S1. *Design d in (S2.1) is a $RD\{n_1n_2, (m_1p_1, \ldots, m_1p_k), (r_1, \ldots, r_k); r\}$ if the following hold simultaneously: (i) A is column-orthogonal; (ii) the r_i's and r are 3 or 4; and (iii) d_0 is a $RD\{n_2, (p_1, \ldots, p_k), (r_1, \ldots, r_k); r\}$.*

Construction S2. Let $c_1 = (c_{1i})$ be a column of n_2 entries with $c_{1i} = \pm 1$, $c_2 = (c_{2i})$ be a column of n_1 entries with $c_{2i} = \pm 1$, $d_{01} = (D_{11}, \ldots, D_{1k_1})$ be a $D\{n_1, (p_1, \ldots, p_{k_1}), (r_1, \ldots, r_{k_1}); r\}$, and $d_{02} = (D_{21}, \ldots, D_{2k_2})$ be a $D\{n_2, (q_1, \ldots, q_{k_2}), (s_1, \ldots, s_{k_2}); s\}$. Construction 2 in Lin (2012) gives

$$d = (D_1, \ldots, D_{k_1+k_2}), \tag{S2.2}$$

where $D_i = D_{1i} \otimes c_1$ for $i = 1, \ldots, k_1$ and $D_{j+k_1} = c_2 \otimes D_{2j}$ for $j = 1, \ldots, k_2$.

Proposition S2. *Design d in (S2.2) is a RD$\{n_1n_2, (p_1, \ldots, p_{k_1}, q_1, \ldots, q_{k_2}), (r_1, \ldots, r_{k_1}, s_1, \ldots, s_{k_2}); \min(r, s)\}$ if (i) $c_1 = \mathbf{1}_{n_2}$ or (ii) $c_2 = 1_{n_1}$, where 1_u is a column of u 1's.*

Construction S3. Let $A = (a_1, \ldots, a_{p_1})$ be a $D(n_1, p_1, r)$ and $B = (b_1, \ldots, b_{p_2})$ be a $D(n_2, p_2, s)$, where $r \geq 3$, $s \geq 3$ and $p_1 \geq p_2$. Let $K = \min(p_1 - 1, p_2)$. The first construction of Section 3.3 in Lin (2012) gives

$$d = (D_1, \ldots, D_K), \tag{S2.3}$$

where, for $k = 1, \ldots, K$,

$$D_k = (c_{k+1,k}, \ldots, c_{p_1,k}, d_{k,k}, \ldots, d_{p_2,k}),$$

with $c_{i,k} = a_i \otimes b_k$ for $i = k+1, \ldots, p_1$ and $d_{j,k} = a_k \otimes b_j$ for $j = k, \ldots, p_2$.

Proposition S3. *Design d in (S2.3) is a RD$\{n_1n_2, (p_1+p_2-1, p_1+p_2-3, \ldots, p_1+p_2-2K+1), (4, \ldots, 4); \min(r, s)\}$ if $r = 4$ or $s = 4$.*

Construction S4. Let $A = (a_1, \ldots, a_{p_1})$ be a $D(n_1, p_1, 5)$ if $p_1 \geq 4$ and be a $D(n_1, p_1, p_1 + 1)$ if $p_1 \leq 3$. Further, let $B = (b_1, \ldots, b_{p_2})$ be a $D(n_2, p_2, 3)$. The second construction in Section 3.3 in Lin (2012) gives

$$d = (D_1, \ldots, D_{p_1}), \tag{S2.4}$$

where, for $k = 1, \ldots, p_1$,

$$D_k = (c_{k,k}, \ldots, c_{p_1,k}, d_{1,k}, \ldots, d_{p_2,k}),$$

with $c_{i,k} = (a_{k-1}a_i) \otimes \mathbf{1}_{n_2}$ for $i = k, \ldots, p_1$, $d_{j,k} = (a_1a_2a_{k-1}) \otimes b_j$ for $j = 1, \ldots, p_2$, and a_0 is a column of n_1 1's.

Proposition S4. *Design d in (S2.4) is a RD$\{n_1n_2, (p_1+p_2, p_1+p_2-1, \ldots, p_2+1), (4, \ldots, 4); 3\}$.*

Propositions S1 - S4 provide conditions for designs in Constructions S1 - S4 to be robust designs of variable resolution. Proposition S1 requires that Construction S1 uses a robust design of variable resolution to build a larger one. Proposition S3 is analogous to, but different from, Proposition S1 in Lin (2012) in that the former requires either A or B in Construction S3 to be of resolution IV. Propositions S2 and S4 reveal that the corresponding original constructions in Lin (2012) in fact provide robust designs of variable resolution. The proof of these propositions is straightforward and thus omitted.

S3 Proof of Proposition 7

Within the group of factors forming $D_1 = (d_{11}, \ldots, d_{1p_2})$, the $p_2 - 1$ two-factor interactions $d_{11}d_{12}, \ldots, d_{11}d_{1p_2}$ must be orthogonal to each other, and also orthogonal to the p_1p_2 main effects. We then have that

$$p_1p_2 + p_2 - 1 \leq n - 1.$$

S4 Proof of Proposition 8

To prove Proposition 8, it is equivalent to only use three groups and show that there does not exist a $RD(n, (p_1, p_2, p_3), (r_1, r_2, r_3); r)$ with $r_i \geq 4$ and $r = 3$ if all $D(n, m, 4)$'s are fold-overs for $m \leq n/2$. Suppose that there exists a $d = (D_1, D_2, D_3) = RD(n, (p_1, p_2, p_3), (r_1, r_2, r_3); 3)$ with $r_i \geq 4$. By definition, d has C_3 in (4.1) equal to 0. By Corollary 1 and all n-run designs of resolution IV under consideration are fold-over, d must be of the form

$$d = \begin{pmatrix} A & B & E_1 \\ -A & -B & E_2 \end{pmatrix},$$

where $D_1 = (A^T, -A^T)^T$, $D_2 = (B^T, -B^T)^T$, and $D_3 = (E_1^T, E_2^T)^T$. If all resolution IV designs of n runs are fold-over, d must be a fold-over design. To see this, by Corollary 1, there must exist a row permutation π such that $A^* = \pi(A)$, $E_2 = -E_1$, and

$$\begin{pmatrix} A^* & E_1 \\ -A^* & -E_1 \end{pmatrix}.$$

Now apply the permutation π to B and denote the resulting design by B^*, we have that

$$d = \begin{pmatrix} A^* & B^* & E_1 \\ -A^* & -B^* & -E_1 \end{pmatrix}. \quad \text{(S4.5)}$$

For d in (S4.5), both $B_{(i,3)}$ in (2.4) and $B_{(i,1),(j,1),(l,1)}$ in (4.2) are 0. Since $r = 3$, we have $B_3 \neq 0$ and thus by (4.1), $C_3 > 0$. This leads to a contradiction, thus we complete the proof.