Generalized Partial Linear Model with Unknown Link and Unknown Baseline Functions for Longitudinal Data (Supplementary Material: conditions and proofs of Theorems)

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S1 Conditions

To show asymptotic results of our estimators, we need to make following assumptions.

1. Let K_1, K_2 ,K and $\mathcal{K}$ be bounded and symmetric function with a support $[-1,1]$ as well as bounded and continuous second derivatives. We assume that K_1, K_2 and K have orders r_1, r_2 and r_0, respectively. For example, $\int_{-1}^{1}\mu^j K_1(\mu)d\mu = \begin{cases} 1 & j=0 \\ 0 & 1\le j\le r_1-1 \\ \neq 0 & j=r_1 \end{cases}$.
2. There exists a sequence $\widehat{\boldsymbol{\beta}}=\widehat{\boldsymbol{\beta}}_n$ such that $\|\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}\|=O_p(n^{-1/2})$.
3. The derivatives $p^{(k_1)}(z)$, $p^{(k_1,k_2)}(z,t)$, $\mu^{(k_1,k_2)}(z,t)$, $\varrho^{(k_1,k_2)}(z,t)$, $\rho^{(k_2)}(t)$,

$V^{(k_2)}(t)$, $k_1 = 1, \cdots, r_1+1, k_2 = 1, \cdots, r_2+1$; $q^{(k)}(w)$, $f^{(k)}(w)$, $m^{(k)}(w)$, $k = 1, \cdots, s+1$ exist and are uniformly bounded over z, t and w.

4. The random vector $\mathbf{X}$ and T are associated with bounded supports.

5. $\frac{\log n}{\sqrt{nh_1^5 h_2}} \to 0$, $\frac{\log n}{\sqrt{nh_1^3 h_2^3}} \to 0$ and $\frac{\log n}{\sqrt{nh^3}} \to 0$.

The assumptions on h_1, h_2 and h in Theorems 1 and 2 can be satisfied, for example, K_1 is a sixth-order kernel, K_2 and K are fourth-order kernels, $h_1 \propto n^{-1/9}$ and $h_2 \propto n^{-1/5}, h \propto n^{-1/5}$. The higher-order kernel for K_1 is needed to assure sufficiently rapid convergence because $\widehat{\mu}_1(z,t)$ is a functional of derivatives of K_1. The high order kernel can be taken from Muller (1984). The initial value of $\boldsymbol{\beta}$ given in Step 1 is a $\sqrt{n}-$consistent estimator, hence, Assumption 2 can be satisfied. Assumption 3, the existence of higher-order derivatives of $p(z)$, $p(z,t)$, $\mu(z,t)$, $\varrho(z,t)$, $\rho(t)$, $V(t)$, $q(w)$, $f(w)$ and $m(w)$, is needed to assure that bias terms associated with the kernel estimators $\mu_n(z,t)$, $\mu_{1n}(z,t)$, $\mu_{2n}(z,t)$, $p_n(z,t)$ and $E_n(w)$ vanish sufficiently fast (Horowitz, 1996). Assumption 4 is commonly used in nonparametric literature (Horowitz, 1996; Carroll *et al.*, 1997) and can be relaxed at the expense of more complex proofs.

S2 Proofs

S2·1 Lemmas

Lemmas 1-3 establish asymptotic forms of the convergence of $\widehat{p}(\widehat{z}, t)$, $\widehat{py}(\widehat{z}, t)$ and their derivatives, which are estimates of $p(z, t)$, $\mu(z, t)p(z, t)$ and their derivatives. These are used in proving Theorem 1 and Lemma 4, which results in a linear approximation to $\frac{\widehat{\mu}_2(\widehat{z},t)\widehat{p}(\widehat{z},t)}{\widehat{\mu}_1(\widehat{z},t)\widehat{p}(\widehat{z},t)}$ that is used in proving Theorems 1 and 2. Because the arguments used to prove Lemma 3 is essentially same with that in Lemma 2 , Lemma 3 is stated without a proof.

Lemma 1: Under the conditions in Supplementary Material S1, we have

$$\frac{(-1)^{k_1+k_2+k_3}}{h_1^{k_1+1}h_2^{k_2+1}}E\left\{K_1^{(k_1)}\left(\frac{Z_{ij}-z}{h_1}\right)K_2^{(k_2)}\left(\frac{t_{ij}-t}{h_2}\right)\right.$$
$$\left.\times S(Z_{ij}, t_{ij})\right\} = \{S(z, t)p(z, t)\}^{(k_1,k_2)} + O(h_1^{r_1} + h_2^{r_2}),$$

where $k_1, k_2 = 0, 1, 2$, and S is a function, the derivatives $S^{(r_1+1,r_2+1)}(z, t)$ exist and are uniformly bounded over $\mathbf{x} \in \Omega_1$ and $t \in \Omega_2$. Ω_1 and Ω_2 are the supports of $\mathbf{X}_{ij}$ and t_{ij}, respectively, $z = \mathbf{x}'\boldsymbol{\beta}$.

Proof of Lemma 1: see Horowitz (1996).

Lemma 2: Define

$$\begin{aligned}
\widehat{p}(z,t) &= \frac{1}{Nh_1h_2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} K_1\left(\frac{\mathbf{X}_{ij}'\widehat{\boldsymbol\beta}-z}{h_1}\right)K_2\left(\frac{t_{ij}-t}{h_2}\right)\\
p_n(z,t) &= \frac{1}{Nh_1h_2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} K_1\left(\frac{\mathbf{X}_{ij}'\boldsymbol\beta-z}{h_1}\right)K_2\left(\frac{t_{ij}-t}{h_2}\right),\\
\widehat{py}(z,t) &= \frac{1}{Nh_1h_2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} Y_{ij}K_1\left(\frac{\mathbf{X}_{ij}'\widehat{\boldsymbol\beta}-z}{h_1}\right)K_2\left(\frac{t_{ij}-t}{h_2}\right),\\
py_n(z,t) &= \frac{1}{Nh_1h_2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} Y_{ij}K_1\left(\frac{\mathbf{X}_{ij}'\boldsymbol\beta-z}{h_1}\right)K_2\left(\frac{t_{ij}-t}{h_2}\right),
\end{aligned}$$

Under the conditions in Supplementary Material S1, as $n\to\infty$, $h_1\to 0$ and $h_2\to 0$, if $\log n/(nh_1^3h_2)^{1/2}\to 0$, we have

$$\widehat{p}(\mathbf{x}'\widehat{\boldsymbol\beta},t) = p_n(\mathbf{x}'\boldsymbol\beta,t) + (\widehat{\boldsymbol\beta}-\boldsymbol\beta)'\Gamma^{(100)}(\mathbf{x}'\boldsymbol\beta,t,\mathbf{x}) + o_p(\widehat{\boldsymbol\beta}-\boldsymbol\beta),$$

$$\widehat{py}(\mathbf{x}'\widehat{\boldsymbol\beta},t) = py_n(\mathbf{x}'\boldsymbol\beta,t) + (\widehat{\boldsymbol\beta}-\boldsymbol\beta)'\Lambda^{(100)}(\mathbf{x}'\boldsymbol\beta,t,\mathbf{x}) + o_p(\widehat{\boldsymbol\beta}-\boldsymbol\beta),$$

uniformly over $\mathbf{x}\in\Omega_1$ and $t\in\Omega_2$, where $\Gamma(z,t,\mathbf{x}) = -p(z,t)\,[\varrho(z,t)-\mathbf{x}]$, and $\Lambda(z,t,\mathbf{x}) = -p(z,t)\,[\varrho(z,t)-\mathbf{x}]\,\mu(z,t)$.

Proof of Lemma 2. We only present the proof of the first equation, and others can be argued in a similar manner. By the mean value theorem of differential calculus

$$\begin{aligned}
\widehat{p}(\mathbf{x}'\widehat{\boldsymbol\beta},t) &= \frac{1}{Nh_1h_2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} K_1\left(\frac{\mathbf{X}_{ij}'\boldsymbol\beta-\mathbf{x}'\boldsymbol\beta}{h_1}\right)K_2\left(\frac{t_{ij}-t}{h_2}\right)\\
&+\frac{1}{Nh_1^2h_2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} K_1^{(1)}\left(\frac{(\mathbf{X}_{ij}-\mathbf{x})'\boldsymbol\beta}{h_1}\right)K_2\left(\frac{t_{ij}-t}{h_2}\right)(\mathbf{X}_{ij}-\mathbf{x})'(\widehat{\boldsymbol\beta}-\boldsymbol\beta) + o_p(\widehat{\boldsymbol\beta}-\boldsymbol\beta)\\
&= p_n(\mathbf{x}'\boldsymbol\beta,t) + g_n^*(\mathbf{x},t)'(\widehat{\boldsymbol\beta}-\boldsymbol\beta) + o_p(\widehat{\boldsymbol\beta}-\boldsymbol\beta). \qquad \text{(S2.1)}
\end{aligned}$$

By Theorem 2.37 of Pollard (1984, p.34),

$$\sup_{y,\mathbf{X}} |g_n^*(\mathbf{x},t) - Eg_n^*(\mathbf{x},t)| \quad = \quad o[(\log n)/(nh_1^3 h_2)^{1/2}]. \tag{S2.2}$$

Using Lemma 1, we get

$$Eg_n^*(\mathbf{x},t) - \Gamma^{(100)}(z,\mathbf{x},t) = O(h_1^{r_1} + h_2^{r_2}). \tag{S2.3}$$

By (S2.2), (S2.3) and the conditions on h_1 and h_2, we have

$$g_n^*(\mathbf{x},t) - \Gamma^{(100)}(z,t,\mathbf{x}) \to 0. \tag{S2.4}$$

By (S2.1), (S2.4) and the conditions on h_1, h_2, Lemma 2 follows.

Lemma 3: Denote $w = \mathbf{x}'\boldsymbol{\beta} + V(t)$, $W_{ij} = \mathbf{X}_{ij}'\boldsymbol{\beta} + V(t_{ij})$, $\widehat{w} = \mathbf{x}'\widehat{\boldsymbol{\beta}} + \widehat{V}(t)$ and $\widehat{W}_{ij} = \mathbf{X}_{ij}'\widehat{\boldsymbol{\beta}} + \widehat{V}(t_{ij})$. Define

$$\widehat{f}y(w) = \frac{1}{Nh}\sum_{i=1}^{n}\sum_{j=1}^{n_i} Y_{ij} K\left(\frac{\widehat{W}_{ij} - w}{h}\right), \quad fy_n(w) = \frac{1}{Nh}\sum_{i=1}^{n}\sum_{j=1}^{n_i} Y_{ij} K\left(\frac{W_{ij} - w}{h}\right),$$

$$\widehat{f}(w) = \frac{1}{Nh}\sum_{i=1}^{n}\sum_{j=1}^{n_i} K\left(\frac{\widehat{W}_{ij} - w}{h}\right), \quad f_n(w) = \frac{1}{Nh}\sum_{i=1}^{n}\sum_{j=1}^{n_i} K\left(\frac{W_{ij} - w}{h}\right).$$

Under the conditions in Supplementary Material S1, as $n \to \infty$, $h \to 0$, if $\log n/(nh^3)^{1/2} \to 0$, we have

$$\begin{aligned}
\widehat{f}y(\widehat{w}) = fy_n(w) &+ (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\Lambda^{(10)}(w,\mathbf{x}) + \Delta^{(1)}(w)\left(\widehat{V}(t) - V(t)\right) \\
&+ \frac{1}{Nh^2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} Y_{ij} K^{(1)}\left(\frac{W_{ij} - w}{h}\right)\left(\widehat{V}(t_{ij}) - V(t_{ij})\right) \\
&+ o_p(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + O_p\left(\frac{\sup_{t\in[0,\tau]}\left(\widehat{V}(t) - V(t)\right)^2}{h}\right),
\end{aligned}$$

$$\widehat{f}(\widehat{w}) = f_n(w) + (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\Gamma^{(10)}(w, \mathbf{x}) - f^{(1)}(w)\left(\widehat{V}(t) - V(t)\right)$$
$$+\frac{1}{Nh^2}\sum_{i=1}^{n}\sum_{j=1}^{n_i} K^{(1)}\left(\frac{W_{ij} - w}{h}\right)\left(\widehat{V}(t_{ij}) - V(t_{ij})\right)$$
$$+o_p(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + O_p\left(\frac{\sup_{t\in[0,\tau]}\left(\widehat{V}(t) - V(t)\right)^2}{h}\right),$$

uniformly over w, where $\Gamma(w, \mathbf{x}) = -f(w)\left[q(w) - \mathbf{x}\right]$, $\Lambda(w, \mathbf{x}) = -f(w)\left[q(w) - \mathbf{x}\right] m(w)$, $\Delta(w) = -f(w)m(w)$, $q(w) = E\{\mathbf{X}_{ij}|W_{ij} = w\}$ and $f(w)$ is the density function of W.

Lemma 4: As $n \to \infty$, if $\log n/(nh_1^3h_2^3)^{1/2} \to 0$ and $\log n/(nh_1^5h_2)^{1/2} \to 0$, then we have

$$\begin{aligned}
\widehat{p}^{(10)}(\mathbf{x}'\widehat{\boldsymbol{\beta}}, t) &= p_n^{(10)}(\mathbf{x}'\boldsymbol{\beta}, t) + (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\Gamma^{(200)}(\mathbf{x}'\boldsymbol{\beta}, t, \mathbf{x}) + o_p(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})\\
\widehat{p}^{(01)}(\mathbf{x}'\widehat{\boldsymbol{\beta}}, t) &= p_n^{(01)}(\mathbf{x}'\boldsymbol{\beta}, t) + (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\Gamma^{(110)}(\mathbf{x}'\boldsymbol{\beta}, t, \mathbf{x}) + o_p(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}),\\
\widehat{py}^{(01)}(\mathbf{x}'\widehat{\boldsymbol{\beta}}, t) &= py_n^{(01)}(\mathbf{x}'\boldsymbol{\beta}, t) + (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\Lambda^{(110)}(\mathbf{x}'\boldsymbol{\beta}, t, \mathbf{x}) + o_p(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}),\\
\widehat{py}^{(10)}(\mathbf{x}'\widehat{\boldsymbol{\beta}}, t) &= py_n^{(10)}(\mathbf{x}'\boldsymbol{\beta}, t) + (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\Lambda^{(200)}(\mathbf{x}'\boldsymbol{\beta}, t, \mathbf{x}) + o_p(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}),
\end{aligned}$$

uniformly over $\mathbf{x} \in \Omega_1$ and $t \in \Omega_2$, where $\Lambda(z, t, \mathbf{x}) = -p(z, t)\left[\varrho(z, t) - \mathbf{x}\right]\mu(z, t)$.

Lemma 5: Let

$$U(z, t) = \mu(z, t)p(z, t),\ U_1(z, t) = \mu^{(10)}(z, t)p(z, t),\ U_2(z, t) = \mu^{(01)}(z, t)p(z, t),$$
$$\widehat{U}_1(z, t) = \frac{\partial\widehat{\mu}(z, t)}{\partial z}\widehat{p}(z, t),\ \widehat{U}_2(z, t) = \frac{\partial\widehat{\mu}(z, t)}{\partial t}\widehat{p}(z, t),$$

where $\widehat{\mu}$ and $\widehat{p}$ are obtained by μ_n and p_n with $\boldsymbol{\beta}$ replaced by $\widehat{\boldsymbol{\beta}}$, respectively.

Denote

$$\Omega_{n1}(t) = -\frac{1}{N^2 h_1 h_2^2 \rho(t)} \sum_{i=1}^{n}\sum_{j=1}^{n_i}\sum_{r=1}^{n}\sum_{k=1}^{n_r} \{Y_{rk} - \mu(Z_{ij},t)\} K_1\left(\frac{Z_{rk}-Z_{ij}}{h_1}\right) K_2^{(1)}\left(\frac{t_{rk}-t}{h_2}\right),$$
$$\Omega_{n2}(t) = \frac{v(t)}{N^2 h_1^2 h_2 \rho(t)} \sum_{i=1}^{n}\sum_{j=1}^{n_i}\sum_{r=1}^{n}\sum_{k=1}^{n_r} \{Y_{rk} - \mu(Z_{ij},t)\} K_1^{(1)}\left(\frac{Z_{rk}-Z_{ij}}{h_1}\right) K_2\left(\frac{t_{rk}-t}{h_2}\right),$$
$$\Omega_{n3}(t) = -\frac{1}{N^2 h_1 h_2 \rho(t)} \sum_{i=1}^{n}\sum_{j=1}^{n_i}\sum_{r=1}^{n}\sum_{k=1}^{n_r} \{Y_{rk} - \mu(Z_{ij},t)\} \left\{\frac{p^{(01)}(Z_{ij},t) - v(t)p^{(10)}(Z_{ij},t)}{p(Z_{ij},t)}\right\} \times K_1\left(\frac{Z_{rk}-Z_{ij}}{h_1}\right) K_2\left(\frac{t_{rk}-t}{h_2}\right),$$
$$\mathcal{Q}(t) = \frac{1}{\rho(t)} E p(Z_{ij},t)\left(\varrho^{(10)}(Z_{ij},t)\mu_2(Z_{ij},t) - \varrho^{(01)}(Z_{ij},t)\mu_1(Z_{ij},t)\right),$$

where $\rho(t)$ is defined in Section 3. Then we have

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{n_i}\widehat{U}_2(\widehat{Z}_{ij},t)}{\sum_{i=1}^{n}\sum_{j=1}^{n_i}\widehat{U}_1(\widehat{Z}_{ij},t)} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{n_i}U_2(Z_{ij},t)}{\sum_{i=1}^{n}\sum_{j=1}^{n_i}U_1(Z_{ij},t)} = \Omega_{n1}(t) + \Omega_{n2}(t) + \Omega_{n3}(t) + \mathcal{Q}(t)'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + o_p(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}).$$

Proof of Lemma 5: Since

$$\widehat{U}_1(z,t) = \widehat{py}^{(10)}(z,t) - \frac{\widehat{py}(z,t)\widehat{p}^{(10)}(z,t)}{\widehat{p}(z,t)},\quad U_1(z,t) = U^{(10)}(z,t) - \frac{U(z,t)p^{(10)}(z,t)}{p(z,t)},$$
$$\widehat{U}_2(z,t) = \widehat{py}^{(01)}(z,t) - \frac{\widehat{py}(z,t)\widehat{p}^{(01)}(z,t)}{\widehat{p}(z,t)},\quad U_2(z,t) = U^{(01)}(z,t) - \frac{U(z,t)p^{(01)}(z,t)}{p(z,t)}.$$

Note that $\mu_2(z,t) = v(t)\mu_1(z,t)$, then we have the expansion,

$$\begin{aligned}
& \frac{\sum_{i=1}^n \sum_{j=1}^{n_i} \widehat{U}_2(\widehat{Z}_{ij},t)}{\sum_{i=1}^n \sum_{j=1}^{n_i} \widehat{U}_1(\widehat{Z}_{ij},t)} - \frac{\sum_{i=1}^n \sum_{j=1}^{n_i} U_2(Z_{ij},t)}{\sum_{i=1}^n \sum_{j=1}^{n_i} U_1(Z_{ij},t)} \\
= \ & \frac{1}{N\rho(t)} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \left\{ \widehat{py}^{(01)}(\widehat{Z}_{ij},t) - U^{(01)}(Z_{ij},t) \right\} \\
& - \frac{v(t)}{N\rho(t)} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \left\{ \widehat{py}^{(10)}(\widehat{Z}_{ij},t) - U^{(10)}(Z_{ij},t) \right\} \\
& + \frac{1}{N\rho(t)} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{U(Z_{ij},t)}{(p(Z_{ij},t))^2} \left\{ p^{(01)}(Z_{ij},t) - v(t)p^{(10)}(Z_{ij},t) \right\} \left(\widehat{p}(\widehat{Z}_{ij},t) - p(Z_{ij},t) \right) \\
& - \frac{1}{N\rho(t)} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{U(Z_{ij},t)}{p(Z_{ij},t)} (\widehat{p}^{(01)}(\widehat{Z}_{ij},t) - p^{(01)}(Z_{ij},t)) \\
& + \frac{1}{N\rho(t)} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{v(t)U(Z_{ij},t)}{p(Z_{ij},t)} (\widehat{p}^{(10)}(\widehat{Z}_{ij},t) - p^{(10)}(Z_{ij},t)) \\
& - \frac{1}{N\rho(t)} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \left\{ \frac{p^{(01)}(Z_{ij},t) - v(t)p^{(10)}(Z_{ij},t)}{p(Z_{ij},t)} \right\} (\widehat{py}(\widehat{Z}_{ij},t) - U(Z_{ij},t)) + o_p(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}).
\end{aligned}$$

Lemma 5 follows by Lemmas 2-4, the conditions on h_1 and h_2 and some tedious computations.

Lemma 6: Define

$$\begin{aligned}
\Upsilon_n(y) &= \frac{1}{N} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \int_{y_0}^{y} \frac{S(u)}{h_p} K_p(\frac{\zeta_i - u}{h_p}) du \\
\vartheta_n(y) &= \frac{1}{N} \sum_{i=1}^{n} \sum_{j=1}^{n_i} I(y_0 \le \zeta_i \le y) S(\zeta_i)
\end{aligned}$$

where $p = 1, 2$ and the derivative $S^{(r_p+1)}(u)$ exists and uniformly bounded over the bounded set of y. If $nh_p^{2r_p} \to 0$, then

$$\Upsilon_n(y) - \vartheta_n(y) = o_p(n^{-1/2}),$$

uniformly over the bounded set of y.

Proof of Lemma 6: It is easy to show

$$E\left[\Upsilon_n(y) - \vartheta_n(y)\right] = O(h_p^{r_p}). \tag{S2.5}$$

Since $E\left[\int_{y_0}^{y} \frac{S(u)}{h_p} K_p(\frac{\xi_i - u}{h_p}) du - I(y_0 \le \xi_i \le y) S(\xi_i)\right]^2 = O(h_p)$, by Theorem 2.37 of Pollard (1984), we have

$$\sup_y \|\Upsilon_n(y) - \vartheta_n(y) - E[\Upsilon_n(y) - \vartheta_n(y)]\| = o(h_p^{1/2}(\log n)/n^{1/2}) \tag{S2.6}$$

almost surely. Combining (S2.5) and (S2.6) with the condition on $h_p, p = 1, 2$, we obtain Lemma 6.

S2·2 Proof of Theorem 1

Proof of Theorem 1 relies on two steps. The first step includes an asymptotic expansion of $\widehat{V}(t)$. Combining asymptotic expansions and the estimating equation on $\boldsymbol{\beta}$, the asymptotic expansion of $\widehat{\boldsymbol{\beta}}$ is obtained, this is the second step.

Step 1. Note that $v(t) = \mu_2(z,t)/\mu_1(z,t)$, by the result in Lemma 5 and (2.7), we have

$$\begin{aligned} \widehat{V}(t) - V(t) = &\int_0^t \Omega_{n1}(u) du + \int_0^t \Omega_{n2}(u) du \\ &+ \int_0^t \Omega_{n3}(u) du + (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})' \pi(t) + o_p(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}). \end{aligned} \tag{S2.7}$$

where $\pi(t) = \int_0^t \mathcal{Q}(t)dt$. Define

$$R_n(t) = R_{n1}(t) - R_{n1}(0),$$

where

$$R_{n1}(t) = \frac{1}{Nh_2} \sum_{r=1}^{n} \sum_{k=1}^{n_r} \frac{\{Y_{rk} - \mu(Z_{rk}, t)\} p(Z_{rk})}{\rho(t)} K_2 \left(\frac{t_{rk} - t}{h_2} \right),$$

and p is the density function of Z. Exchanging the summation in Ω_{n1} and using the conditions on h_1, we get

$$\Omega_{n1}(t) = -\frac{1}{Nh_2^2 \rho(t)} \sum_{r=1}^{n} \sum_{k=1}^{n_r} \{Y_{rk} - \mu(Z_{rk}, t)\} p(Z_{rk}) K_2^{(1)} \left(\frac{t_{rk} - t}{h_2} \right) + O_p(h_1^{r_1}).$$

Hence, by the conditions on h_1 and h_2

$$\begin{aligned} \int_0^t \Omega_{n1}(t) &= \frac{1}{Nh_2} \sum_{r=1}^{n} \sum_{k=1}^{n_r} \int_0^t \frac{\{Y_{rk} - \mu(Z_{rk}, t)\} p(Z_{rk})}{\rho(t)} dK_2 \left(\frac{t_{rk} - t}{h_2} \right) + O_p(h_1^{r_1}) \\ &= R_n(t) - \frac{1}{Nh_2} \sum_{r=1}^{n} \sum_{k=1}^{n_r} \int_0^t K_2 \left(\frac{t_{rk} - t}{h_2} \right) \\ &\quad \times \left[\frac{-\mu_2(Z_{rk}, t)}{\rho(t)} - \frac{\{Y_{rk} - \mu(Z_{rk}, t)\} \rho^{(1)}(t)}{\rho^2(t)} \right] p(Z_{rk}) dt + O_p(h_1^{r_1}). \end{aligned}$$

Similarly, we get

$$\begin{aligned} \int_0^t \Omega_{n2}(t) &= \int_0^t \frac{v(t)}{Nh_2 \rho(t)} \sum_{r=1}^{n} \sum_{k=1}^{n_r} \left[\{Y_{rk} - \mu(Z_{rk}, t)\} p^{(1)}(Z_{rk}) - \mu_1(Z_{rk}, t) p(Z_{rk}) \right] \\ &\quad \times K_2 \left(\frac{t_{rk} - t}{h_2} \right) dt + O_p(h_1^{r_1}) \\ \int_0^t \Omega_{n3}(t) &= -\int_0^t \frac{1}{Nh_2 \rho(t)} \sum_{r=1}^{n} \sum_{k=1}^{n_r} \{Y_{rk} - \mu(Z_{rk}, t)\} \left\{ \frac{p^{(01)}(Z_{rk}, t) - v(t) p^{(10)}(Z_{rk}, t)}{p(Z_{rk}, t)} \right\} \\ &\quad \times p(Z_{rk}) K_2 \left(\frac{t_{rk} - t}{h_2} \right) dt + O_p(h_1^{r_1}). \end{aligned}$$

Note that $v(t) = \mu_2(z,t)/\mu_1(z,t)$, hence,

$$\begin{aligned}
&\int_0^t \Omega_{n1}(t) + \int_0^t \Omega_{n2}(t)dt + \int_0^t \Omega_{n3}(t)dt \\
&= R_n(t) + \frac{1}{Nh_2}\sum_{r=1}^{n}\sum_{k=1}^{n_r}\int_0^t K_2\left(\frac{t_{rk}-t}{h_2}\right)\frac{1}{\rho(t)}\left\{Y_{rk}-\mu(Z_{rk},t)\right\} \\
&\times\left\{\frac{\rho^{(1)}(t)p(Z_{rk})}{\rho(t)} + v(t)p^{(1)}(Z_{rk}) - \left\{\frac{p^{(01)}(Z_{rk},t)-v(t)p^{(10)}(Z_{rk},t)}{p(Z_{rk},t)}\right\}p(Z_{rk})\right\}dt + O_p(h_1^{r_1}) \\
&= R_n(t) + \frac{1}{N}\sum_{r=1}^{n}\sum_{k=1}^{n_r} I(t_{rk}\le t)\frac{\xi(Z_{rk},t_{rk})}{\rho(t_{rk})}\left\{Y_{rk}-\mu(Z_{rk},t_{rk})\right\} + O_p(h_1^{r_1}+h_2^{r_2}) \qquad \text{(S2.8)}
\end{aligned}$$

where $\xi(z,t)$ is defined in Section 3.

Step 2. Denote $w = w_0$ and $\widehat{w} = \widehat{V}(t) + \mathbf{x}'\widehat{\boldsymbol{\beta}}$. First, it follows from (2.8) that $\widehat{E}(Y|t,\mathbf{x}) \equiv \widehat{E}(y|\widehat{w}) = \widehat{py}(\widehat{w})/\widehat{p}(\widehat{w})$. Applying the results in Lemma 3, we obtain

$$\begin{aligned}
\widehat{E}(Y|\widehat{w}) - m(w) &= \frac{\widehat{py}(\widehat{w}) - m(w)\widehat{p}(\widehat{w})}{\widehat{p}(\widehat{w})} \\
&= \frac{\sum_{\ell=1}^{n}\sum_{s=1}^{n_\ell}\left(Y_{\ell s}-m(w)\right)K\left(\frac{V(t_{\ell s})+\mathbf{X}_{\ell s}'\boldsymbol{\beta}-w}{h}\right)}{Nhf_n(w)} \\
&+\frac{(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)'\left\{\Lambda^{(10)}(w,\mathbf{x}) - m(w)\Gamma^{(10)}(w,\mathbf{x})\right\}}{f(w)} \\
&+\frac{1}{Nh^2f(w)}\sum_{\ell=1}^{n}\sum_{s=1}^{n_\ell}\left(Y_{\ell s}-m(w)\right)K^{(1)}\left(\frac{V(t_{\ell s})+\mathbf{X}_{\ell s}'\boldsymbol{\beta}-w}{h}\right)\left(\widehat{V}(t_{\ell s})-V(t_{\ell s})\right) \\
&-m^{(1)}(w)\left(\widehat{V}(t)-V(t)\right) + o_p(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}) + O_p\left(\frac{\sup_{t\in[0,\tau]}\left(\widehat{V}(t)-V(t)\right)^2}{h}\right) \qquad \text{(S2.9)}
\end{aligned}$$

uniformly over $\mathbf{x}$ and t.

Second, noting that $N = O(n)$, the estimate $\widehat{\boldsymbol{\beta}}$ is the root of the following equation,

$$\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\{Y_{ij}-\widehat{E}(Y|\widehat{W}_{ij})\}\mathbf{X}_{ij} \;=\; 0,$$

which can be rewritten as follows,

$$\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\{Y_{ij}-m(W_{ij})\}\mathbf{X}_{ij}-\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\{\widehat{E}(Y|\widehat{W}_{ij})-m(W_{ij})\}\mathbf{X}_{ij}=0. \tag{S2.10}$$

Using (S2.9), exchanging the order of two summations, and applying Condition on h, we obtain

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\{\widehat{E}(Y|\widehat{W}_{ij})-m(W_{ij})\}\mathbf{X}_{ij}\\
&=\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\left(Y_{ij}-m(W_{ij})\right)q(W_{ij})\\
&-\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\mathbf{X}_{ij}m^{(1)}(W_{ij})\left\{q(W_{ij})-\mathbf{X}_{ij}\right\}'(\widehat{\boldsymbol\beta}-\boldsymbol\beta_0)\\
&-\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\left\{\left(Y_{ij}-m(W_{ij})\right)q^{(1)}(W_{ij})-m^{(1)}(W_{ij})\left[q(W_{ij})-\mathbf{X}_{ij}\right]\right\}\left(\widehat{V}(t_{ij})-V(t_{ij})\right)\\
&+O_p(h^{r_0})+o_p(\widehat{\boldsymbol\beta}-\boldsymbol\beta)+O_p\left(\frac{\sup_{t\in[0,\tau]}\left(\widehat{V}(t)-V(t)\right)^2}{h}\right).
\end{aligned} \tag{S2.11}$$

Substituting (S2.7) and (S2.8) into (S2.11), we get

$$\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\{\widehat{E}(Y|\widehat{W}_{ij})-m(W_{ij})\}\mathbf{X}_{ij}$$
$$=\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}(Y_{ij}-m(W_{ij}))\left\{q(W_{ij})+\frac{\xi(Z_{ij},t_{ij})}{\rho(t_{ij})}+\frac{\kappa(t_{ij})p(Z_{ij})}{\rho(t_{ij})}\right\}$$
$$+\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\left\{m^{(1)}(W_{ij})\left[q(W_{ij})-\mathbf{X}_{ij}\right]\pi'(t_{ij})-\mathbf{X}_{ij}m^{(1)}(W_{ij})\left\{q(W_{ij})-\mathbf{X}_{ij}\right\}'\right\}(\widehat{\boldsymbol\beta}-\boldsymbol\beta)$$
$$+o_p(n^{-1/2}+\|\widehat{\boldsymbol\beta}-\boldsymbol\beta\|)+O_p\left(\frac{\sup_{t\in[0,\tau]}\left(\widehat{V}(t)-V(t)\right)^2}{h}\right), \qquad \text{(S2.12)}$$

where $\kappa(t)=E\left\{\{m^{(1)}(W)\left[q(W)-q(W,t)\right]\}f(t_{k\ell}|W)\right\}$, and $f(\cdot|w)$ is the conditional density of T given $W=w$. Substituting (S2.12) into (S2.10), Theorem 1 follows.

S2·3 Proof of Theorem 2

By (S2.7), (S2.8), the conditions on h_1, b and Theorem 1, we have

$$\widehat{V}(t)-V(t)=R_{n1}(t)+\frac{1}{Nb}\sum_{r=1}^{n}\sum_{k=1}^{n_r}\int_0^t\mathcal{K}\left(\frac{t_{rk}-t}{b}\right)\frac{1}{\rho(t)}\left\{Y_{rk}-\mu(Z_{rk},t_{rk})\right\}\xi(Z_{rk},t)dt$$
$$+\frac{1}{Nb}\sum_{r=1}^{n}\sum_{k=1}^{n_r}\int_0^t\mathcal{K}\left(\frac{t_{rk}-t}{b}\right)\frac{1}{\rho(t)}\mu^{(01)}(Z_{rk},t)(t_{rk}-t)\xi(Z_{rk},t)dt$$
$$+\frac{1}{Nb}\sum_{r=1}^{n}\sum_{k=1}^{n_r}\int_0^t\mathcal{K}\left(\frac{t_{rk}-t}{b}\right)\frac{1}{\rho(t)}\mu^{(02)}(Z_{rk},t)(t_{rk}-t)^2\xi(Z_{rk},t)dt/2$$
$$+O_p(h_1^{r_1})+o_p(b^2)$$
$$=R_{n1}(t)+Q_n(t)+a_0b^2\tau(t)+O_p(h_1^{r_1})+o_p(b^2), \qquad \text{(S2.13)}$$

where $\tau(t) = \int_0^t \frac{\varphi_1^{(10)}(t,t)+\varphi_2(t)/2}{\rho(t)} dt$, $\varphi_1(t,t)$ and $\varphi_2(t)$ are defined in Section 3,

$$Q_n(t) = \frac{1}{Nb} \sum_{r=1}^{n} \sum_{k=1}^{n_r} \int_0^t \mathcal{K}\left(\frac{t_{rk}-t}{b}\right) \frac{1}{\rho(t)} \left\{Y_{rk} - \mu(Z_{rk}, t_{rk})\right\} \xi(Z_{rk}, t) dt.$$

Since

$$\begin{aligned} Var\{Q_n(t)\} &= \frac{1}{N} E\left(Var\left[Y_{rk}|Z_{rk}, t_{rk}\right] \left\{ \int_0^t \mathcal{K}\left(\frac{t_{rk}-t}{b}\right) \frac{\xi(Z_{rk}, t)}{b\rho(t)} dt \right.\right. \\ &\qquad \left.\left. - I(0 \le t_{rk} \le t) \frac{\xi(Z_{rk}, t_{rk})}{\rho(t_{rk})} + I(0 \le t_{rk} \le t) \frac{\xi(Z_{rk}, t_{rk})}{\rho(t_{rk})} \right\}^2 \right), \end{aligned}$$

and $E\left\{\int_0^t \mathcal{K}\left(\frac{t_{rk}-t}{b}\right) \frac{\xi(Z_{rk},t)}{b\rho(t)} dt - I(0 \le t_{rk} \le t) \frac{\xi(Z_{rk},t_{rk})}{\rho(t_{rk})}\right\}^2 = O(b)$, we have

$$\begin{aligned} Var\{Q_n(t)\} &= O(\frac{b}{N}) + \frac{1}{N} E\left(Var\left[Y_{rk}|Z_{rk}, t_{rk}\right] \left\{ I(0 \le t_{rk} \le t) \frac{\xi(Z_{rk}, t_{rk})}{\rho(t_{rk})} \right\}^2 \right) \\ &= O(\frac{1}{N}). \end{aligned}$$

This coupled with $E\{Q_n(t)\} = 0$ results in

$$Q_n(t) = O(n^{-1/2}). \tag{S2.14}$$

Now, we consider $R_{n1}(t)$. It can be shown that

$$ER_{n1}(t) = \frac{b^2}{\rho(t)} E\left\{\mu^{(01)}(Z,t) p^{(01)}(Z,t) + \mu^{(02)}(Z,t) p(Z,t)/2\right\} \int x^2 \mathcal{K}(x) dx, \tag{S2.15}$$

and

$$\begin{aligned} Var\{R_{n1}(t)\} &= \frac{1}{Nb^2} E\left\{ \frac{\left\{Y_{rk} - \mu(Z_{rk}, t)\right\} p(Z_{rk})}{\rho(t)} \mathcal{K}\left(\frac{t_{rk}-t}{b}\right) \right\}^2 (1 + o_p(1)) \\ &= \frac{1}{Nb} \frac{E\left[H(Z,t) p(Z,t) p(Z)\right]}{\rho^2(t)} \int_x \mathcal{K}^2(x) dx (1 + o_p(1)), \end{aligned} \tag{S2.16}$$

where $H(z,t) = \mathrm{Var}(Y_{rk}|Z_{rk} = z, t_{rk} = t)$. Hence, Theorem 2 follows from (S2.13), (S2.14), (S2.15) and (S2.16).