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A FLUCTUATION LIMIT THEOREM OF BRANCHING PROCESSES AND ITS APPLICATIONS

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Abstract: We prove that the fluctuation limit of a sequence of Galton-Watson branching processes with immigration can be an Ornstein-Uhlenbeck type process under some assumptions on the offspring and the immigration laws. The asymptotic properties of the conditional least square estimators of the offspring mean and the immigration mean are studied when the limit process is an Ornstein-Uhlenbeck diffusion process.

Key words and phrases: Conditional least square estimator, fluctuation limit, Galton-Watson branching process with immigration, Lévy process, Ornstein-Uhlenbeck type process.

1. Introduction

Let $\{\zeta(l,j)\}_{l,j\geq 1}$ and $\{\eta(l)\}_{l\geq 1}$ be independent sequences of independently identically distributed nonnegative integer valued random variables. A Galton-Watson branching process with immigration (GWI process) $\{Y(l)\}_{l\geq 0}$ is defined as

$$Y(l) = \sum_{j=1}^{Y(l-1)} \zeta(l,j) + \eta(l), \qquad l = 1, 2, \ldots,$$

where $Y(0)$ is a nonnegative integer valued random variable independent of $\{\zeta(l,j)\}_{l,j\geq 1}$ and $\{\eta(l)\}_{l\geq 1}$. We call the expectations $m = \mathbb{E}[\zeta(l,j)]$ and $\lambda = \mathbb{E}[\eta(l)]$ the offspring mean and immigration mean, respectively, if they exist. Meanwhile, the GWI process $\{Y(l)\}_{l\geq 0}$ is called subcritical, critical, and supercritical, respectively, when the offspring mean $m < 1$, $= 1$, and > 1. When a sequence of GWI processes $\{Y_k(l)\}_{l\geq 0}$, for $k = 1, 2, 3, \ldots$, is given, it is called nearly critical if the corresponding offspring mean sequence $m_k = \mathbb{E}[\zeta_k(l,j)] \to 1$ as $k \to \infty$. See Arthreya and Ney (1972) for more details.

A spectrally positive homogeneous Ornstein-Uhlenbeck (O-U) type process is defined as a real-valued cádlág Markov process $\{X(t)\}_{t\geq 0}$ with generator A given by

$$Af(x) = -pxf'(x) + qf''(x) + \int_{0+}^{\infty} \{f(x+u) - f(x) - uf'(x)\}\mu_0(du),$$

where p and q are two nonnegative constants and $\mu_0(du)$ is a $\sigma-$finite measure on $(0,\infty)$ such that $\int_{0+}^{\infty}(u\wedge u^2)\mu_0(du)<\infty$. A realization of this O-U type process can be given as the unique solution to the stochastic differential equation

$$dX(t)=dL(t)-pX(t)dt, \qquad t\geq 0,$$

where $\{L(t)\}_{t\geq 0}$ is a spectrally positive cádlág Lévy process with $L(0)=0$ and its increment distribution determined by

$$\mathbb{E}[\exp\{-\lambda(L(r+t)-L(r))\}]=\exp\left\{t\Big(q\lambda^2+\int_{0+}^{\infty}(e^{-\lambda u}-1+\lambda u)\mu_0(du)\Big)\right\}$$

for any $\lambda\geq 0$. The connection between O-U type processes and Lévy processes was found in the study of the limit distributions for sums of certain random variables in Sato and Yamazato (1984) and Wolfe (1982). See also Bertoin (1996) for more on Lévy processes.

Studying the functional weak limit theorem for branching processes has an extended history. In Feller (1951), a procedure for obtaining diffusion processes as limits of a sequence of Galton-Watson processes was formulated. Kawazu and Watanabe (1971) characterized the continuous state branching processes with immigration (CBI processes) by its Laplace transformation and proved that a sequence of GWI processes converges in finite dimensional distributions to a stochastically continuous and conservative continuous time CBI process under some suitable conditions. Li (2006) extended this result to the case of weak convergence in the Skorokhod space. In addition, Grimvall (1974), Sriram (1994), and Wei and Winnicki (1987, 1990) established the relationship between GWI processes and CBI processes in some special cases.

By virtue of Laplace transforms, Li (2000) considered the fluctuation limit theorem for branching processes. He proved that under certain conditions the fluctuation limit of a sequence of continuous time discrete state branching processes with Poisson immigration is an O-U type process. Ispány, Pap, and Zuijlen (2005) proved that under some suitable moment conditions the fluctuation limit of a sequence of GWI processes is a continuous inhomogeneous O-U type process driven by a time changed Wiener process by means of the Martingale Central Limit Theorem and the Continuous Mapping Theorem. It was also proved in Li (2009) that the fluctuation limit of a sequence of Jiřina processes with immigration, the discrete time CBI processes, is an O-U type process under some moment conditions.

The main purpose of this paper is to establish the functional fluctuation limit of a sequence of GWI processes to a homogeneous O-U type process when some constraints about the moment generating functions for the offspring processes $\{\zeta_k(\cdot,\cdot)\}_{k\geq 1}$ and the immigration processes $\{\eta_k(\cdot)\}_{k\geq 1}$ are satisfied. The main result of this paper is different from the existing ones in Li (2000, 2006) and Ispány, Pap, and Zuijlen (2005) mentioned above, and we also employ the different method to prove it. We present some examples in Section 3.

The functional limit theorems of branching processes can be used in such fields as queueing theory, mathematical finance, and statistical inference for stochastic processes. Indeed, based on the weak convergence results of GWI processes, many asymptotic properties can be obtained for the estimators of the offspring mean m and the immigration mean λ. For example, using their functional weak limit theorem, Wei and Winnicki (1987, 1990) proved that, in the critical case, the various conditional least square (CLS) estimators for m are not asymptotically normal but consistent, and that the CLS estimator for λ is not consistent. Sriram (1994) established a functional weak limit theorem for the critical branching process, and successfully illustrated the invalidity of the parametric bootstrap method for critical GWI processes. Ispány, Pap, and Zuijlen (2005) discussed the CLS estimators' asymptotic properties for a sequence of nearly critical branching processes using their functional fluctuation limit theorems. We give, in Section 4, an application of our functional limit theory to the asymptotic properties of the CLS estimators for a sequence of GWI processes that converge to an O-U diffusion process.

The paper is organized as follows. In Section 2, we prove that the functional fluctuation limit of a sequence of GWI processes is an O-U type process when the offspring and the immigration laws are subject to constraints. We present some examples in Section 3, and discuss the asymptotic properties of the CLS estimators for offspring mean and immigration mean in Section 4. Some of the technical details are reported in the appendix.

2. Main Results

Let $\{Y_k(l)\}_{l\geq 0},\ \ k=1,2,\ldots$ be a sequence of GWI processes whose one-step transition probability is determined by

$$\mathbb{E}_x[e^{-\lambda Y_k(1)}]=\mathbb{E}[e^{-\lambda\left\{\sum_{j=1}^{x}\zeta_k(1,j)+\eta_k(1)\right\}}]=G_k(e^{-\lambda})^x H_k(e^{-\lambda}),$$

where $G_k(\cdot)$ and $H_k(\cdot)$ are the moment generating functions of $\zeta_k(1,j)$ and $\eta_k(1)$, respectively. We assume

(H0): $H_k^{'}(1)=ka(1-G_k^{'}(1))$, $\mathbb{E}[Y_k(0)]=ka$,
(H1): $k^2(1-G_k^{'}(1))\to\alpha$,
(H2): $k^2G_k^{''}(e^{i\lambda k^{-1/2}})\to\Upsilon_1(\lambda)$,
(H3): $kH_k^{''}(e^{i\lambda k^{-1/2}})\to\Upsilon_2(\lambda)$,

as $k \to \infty$. Here a is a positive constant, α is a nonnegative constant, $i = \sqrt{-1}$, and Υ_j, $j = 1, 2$, are continuous complex-valued functions of λ. Note that we use the notation $G_k^{'}(x_0) = \partial G_k(x)/\partial x|_{x=x_0}$, $G_k^{''}(x_0) = \partial^2 G_k(x)/\partial x^2|_{x=x_0}$, and likewise in this paper. Under assumptions (H2)−(H3), both $\Upsilon_1(\lambda)$ and $\Upsilon_2(\lambda)$ are characteristic functions of finite measures on $[0, \infty)$. Then there is a probability measure μ_1 on $[0, \infty)$ such that

$$\frac{a\alpha + a\Upsilon_1(\lambda) + \Upsilon_2(\lambda)}{a\alpha + a\Upsilon_1(0) + \Upsilon_2(0)} = \int_0^\infty e^{i\lambda u}\mu_1(du).$$

Let $b = \mu_1(\{0\})$, $\mu(du) = \mu_1(du)/u^2$ for $u > 0$, and $c = a\alpha + a\Upsilon_1(0) + \Upsilon_2(0)$. Let $C_c^\infty(\mathbb{R})$ be the set of infinitely differentiable functions with compact support on $\mathbb{R}$.

To formulate our fluctuation limit theorem, we define the random step functions

$$X_k(t) = k^{-1/2}\Big\{Y_k(\lfloor k^2 t\rfloor) - ka\Big\}, \quad t \ge 0, \quad k = 1, 2, \ldots,$$

where $\lfloor x\rfloor$ denotes the integer part of x. The following is the main result of the paper.

Theorem 1. *If assumptions* (H0)−(H3) *hold and* $X_k(0)$ *converges weakly to* $X(0)$ *as* $k \to \infty$*, then the sequence* $\{X_k(\cdot)\}_{k\ge 0}$ *converges weakly in the Skorohod space* $D_{\mathbb{R}}[0, \infty)$ *to a generalized O-U type process with a generator* A *that satisfies*

$$\begin{aligned} Af(x) = &-\alpha x f^{'}(x) + \frac{1}{2}(a\alpha + bc)f^{''}(x) \\ &+c\int_{0+}^\infty \{f(x+u) - f(x) - uf^{'}(x)\}\mu(du) \end{aligned} \tag{2.1}$$

for all $f \in C_c^\infty(\mathbb{R})$.

The proof of the theorem needs the following two lemmas, whose proofs are given in the appendix. Let $\xi_k(x) = ka + \sqrt{k}x$, which is treated as non-negative integers by properly choosing x, and

$$Z_k(x_k) = k^{-1/2}\{Y_k(1) - ka - \sqrt{k}x_k\} = k^{-1/2}\{Y_k(1) - \xi_k(x_k)\}.$$

Lemma 1. *If* $x_k \to \infty$ *as* $k \to \infty$*, then*

$$\mathbb{E}_{\xi_k(x_k)}\Big[\int_0^1 (1-w)k^2 f^{''}(x_k + wZ_k(x_k))Z_k^2(x_k)dw\Big] \to 0.$$

Lemma 2. *If* $x_k \to x < \infty$ *as* $k \to \infty$*, then*

$$\begin{aligned} &\mathbb{E}_{\xi_k(x_k)}\Big[\int_0^1 (1-w)k^2 f^{''}(x_k + wZ_k(x_k))Z_k^2(x_k)dw\Big] \\ &\quad\to \frac{1}{2}(a\alpha + bc)f^{''}(x) + c\int_{0+}^\infty \{f(x+u) - f(x) - uf^{'}(x)\}\mu(du). \end{aligned}$$

Proof of Theorem 1. Note that $X_k(\cdot)$ is a Markov chain taking values in $E_k = [-\sqrt{k}a, +\infty)$. Define

$$T_k f(x) = \mathbb{E}_{\xi_k(x)}\Big[f\Big(k^{-1/2}(Y_k(1) - ka)\Big)\Big].$$

By Duffie, Filipović and Schachermayer (2003) Theorem 2.7, $C_c^\infty(\mathbb{R})$ is a core for the generator A of an O-U type process. So by Ethier and Kurtz (1986, p.31, p.233), it is sufficient to show that if $E_k \ni x_k \to x \in \mathbb{R} \cup \{\infty\}$, then

$$\lim_{k\to\infty}\Big|k^2[T_k f(x_k) - f(x_k)] - Af(x_k)\Big| = 0, \quad \forall f \in C_c^\infty(\mathbb{R}).$$

By Taylor's expansion,

$$\begin{aligned} k^2\Big[T_k f(x_k) - f(x_k)\Big] &= k^2\mathbb{E}_{\xi_k(x_k)}\Big[f\Big(k^{-1/2}(Y_k(1) - ka)\Big) - f(x_k)\Big] \\ &= \mathbb{E}_{\xi_k(x_k)}\Big[k^2 f'(x_k)Z_k(x_k) + \int_0^1 (1-w)k^2 f''(x_k + wZ_k(x_k))Z_k^2(x_k)dw\Big]. \end{aligned}$$

Because $k^2\mathbb{E}_{\xi_k(x_k)}[Z_k(x_k)] = k^2 x_k[G_k'(1) - 1] \to -\alpha x \mathbf{1}_{\{x_k\to x<\infty\}}$ and $Af(x_k) \to Af(x)\mathbf{1}_{\{x_k\to x<\infty\}}$, using Lemma 1 and Lemma 2, one obtains Theorem 1.

Remark 1. The scaling used in this theorem is k^2 in time, rather than the commonly used k. But one can prove similarly that

$$X_k(t) = k^{-s}(Y_k(\lfloor k^d t\rfloor) - k^r a)$$

converges to an O-U type process when positive constants d, r, and s are properly chosen, and similar assumptions to (H0)−(H3) are given.

Remark 2. The limit process would be an O-U diffusion process if we assumed that $\Upsilon_1(\lambda)$ and $\Upsilon_2(\lambda)$ in assumptions (H2) and (H3) are positive constants. One typical situation that lead to positive real values for $\Upsilon_1(\lambda)$ and $\Upsilon_2(\lambda)$ is when both the offspring number and the immigration number take nonnegative integer values no larger than 2. Examples that converge to O-U type processes according to our weak convergence theory, either when $\Upsilon_1(\lambda)$ and $\Upsilon_2(\lambda)$ are real values or complex functions of λ, are given in Section 3.

Remark 3. Since our limit processes are of O-U type, which are special cases of the affine processes suggested by Duffie, Filipović and Schachermayer (2003) we can try to do some option pricing work based on our convergence result. Specially, it is interesting to consider the pricing and hedging problems under model (3.1), namely the Vasicek model with Poisson jumps, by considering the convergence problem of the prices under the corresponding GWI processes.

Remark 4 In Section 4, we present studies of this fluctuation limit theorem for statistical inference for the parameters of the GWI process.

3. Examples

In this section, we give some examples of convergence to O-U type processes according to Theorem 1. We use g_k and h_k to denote the probability distribution corresponding to the moment generation functions G_k and H_k for the offspring processes and the immigration processes, respectively. Also write $\{B_t\}_{t\geq 0}$ for the standard Brownian motion.

3.1. Limit processes without jumps

The first two examples are extremely simple cases with only two possible states for both the offspring processes and the immigration processes.

(I) Let the offspring processes be a sequence of independently identically distributed $0-1$ random variables with $g_k(0) = \alpha k^{-2}$ and $g_k(1) = 1-\alpha k^{-2}$, and the immigration processes be a sequence of independently identically distributed $0-1$ random variables with $h_k(0) = 1 - a\alpha k^{-1}$ and $h_k(1) = a\alpha k^{-1}$. Here, a and α are positive constants. Then $G_k^{'}(1) = 1 - \alpha/k^2$ and $H_k^{'}(1) = a\alpha/k$. So $H_k^{'}(1) = ka(1 - G_k^{'}(1))$, and we have

$$k^2(1 - G_k^{'}(1)) = \alpha,\ k^2 G_k^{''}(e^{i\lambda k^{-1/2}}) = 0,\ kH_k^{''}(e^{i\lambda k^{-1/2}}) = 0,$$

$$\frac{a\alpha + a\Upsilon_1(\lambda) + \Upsilon_2(\lambda)}{a\alpha + a\Upsilon_1(0) + \Upsilon_2(0)} = \frac{a\alpha + a\times 0 + 0}{a\alpha + a\times 0 + 0} = 1 \Rightarrow b = 1,\ \mu = 0.$$

Hence,

$$X_k(t) = k^{-1/2}\Big(Y_k(\lfloor k^2 t\rfloor) - ka\Big) \to X(t),$$

where $\{X(t)\}_{t\geq 0}$ is an O-U diffusion process with generator $Af(x) = -\alpha x f^{'}(x) + a\alpha f^{''}(x)$ or, equivalently, that satisfies the stochastic differential equation

$$dX(t) = -\alpha X(t)dt + \sqrt{2a\alpha}dB_t.$$

(II) Let the offspring processes be a sequence of independently identically distributed $0-1$ random variables with $g_k(0) = \alpha k^{-2}$ and $g_k(1) = 1-\alpha k^{-2}$, and the immigration processes be a sequence of independently identically distributed $0-2$ random variables with $h_k(0) = 1 - a\alpha k^{-1}/2$ and $h_k(2) = a\alpha k^{-1}/2$. Here, a and α are positive constants. Then $G_k^{'}(1) = 1 - \alpha/k^2$ and $H_k^{'}(1) = a\alpha/k$. So $H_k^{'}(1) = ka(1 - G_k^{'}(1))$, and we have

$$k^2(1 - G_k^{'}(1)) = \alpha,\ k^2 G_k^{''}(e^{i\lambda k^{-1/2}}) = 0,\ kH_k^{''}(e^{i\lambda k^{-1/2}}) = a\alpha,\ b = 1,\ \mu = 0.$$

Hence,

$$X_k(t) = k^{-1/2}\Big(Y_k(\lfloor k^2 t\rfloor) - ka\Big) \to X(t),$$

where $\{X(t)\}_{t\geq 0}$ is an O-U diffusion process that satisfies the stochastic differential equation

$$dX(t) = -\alpha X(t)dt + \sqrt{3a\alpha}dB_t.$$

(III) Let the offspring processes be a sequence of independently identically distributed random variables with $g_k(0) = 3\alpha k^{-2}/4$, $g_k(1) = 1 - \alpha k^{-2}$, and $g_k(2) = \alpha k^{-2}/4$, and the immigration processes be a sequence of independently identically distributed $0-1$ random variables with $h_k(0) = 1 - a\alpha k^{-1}/2$ and $h_k(1) = a\alpha k^{-1}/2$. Here, a and α are positive constants. Then $G_k^{'}(1) = 1 - \alpha/2k^2$ and $H_k^{'}(1) = a\alpha/(2k)$. So $H_k^{'}(1) = ka(1 - G_k^{'}(1))$, and we have

$$k^2(1 - G_k^{'}(1)) = \frac{1}{2}\alpha,\ k^2 G_k^{''}(e^{i\lambda k^{-1/2}}) = \frac{1}{2}\alpha,\ kH_k^{''}(e^{i\lambda k^{-1/2}}) = 0,\ b = 1,\ \mu = 0.$$

Hence,

$$X_k(t) = k^{-1/2}\Big(Y_k(\lfloor k^2 t\rfloor) - ka\Big) \to X(t),$$

where $\{X(t)\}_{t\geq 0}$ is an O-U diffusion process that satisfies the following stochastic differential equation

$$dX(t) = -\frac{1}{2}\alpha X(t)dt + \sqrt{\frac{3a\alpha}{2}}dB_t.$$

(IV) Let the offspring processes follow the same law as in (I) and the immigration processes be a sequence of independently identically distributed Poisson random variables with mean $H_k^{'}(1) = ka(1 - G_k^{'}(1)) = a\alpha/k$. Then

$$kH_k^{''}(e^{i\lambda k^{-1/2}}) = \frac{(a\alpha)^2}{k} e^{(a\alpha/k)(i\lambda k^{-1/2}-1)} \to 0$$

and one has the same limit process $\{X(t)\}_{t\geq 0}$ as in (I).

3.2. Limit processes with jumps

(I) Let the probability law corresponding to the independently identically distributed offspring processes be

$$g_k(0) = \frac{2\alpha_1}{3k^2} - \frac{\alpha_2}{k^3},\quad g_k(1) = 1 - \frac{\alpha_1}{k^2},\quad g_k(2) = \frac{\alpha_1}{3k^2},\quad g_k(\lfloor\sqrt{k}\rfloor) = \frac{\alpha_2}{k^3},$$

and the immigration sequences be determined by distribution law

$$h_k(0) = 1 - \frac{a\alpha_1}{4k} + \frac{a\alpha_2(2\lfloor\sqrt{k}\rfloor - 1)}{k^2},\quad h_k(1) = \frac{a\alpha_1}{6k} - \frac{2a\alpha_2\lfloor\sqrt{k}\rfloor}{k^2},$$

$$h_k(2)=\frac{a\alpha_1}{12k},\quad h_k(\lfloor\sqrt{k}\rfloor)=\frac{a\alpha_2}{k^2},$$

where a, α_1 and α_2 are positive constants. Then

$$\begin{aligned}
k^2(1-G_k'(1)) &= \frac{\alpha_1}{3}-\frac{\alpha_2\lfloor\sqrt{k}\rfloor}{k}\to\frac{\alpha_1}{3},\\
k^2G_k''(e^{i\lambda k^{-1/2}}) &= \frac{2}{3}\alpha_1+\alpha_2\frac{\lfloor\sqrt{k}\rfloor(\lfloor\sqrt{k}\rfloor-1)}{k}e^{i\lambda(\lfloor\sqrt{k}\rfloor-2)/\sqrt{k}}\to\frac{2}{3}\alpha_1+\alpha_2e^{i\lambda},\\
kH_k''(e^{i\lambda k^{-1/2}}) &= \frac{a\alpha_1}{6}+\lfloor\sqrt{k}\rfloor(\lfloor\sqrt{k}\rfloor-1)\frac{a\alpha_2}{k}e^{i\lambda(\lfloor\sqrt{k}\rfloor-2)/\sqrt{k}}\to\frac{a\alpha_1}{6}+a\alpha_2e^{i\lambda},\\
b+e^{i\lambda}\mu(\{1\}) &= \frac{7\alpha_1/6}{7\alpha_1/6+2\alpha_2}+\frac{2\alpha_2}{7\alpha_1/6+2\alpha_2}e^{i\lambda}.
\end{aligned}$$

So the corresponding limit process has the realization

$$dX(t)=-3^{-1}\alpha_1X(t)dt+d\left[\sqrt{3a\alpha_1/2}B_t+N_t-2a\alpha_2t\right],\tag{3.1}$$

where $\{N_t\}_{t\geq0}$ is a Poisson process with rate parameter $2a\alpha_2$ independent of the Brownian motion $\{B_t\}_{t\geq0}$.

(II) We note as in Li (2000) that when α in (2.1) is 0, the limit process is a Lévy process. Let the offspring law be

$$g_k(0)=\frac{2\alpha_1}{3k^{5/2}}-\frac{\alpha_2}{k^3},\quad g_k(1)=1-\frac{\alpha_1}{k^{5/2}},\quad g_k(2)=\frac{\alpha_1}{3k^{5/2}},\quad g_k(\lfloor\sqrt{k}\rfloor)=\frac{\alpha_2}{k^3},$$

and the immigration law be

$$h_k(0)=1-a\alpha_1/(3k^{3/2})+a\alpha_2\lfloor\sqrt{k}\rfloor k^{-2},\quad h_k(1)=a\alpha_1/(3k^{3/2})-a\alpha_2\lfloor\sqrt{k}\rfloor k^{-2},$$

where a,α_1,α_2 are positive constants. Then

$$\begin{aligned}
kH_k''(e^{i\lambda k^{-1/2}})=0,\ k^2(1-G_k'(1)) &= \frac{\alpha_1}{3\sqrt{k}}-\frac{\alpha_2\lfloor\sqrt{k}\rfloor}{k}\to0,\\
k^2G_k''(e^{i\lambda k^{-1/2}}) &= \frac{2\alpha_1}{3\sqrt{k}}+\alpha_2\frac{\lfloor\sqrt{k}\rfloor(\lfloor\sqrt{k}\rfloor-1)}{k}e^{i\lambda(\lfloor\sqrt{k}\rfloor-2)/\sqrt{k}}\to\alpha_2e^{i\lambda},\\
b+e^{i\lambda}\mu(\{1\}) &= e^{i\lambda}.
\end{aligned}$$

So the corresponding limit process has the realization

$$X_k(t)\xrightarrow{\mathcal{D}}X(t)=N_t-a\alpha_2t,$$

where $\{N_t\}_{t\geq0}$ is a Poisson process with rate parameter $a\alpha_2$.

4. Applications to Statistical Inference for GWI Processes

Consider a sequence of GWI processes satisfying Theorem 1. When the immigration mean $\lambda_k = H_k^{'}(1) = \mathbb{E}[\eta_k(l)]$ is known,

$$\sum_{l=1}^{k^2} \Big[Y_k(l) - \mathbb{E}[Y_k(l)|\mathcal{F}_{l-1}^{(k)}]\Big]^2 = \Theta(G_k^{'}(1), H_k^{'}(1)), \tag{4.1}$$

where $\Theta(u,v) = \sum_{l=1}^{k^2} [Y_k(l) - uY_k(l-1) - v]^2$. Then the CLS estimator $\hat{m}_k$ of the offspring mean $m_k = G_k^{'}(1) = \mathbb{E}[\zeta_k(l,j)]$ is

$$\hat{m}_k = \frac{\sum_{l=1}^{k^2} Y_k(l-1)\{Y_k(l) - H_k^{'}(1)\}}{\sum_{l=1}^{k^2}\{Y_k(l-1)\}^2},$$

which minimizes $\Theta(u, H_k^{'}(1))$. If the immigration mean λ_k is unknown, then the joint CLS estimators for (m_k, λ_k), which could be gained by minimizing $\Theta(u,v)$ with respect to u and v, have the form

$$\tilde{m}_k = \frac{\sum_{l=1}^{k^2} Y_k(l-1)\{Y_k(l) - \bar{Y}_k\}}{\sum_{l=1}^{k^2}\{Y_k(l-1) - \bar{Y}_k^*\}^2}, \quad \tilde{\lambda}_k = \bar{Y}_k - \tilde{m}_k \bar{Y}_k^*,$$

where $\bar{Y}_k = k^{-2}\sum_{l=1}^{k^2} Y_k(l)$, $\bar{Y}_k^* = k^{-2}\sum_{l=1}^{k^2} Y_k(l-1)$.

Based on Theorem 1, the following three asymptotic properties could be established by using the Continuous Mapping Theory and the Martingale Transform Theorem. This method has been suggested in Ispány, Pap, and Zuijlen (2003a,b, 2005) and Strasser (1986).

Theorem 2. *Suppose that, for a sequence of GWI processes $\{Y_k(l)\}_{l\geq 0}$, $k = 1,2,3,\ldots$, Theorem 1 holds with an O-U diffusion process $\{X(t)\}_{t\geq 0}$ as the limit process, with generator*

$$Af(x) = -\alpha x f^{'}(x) + a\alpha f^{''}(x), \ \textit{for all } f(\cdot) \in C_c^{\infty}(\mathbb{R}),$$

where a, α are two positive constants and $E[X(0)^2] < \infty$. Then

$$k^2(\tilde{m}_k - m_k) \xrightarrow{\mathcal{D}} \frac{N(1) - M(1)\int_0^1 X(u)du}{\int_0^1 X^2(u)du - \{\int_0^1 X(u)du\}^2}, \tag{4.2}$$

$$k(\tilde{\lambda}_k - \lambda_k) \xrightarrow{\mathcal{D}} \frac{aM(1)\int_0^1 X(u)du - aN(1)}{\int_0^1 X^2(u)du - \{\int_0^1 X(u)du\}^2},$$

$$k^{5/2}(\hat{m}_k - m_k) \xrightarrow{\mathcal{D}} a^{-1}M(1),$$

where $M(t) = X(t) - X(0) + \alpha\int_0^t X(u)du$, $N(t) = e^{-\alpha t}X(0)M(t) + X(0)\int_0^t M(u) de^{-\alpha u} + e^{\alpha(1-t)}\Gamma(t) + \int_0^t \Gamma(u)de^{\alpha(1-u)}$. Here $S(t) = e^{\alpha(t-1)}X(t) - e^{-\alpha}X(0)$, $\Gamma(t) = \int_0^t S(t)dM(t)$.

Proof of Theorem 2. Let

$$X_{k,l} = k^{-1/2}(Y_k(l) - ka),$$
$$M_{k,l} = X_{k,l} - E[X_{k,l}|\mathcal{F}_{l-1}^{(k)}] = k^{-1/2}\{Y_k(l) - G_k^{'}(1)Y_k(l-1) - H_k^{'}(1)\}.$$

Then

$$\tilde{m}_k - G_k^{'}(1) = \frac{k\left[\sum_{l=1}^{k^2} X_{k,l-1}M_{k,l} - (1/k^2)\sum_{l=1}^{k^2} X_{k,l-1} \times \sum_{l=1}^{k^2} M_{k,l}\right]}{k^3\left[(1/k^2)\sum_{l=1}^{k^2} X_{k,l-1}^2 - \left((1/k^2)\sum_{l=1}^{k^2} X_{k,l-1}\right)^2\right]}. \quad (4.3)$$

We need to consider the weak convergence of each item in the last equality.

By using the Continuous Mapping Theory, we have

$$\frac{1}{k^2}\sum_{l=1}^{k^2} X_{k,l-1} = \int_0^1 X_k(u)du \xrightarrow{\mathcal{D}} \int_0^1 X(u)du,$$
$$\frac{1}{k^2}\sum_{l=1}^{k^2} X_{k,l-1}^2 = \int_0^1 X_k^2(u)du \xrightarrow{\mathcal{D}} \int_0^1 X^2(u)du,$$
$$M_k(t) = \sum_{l=1}^{\lfloor k^2 t\rfloor} M_{k,l} = X_k(t) - X_k(0) + \left\{1 - G_k^{'}(1)\right\}k^2\int_0^{\lfloor k^2 t\rfloor/k^2} X_k(u)du$$
$$\xrightarrow{\mathcal{D}} X(t) - X(0) + \alpha\int_0^t X(u)du = M(t).$$

Here, $M(t)$ is obviously a square-integrable martingale.

When the limit process is an O-U diffusion process, by similar reasoning as in Lemma 1, we can see that $\{M_{k,l}\}_{l\geq 1}$ satisfies

$$\sum_{l=1}^{\lfloor k^2 t\rfloor} \mathbb{E}[|M_{k,l}|\mathbf{1}_{\{|M_{k,l}|>\varepsilon\}}|\mathcal{F}_{k,l-1}]\xrightarrow{P_n}0, \quad \forall\epsilon > 0.$$

Define $S_k(t) = S_{k,\lfloor k^2 t\rfloor} = \sum_{l=1}^{\lfloor k^2 t\rfloor} G_k^{'}(1)^{k^2-l}M_{k,l}$. Then

$$S_k(t) = G_k^{'}(1)^{k^2-\lfloor k^2 t\rfloor}X_{k,\lfloor k^2 t\rfloor} - G_k^{'}(1)^{k^2}X_{k,0}\xrightarrow{\mathcal{D}}e^{\alpha(t-1)}X(t) - e^{-\alpha}X(0) = S(t),$$

and, by the Martingale Transform Theorem,

$$\Gamma_k(t) = \Gamma_{k,\lfloor k^2 t\rfloor} = \sum_{l=1}^{\lfloor k^2 t\rfloor} S_{k,l-1}M_{k,l}\xrightarrow{\mathcal{D}}\int_0^t S(t)dM(t) = \Gamma(t).$$

Using the Continuous Mapping Theorem, we get

$$\sum_{l=1}^{\lfloor k^2t\rfloor} X_{k,l-1}M_{k,l} \xrightarrow{\mathcal{D}} e^{-\alpha t}X(0)M(t) + X(0)\int_0^t M(u)de^{-\alpha u}$$
$$+e^{\alpha(1-t)}\Gamma(t) + \int_0^t \Gamma(u)de^{\alpha(1-u)} = N(t).$$

A further application of the Continuous Mapping Theory to (4.3) completes the proof of (4.2).

Similarly, we can prove that

$$\begin{aligned} &k(\tilde{\lambda}_k - H_k'(1)) \\ &= k\Big[k^{3/2}\textstyle\sum_{l=1}^{k^2} M_{k,l}\cdot \frac{1}{k^2}\sum_{l=1}^{k^2} X_{k,l-1}^2 + k^2a\sum_{l=1}^{k^2} M_{k,l}\cdot\frac{1}{k^2}\sum_{l=1}^{k^2} X_{k,l-1} \\ &\quad -k^{3/2}\cdot\textstyle\frac{1}{k^2}\sum_{l=1}^{k^2} X_{k,l-1}\cdot\sum_{l=1}^{k^2} X_{k,l-1}M_{k,l} - k^2a\sum_{l=1}^{k^2} X_{k,l-1}M_{k,l}\Big]\Big/ \\ &\quad \Big[k^3\Big\{\textstyle\frac{1}{k^2}\sum_{l=1}^{k^2} X_{k,l-1}^2 - \Big(\frac{1}{k^2}\sum_{l=1}^{k^2} X_{k,l-1}\Big)^2\Big\}\Big] \\ &\xrightarrow{\mathcal{D}} \frac{aM(1)\int_0^1 X(u)du - aN(1)}{\int_0^1 X^2(u)du - \{\int_0^1 X(u)du\}^2}, \end{aligned}$$

and

$$\begin{aligned} &k^{5/2}(\hat{m}_k - G_k'(1)) \\ &= k^{5/2}\cdot\frac{k\sum_{l=1}^{k^2} X_{k,l-1}M_{k,l} + k^{3/2}a\sum_{l=1}^{k^2} M_{k,l}}{k^3\cdot k^{-2}\sum_{l=1}^{k^2} X_{k,l-1}^2 + 2k^{7/2}a\cdot k^{-2}\sum_{l=1}^{k^2} X_{k,l-1} + k^4a^2} \xrightarrow{\mathcal{D}} a^{-1}M(1). \end{aligned}$$

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Appendix

Proof of Lemma 1. When $x_k \to \infty$, without loss of generality we can assume that $|x_k| > \delta$. We first prove

$$\Big|\mathbb{E}_{\xi_k(x_k)}\Big[\int_0^1 (1-w)k^2 f''(x_k + wZ_k(x_k))Z_k^2(x_k)dw\Big]\Big|$$

$$\le \frac{1}{2}||f^{''}||\mathbb{E}_{\xi_k(x_k)}\Big[k\Big\{Y_k(1)-\xi_k(x_k)G_k^{'}(1)-H_k^{'}(1)\Big\}^2\mathbf{1}_{U_k^0}\Big],$$

where

$$U_k^0=\{|Y_k(1)-\xi_k(x_k)G_k^{'}(1)-H_k^{'}(1)|\ge k^{1/2}(|x_k|G_k^{'}(1)-\delta)\}.$$

Note we assumed the support set for an arbitrary function $f\in C_c^{\infty}(\mathbb{R})$ is $[-\delta,\delta]$, which contains the set where $f(x)\neq 0$ and $f^{''}(x)\neq 0$, so

$$\begin{aligned}&\{|f''(x_k+wZ_k(x_k))|>0,\ \text{when } 0<w<1\}\\&\quad\subset\{-\delta<x_k+wZ_k(x_k)<\delta,\ \text{when } 0<w<1\}\\&\quad\subset\begin{cases}\{Z_k<\delta-x_k,(\delta-x_k)/Z_k(x_k)\le w\le 1\}, & x_k>\delta\\ \{Z_k>-\delta-x_k,(-\delta-x_k)/Z_k(x_k)\le w\le 1\}, & x_k<-\delta\end{cases}.\end{aligned}$$

When $x_k>\delta$, we have

$$\begin{aligned}&\Big|\mathbb{E}_{\xi_k(x_k)}\Big[\int_0^1(1-w)k^2f''(x_k+wZ_k(x_k))Z_k^2(x_k)dw\Big]\Big|\\
&\le \mathbb{E}_{\xi_k(x_k)}\Big[\int_0^1(1-w)k^2|f''(x_k+wZ_k(x_k))|Z_k^2(x_k)\mathbf{1}_{\{|f''(x_k+wZ_k(x_k))|>0\}}dw\Big]\\
&\le \mathbb{E}_{\xi_k(x_k)}\Big[\int_{(\delta-x_k)/Z_k(x_k)}^1(1-w)k^2||f^{''}||Z_k^2(x_k)\mathbf{1}_{\{Z_k(x_k)<\delta-x_k\}}dw\Big]\\
&=\frac{1}{2}||f''||\mathbb{E}_{\xi_k(x_k)}\Big[k^2\Big\{Z_k(x_k)-\delta+x_k\Big\}^2\mathbf{1}_{\{Z_k(x_k)<\delta-x_k\}}\Big]\\
&=\frac{1}{2}||f''||\mathbb{E}_{\xi_k(x_k)}\Big[k^2\Big\{k^{-1/2}Y_k(1)-k^{-1/2}\xi_k(x_k)-\delta+x_k\Big\}^2\mathbf{1}_{U_k^1}\Big]\\
&=\frac{1}{2}||f''||\mathbb{E}_{\xi_k(x_k)}\Big[k^2\Big\{k^{-1/2}Y_k(1)-\delta-k^{-1/2}(ka+k^{1/2}x_k)+x_k\Big\}^2\mathbf{1}_{U_k^1}\Big]\\
&=\frac{1}{2}||f''||\mathbb{E}_{\xi_k(x_k)}\Big[k^2\Big\{k^{-1/2}Y_k(1)-\delta-k^{-1/2}(ka+k^{1/2}x_k)G_k^{'}(1)\\
&\qquad\qquad +k^{-1/2}ka(G_k^{'}(1)-1)+x_kG_k^{'}(1)\Big\}^2\mathbf{1}_{U_k^1}\Big]\\
&=\frac{1}{2}||f''||\mathbb{E}_{\xi_k(x_k)}\Big[k^2\Big\{k^{-1/2}Y_k(1)-k^{-1/2}\xi_k(x_k)G_k^{'}(1)\\
&\qquad\qquad -k^{-1/2}H_k^{'}(1)-(\delta-x_kG_k^{'}(1))\Big\}^2\mathbf{1}_{U_k^2}\Big]\\
&\le\frac{1}{2}||f^{''}||\mathbb{E}_{\xi_k(x_k)}\Big[k^2\Big\{k^{-1/2}Y_k(1)-k^{-1/2}\xi_k(x_k)G_k^{'}(1)-k^{-1/2}H_k^{'}(1)\Big\}^2\mathbf{1}_{U_k^2}\Big]\\
&\le\frac{1}{2}||f^{''}||\mathbb{E}_{\xi_k(x_k)}\Big[k\Big\{Y_k(1)-\xi_k(x_k)G_k^{'}(1)-H_k^{'}(1)\Big\}^2\mathbf{1}_{U_k^0}\Big],\end{aligned}$$

where

$$U_k^1=\big\{Y_k(1)-\xi_k(x_k)<k^{1/2}(\delta-x_k)\big\},$$

$$U_k^2 = \big\{Y_k(1) - \xi_k(x_k)G_k^{'}(1) - H_k^{'}(1) < k^{1/2}(\delta - x_k G_k^{'}(1))\big\}.$$

By the same arguments, (A.1) also holds when $x_k < -\delta$.

Now we prove that

$$\lim_{k\to\infty} \mathbb{E}_{\xi_k(x_k)}\Big[k\Big\{Y_k(1) - \xi_k(x_k)G_k^{'}(1) - H_k^{'}(1)\Big\}^2 \mathbf{1}_{U_k^0}\Big] = 0.$$

According to the definition of GWI processes, let

$$W_k = \sum_{j=1}^{\xi_k(x_k)} \zeta(1,j), \qquad \eta_k = \eta_k(1).$$

Then $E[e^{-\lambda W_k}] = G_k(e^{-\lambda})^{\xi_k(x_k)}$, $E(e^{-\lambda\eta_k}) = H_k(e^{-\lambda})$. Let

$$\begin{aligned}
I_k^1 &= \{|W_k + \eta_k - \xi_k(x_k)G_k^{'}(1) - H_k^{'}(1)| \geq k^{1/2}(|x_k|G_k^{'}(1) - \delta)\},\\
I_k^2 &= \{|W_k - \xi_k(x_k)G_k^{'}(1)| \geq \frac{1}{2}k^{1/2}(|x_k|G_k^{'}(1) - \delta)\},\\
I_k^3 &= \{|\eta_k - H_k^{'}(1)| \geq \frac{1}{2}k^{1/2}(|x_k|G_k^{'}(1) - \delta)\}.
\end{aligned}$$

Then based on the assumptions (H2) and (H3) and a Taylor expansion, one obtains that

$$\begin{aligned}
&\mathbb{E}_{\xi_k(x_k)}\Big[k\Big\{Y_k(1) - \xi_k(x_k)G_k^{'}(1) - H_k^{'}(1)\Big\}^2 \mathbf{1}_{U_k^0}\Big]\\
&= \mathbb{E}\Big[k\Big\{W_k + \eta_k - \xi_k(x_k)G_k^{'}(1) - H_k^{'}(1)\Big\}^2 \mathbf{1}_{I_k^1}\Big]\\
&\leq 2\mathbb{E}\left[k\left[\Big\{W_k - \xi_k(x_k)G_k^{'}(1)\Big\}^2 + \Big\{\eta_k - H_k^{'}(1)\Big\}^2\right]\mathbf{1}_{I_k^2}\right]\\
&\quad +2\mathbb{E}\left[k\left[\Big\{W_k - \xi_k(x_k)G_k^{'}(1)\Big\}^2 + \Big\{\eta_k - H_k^{'}(1)\Big\}^2\right]\mathbf{1}_{I_k^3}\right]\\
&\leq 2\mathbb{E}\Big[k\Big\{W_k - \xi_k(x_k)G_k^{'}(1)\Big\}^2 \mathbf{1}_{I_k^2}\Big] \times \left[1 + \frac{4E\Big[\Big\{\eta_k - H_k^{'}(1)\Big\}^2\Big]}{k\Big\{|x_k|G_k^{'}(1) - \delta\Big\}^2}\right]\\
&\quad +2\mathbb{E}\Big[k\Big\{\eta_k - H_k^{'}(1)\Big\}^2 \mathbf{1}_{I_k^3}\Big] \times \left[1 + \frac{4E\Big[\Big\{W_k - \xi_k(x_k)G_k^{'}(1)\Big\}^2\Big]}{k\Big\{|x_k|G_k^{'}(1) - \delta\Big\}^2}\right]\\
&\leq 2\mathbb{E}\Big[\Big\{W_k - \xi_k(x_k)G_k^{'}(1)\Big\}^4\Big] \frac{4k}{k\Big\{|x_k|G_k^{'}(1) - \delta\Big\}^2} \times \left[1 + \frac{4E\Big[\Big\{\eta_k - H_k^{'}(1)\Big\}^2\Big]}{k\Big\{|x_k|G_k^{'}(1) - \delta\Big\}^2}\right]
\end{aligned}$$

$$+2\mathbb{E}\Big[\Big\{\eta_k - H_k'(1)\Big\}^4\Big]\frac{4k}{k\Big\{|x_k|G_k'(1)-\delta\Big\}^2} \times \Big[1+\frac{4E\Big[\Big\{W_k-\xi_k(x_k)G_k'(1)\Big\}^2\Big]}{k\Big\{|x_k|G_k'(1)-\delta\Big\}^2}\Big].$$

Since the right side of the above inequality goes to 0 as $k \to \infty$, the above inequality implies that Lemma 1 holds.

Proof of Lemma 2. Because $\mathbb{E}_{\xi_k(x_k)}[Z_k^2(x_k)] < \infty$ and $f \in C_c^\infty(\mathbb{R})$, we have by Fubini's Theorem and the uniformly continuity of f'' that

$$\begin{aligned}&\lim_{k\to\infty}\mathbb{E}_{\xi_k(x_k)}\Big[\int_0^1(1-w)\Big\{k^2f''(x_k+wZ_k(x_k))Z_k^2(x_k)\Big\}dw\Big]\\&\quad=\lim_{k\to\infty}\int_0^1(1-w)\mathbb{E}_{\xi_k(x_k)}\Big[k^2f''(x+wZ_k(x_k))Z_k^2(x_k)\Big]dw.\end{aligned}$$

We first prove that when $k \to \infty$,

$$-k^2\frac{d^2}{d\lambda^2}\mathbb{E}_{\xi_k(x_k)}[e^{i\lambda Z_k(x_k)}] \to a\alpha + c\Big\{b+\int_{0+}^\infty u^2e^{i\lambda u}\mu(du)\Big\}. \tag{A.1}$$

Note condition (H1) and (H2) ensure that $G_k(e^{i\lambda k^{-1/2}}) \neq 0$. Let $L_k(\lambda) = -i\lambda k^{-1/2}\xi_k(x_k) + \xi_k(x_k)\log G_k(e^{i\lambda k^{-1/2}})$. By using the Dominated Convergence Theorem,

$$\begin{aligned}&\mathbb{E}_{\xi_k(x_k)}[k^2e^{i\lambda Z_k(x_k)}Z_k^2(x_k)]\\&\quad=-k^2\frac{d^2}{d\lambda^2}\mathbb{E}_{\xi_k(x_k)}[e^{i\lambda Z_k(x_k)}] = -k^2\frac{d^2}{d\lambda^2}\Big[e^{L_k(\lambda)}H_k(e^{i\lambda k^{-1/2}})\Big]\\&\quad=-k^2L_k''(\lambda)e^{L_k(\lambda)}H_k(e^{i\lambda k^{-1/2}}) - k^2(L_k'(\lambda))^2e^{L_k(\lambda)}H_k(e^{i\lambda k^{-1/2}})\\&\qquad-2k^2L_k'(\lambda)e^{L_k(\lambda)}H_k'(e^{i\lambda k^{-1/2}})(\frac{ie^{i\lambda k^{-1/2}}}{k^{1/2}})\\&\qquad-k^2e^{L_k(\lambda)}H_k''(e^{i\lambda k^{-1/2}})e^{2i\lambda k^{-1/2}}(-\frac{1}{k})\\&\qquad-k^2e^{L_k(\lambda)}H_k'(e^{i\lambda k^{-1/2}})e^{i\lambda k^{-1/2}}(-\frac{1}{k}).\end{aligned}$$

So we can deduce that, as $k \to \infty$,

$$\begin{aligned}&-k^2\frac{d^2}{d\lambda^2}\mathbb{E}_{\xi_k(x_k)}\Big[e^{i\lambda Z_k(x_k)}\Big]\\&\quad=-k^2\Big[-\frac{\xi_k(x_k)}{k}\frac{e^{2i\lambda k^{-1/2}}}{G_k(e^{i\lambda k^{-1/2}})}G_k''(e^{i\lambda k^{-1/2}})\Big]e^{L_k(\lambda)}H_k(e^{i\lambda k^{-1/2}})\\&\qquad+a\alpha-k^2e^{L_k(\lambda)}H_k''(e^{i\lambda k^{-1/2}})e^{2i\lambda k^{-1/2}}(-\frac{1}{k})\end{aligned}$$

$$
\begin{aligned}
&-k^2 e^{L_k(\lambda)} H_k'(e^{i\lambda k^{-1/2}}) e^{i\lambda k^{-1/2}} (-\frac{1}{k}) + o(1) \\
&= k^2 a e^{2i\lambda k^{-1/2}} G_k''(e^{i\lambda k^{-1/2}}) + a\alpha + kH_k''(e^{i\lambda k^{-1/2}}) e^{2i\lambda k^{-1/2}} \\
&\quad + kH_k'(e^{i\lambda k^{-1/2}}) + o(1) \\
&\to 2a\alpha + a\Upsilon_1(\lambda) + \Upsilon_2(\lambda). \qquad \text{(A.2)}
\end{aligned}
$$

What is more,

$$
\begin{aligned}
&kH_k''(e^{i\lambda k^{-1/2}}) e^{2i\lambda k^{-1/2}} + kH_k'(e^{i\lambda k^{-1/2}}) e^{i\lambda k^{-1/2}} + k^2 a e^{2i\lambda k^{-1/2}} G_k''(e^{i\lambda k^{-1/2}}) \\
&= -k^2 \frac{d^2}{d\lambda^2} H_k(e^{i\lambda k^{-1/2}}) + k^2 a e^{2i\lambda k^{-1/2}} G_k''(e^{i\lambda k^{-1/2}}) \\
&= \sum_{m=0}^{\infty} k m^2 e^{i\lambda k^{-1/2} m} h_k(m) + k^2 a \sum_{m=0}^{\infty} m(m-1) e^{i\lambda k^{-1/2} m} g_k(m) \\
&= \sum_{m=0}^{\infty} e^{i\lambda k^{-1/2} m} \left[k m^2 h_k(m) + k^2 a m(m-1) g_k(m) \right],
\end{aligned}
$$

which is proportional to the characteristic function of a nonnegative integer valued process with factor $kH_k''(1) + kH_k'(1) + k^2 a G_k''(1)$. Here $g_k(\cdot)$ and $h_k(\cdot)$ are the distribution laws corresponding to the generating functions $G_k(\cdot)$ and $H_k(\cdot)$, respectively. Observe that $kH_k''(1)+kH_k'(1)+k^2aG_k''(1) \to c$. So, according to the Lévy Continuity Theorem, these ensure the existence of a probability measure μ_1 such that

$$
\frac{-k^2 (d^2/d\lambda^2) \mathbb{E}_{\xi_k(x_k)}[e^{i\lambda Z_k(x_k)}] - a\alpha}{kH_k''(1) + kH_k'(1) + k^2 a G_k''(1)} \to \int_0^\infty e^{i\lambda u} \mu_1(du),
$$

and hence (A.1) holds. Alternatively, based on (A.2) and (H2)−(H3), we know Υ_j, j=1,2, are characteristic functions of finite measures on $[0,\infty)$, and hence there exists a probability measure μ_1 on $[0,\infty)$ such that

$$
\{-k^2 \frac{d^2}{d\lambda^2} \mathbb{E}_{\xi_k(x_k)}[e^{i\lambda Z_k(x_k)}] - a\alpha\} c^{-1} \to \{a\alpha + a\Upsilon_1(\lambda) + \Upsilon_2(\lambda)\} c^{-1} = \int_0^\infty e^{i\lambda u} \mu_1(du).
$$

Therefore, (A.1) follows.

Next, it is an elementary matter to see that Q_k converges weakly to Q based on this convergence result, where $Q_k, k = 1, 2, \ldots$ and Q are measures defined as follows: for any measurable set $\mathcal{O} \subset \mathbb{R}$,

$$
\begin{aligned}
Q_k(\mathcal{O}) &= \mathbb{E}_{\xi_k(x_k)} \left[k^2 Z_k^2(x_k) \mathbf{1}_{\mathcal{O}}(Z_k(x_k)) \right] / \mathbb{E}_{\xi_k(x_k)} \left[k^2 Z_k^2(x_k) \right], \text{ and} \\
Q(\mathcal{O}) &= \Big[(a\alpha + bc) \mathbf{1}_{\mathcal{O}}(0) + c \int_{\mathcal{O} \cap (0,+\infty)} u^2 \mu(du) \Big] (a\alpha + c)^{-1}.
\end{aligned}
$$

This implies, for all $f \in C_c^\infty(\mathbb{R})$,

$$\mathbb{E}_{\xi_k(x_k)}\left[k^2 f(Z_k(x_k))Z_k^2(x_k)\right] \to (a\alpha + bc)f(0) + c\int_{0+}^\infty u^2 f(u)\mu(du).$$

Finally, applying the Dominated Convergence Theorem again, we have

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^1 (1-w)\mathbb{E}_{\xi_k(x_k)}\Big[k^2 f^{''}(x + wZ_k(x_k))Z_k^2(x_k)\Big]dw \\
&= \int_0^1 (1-w)\lim_{k\to\infty}\mathbb{E}_{\xi_k(x_k)}\Big[k^2 f^{''}(x + wZ_k(x_k))Z_k^2(x_k)\Big]dw \\
&= \int_0^1 (1-w)\Big\{(a\alpha + bc)f^{''}(x) + c\int_{0+}^\infty u^2 f^{''}(x+wu)\mu(du)\Big\}d\omega \\
&= \frac{1}{2}(a\alpha + bc)f^{''}(x) + c\int_{0+}^\infty\int_0^1 (1-w)u^2 f^{''}(x+wu)dw\mu(du) \\
&= \frac{1}{2}(a\alpha + bc)f^{''}(x) + c\int_{0+}^\infty \{f(x+u) - f(x) - uf^{'}(x)\}\mu(du),
\end{aligned}$$

and hence Lemma 2 follows.

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