ON SOME MATÉRN COVARIANCE FUNCTIONS FOR SPATIO-TEMPORAL RANDOM FIELDS

Ryan H.L. Ip and W.K. Li

Charles Sturt University and The University of Hong Kong

Supplementary Material

This supplementary material includes the proofs of the theories and detailed simulation results.

S1 Proofs

The proofs for the theories in Section 2 of the paper are given below.

Lemma 1. *Let* $\mathcal{A}_1 = \{\omega > c, \tau > c\}$, $\mathcal{A}_2 = \{0 \le \omega < c, \tau > c\}$, $\mathcal{A}_3 = \{\omega > c, 0 \le \tau < c\}$, $\mathcal{A}_4 = \{\omega > c_1, 0 \le \tau < c\}$, $\mathcal{A}_5 = \{0 \le \omega < c, \tau > c_2\}$, *and* c_1, c_2 *be two constants satisfying* $0 < c_1, c_2 < c$ *such that* $c_1^2 + c_2^2 < c^2$. *For isotropic spectral densities, the condition*

$$\iint_{\|\boldsymbol{\omega},\tau\|>c} \left\{ \frac{f_1(\boldsymbol{\omega},\tau) - f_0(\boldsymbol{\omega},\tau)}{f_0(\boldsymbol{\omega},\tau)} \right\}^2 \mathrm{d}\boldsymbol{\omega}\mathrm{d}\tau < \infty$$

holds if

$$\iint_{\mathcal{A}} \omega^{d-1} \left\{ \frac{f_1(\omega,\tau) - f_0(\omega,\tau)}{f_0(\omega,\tau)} \right\}^2 \mathrm{d}\omega\mathrm{d}\tau < \infty$$

holds where $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_5$.

Proof. For simplicity of notation, let

$$h(\boldsymbol{\omega}, \tau) = \left\{ \frac{f_1(\boldsymbol{\omega}, \tau) - f_0(\boldsymbol{\omega}, \tau)}{f_0(\boldsymbol{\omega}, \tau)} \right\}^2 .$$

For any sets $\mathcal{B}_1$ and $\mathcal{B}_2$ with $\mathcal{B}_1 \subseteq \mathcal{B}_2$, we have $\iint_{\mathcal{B}_1} h(\boldsymbol{\omega}, \tau)\, \mathrm{d}\boldsymbol{\omega}\mathrm{d}\tau \leq \iint_{\mathcal{B}_2} h(\boldsymbol{\omega}, \tau)\, \mathrm{d}\boldsymbol{\omega}\mathrm{d}\tau$ since $h(\boldsymbol{\omega}, \tau) \geq 0$. As $\|\boldsymbol{\omega}, \tau\| = \sqrt{\omega^2 + \tau^2}$, it follows that $\|\boldsymbol{\omega}, \tau\| > c$ is equivalent to $\omega^2 + \tau^2 > c^2$. For any $\omega, \tau \geq 0$, it can be verified that $\{\omega^2 + \tau^2 > c^2\} \subseteq \mathcal{A}$ for all c_1 and c_2 satisfying $0 < c_1, c_2 < c$ and $c_1^2 + c_2^2 < c^2$. Hence, we have

$$\iint_{\|\boldsymbol{\omega}, \tau\| > c} h(\boldsymbol{\omega}, \tau)\, \mathrm{d}\boldsymbol{\omega}\mathrm{d}\tau \leq \iint_{\mathcal{A}} h(\boldsymbol{\omega}, \tau)\, \mathrm{d}\boldsymbol{\omega}\mathrm{d}\tau.$$

Finally, since f is isotropic, it can be verified from Prudnikov et al. (1986, Chapter 3) and Gradshteyn and Ryzhik (2007, Chapter 4) that

$$\iint_{\mathcal{A}} h(\boldsymbol{\omega}, \tau)\, \mathrm{d}\boldsymbol{\omega}\mathrm{d}\tau = \iint_{\mathcal{A}} \omega^{d-1} h(\omega, \tau)\, \mathrm{d}\omega\mathrm{d}\tau.$$

□

Proof of Theorem 1

We begin with the sufficiency parts of (a) and (b). It can be seen that $f_0(\omega, \tau) \|\boldsymbol{\omega}, \tau\|^{2\nu}$ is bounded away from zero and infinity. Let $\varepsilon \in (0, 1]$. The sufficiency part of Theorem 1(a) can be obtained by simply putting $\varepsilon = 1$. When $\varepsilon = 1$, by direct calculation,

$$\gamma = \frac{\sigma^2 \alpha^{2\nu-d} \beta^{2\nu-1} \Gamma(\nu)^2}{\Gamma\left(\nu - \frac{d}{2}\right) \Gamma\left(\nu - \frac{1}{2}\right)}.$$

If $\gamma_0/\varepsilon_0^{2\nu} = \gamma_1/\varepsilon_1^{2\nu}$,

$$\begin{aligned}
& \left| \frac{f_1(\omega,\tau) - f_0(\omega,\tau)}{f_0(\omega,\tau)} \right| \\
= & \left| \frac{\gamma_1 \left(\alpha_0^2\beta_0^2 + \beta_0^2\omega^2 + \alpha_0^2\tau^2 + \varepsilon_0^2\omega^2\tau^2\right)^\nu}{\gamma_0 \left(\alpha_1^2\beta_1^2 + \beta_1^2\omega^2 + \alpha_1^2\tau^2 + \varepsilon_1^2\omega^2\tau^2\right)^\nu} - 1 \right| \\
= & \left| \frac{(\gamma_1/\varepsilon_1^{2\nu}) \left(\frac{\alpha_0^2\beta_0^2}{\varepsilon_0^2} + \frac{\beta_0^2}{\varepsilon_0^2}\omega^2 + \frac{\alpha_0^2}{\varepsilon_0^2}\tau^2 + \omega^2\tau^2 \right)^\nu}{(\gamma_0/\varepsilon_0^{2\nu}) \left(\frac{\alpha_1^2\beta_1^2}{\varepsilon_1^2} + \frac{\beta_1^2}{\varepsilon_1^2}\omega^2 + \frac{\alpha_1^2}{\varepsilon_1^2}\tau^2 + \omega^2\tau^2 \right)^\nu} - 1 \right| \\
\leq & \frac{\left| \left(\frac{\alpha_0^2\beta_0^2}{\varepsilon_0^2} + \frac{\beta_0^2}{\varepsilon_0^2}\omega^2 + \frac{\alpha_0^2}{\varepsilon_0^2}\tau^2 + \omega^2\tau^2 \right)^\nu - \left(\frac{\alpha_1^2\beta_1^2}{\varepsilon_1^2} + \frac{\beta_1^2}{\varepsilon_1^2}\omega^2 + \frac{\alpha_1^2}{\varepsilon_1^2}\tau^2 + \omega^2\tau^2 \right)^\nu \right|}{(\omega^2\tau^2)^\nu} \\
= & \left| \left(\frac{\alpha_0^2\beta_0^2}{\varepsilon_0^2\omega^2\tau^2} + \frac{\beta_0^2}{\varepsilon_0^2\tau^2} + \frac{\alpha_0^2}{\varepsilon_0^2\omega^2} + 1 \right)^\nu - \left(\frac{\alpha_1^2\beta_1^2}{\varepsilon_1^2\omega^2\tau^2} + \frac{\beta_1^2}{\varepsilon_1^2\tau^2} + \frac{\alpha_1^2}{\varepsilon_1^2\omega^2} + 1 \right)^\nu \right|. \qquad \text{(S1.1)}
\end{aligned}$$

Using the approximation $(1+y)^\gamma = 1 + \gamma y + O(y^2)$ as $y \to 0$, (S1.1) can be written as

$$\left| \left(\frac{\nu\alpha_0^2\beta_0^2}{\varepsilon_0^2\omega^2\tau^2} - \frac{\nu\alpha_1^2\beta_1^2}{\varepsilon_1^2\omega^2\tau^2} \right) + \left(\frac{\nu\beta_0^2}{\varepsilon_0^2\tau^2} - \frac{\nu\beta_1^2}{\varepsilon_1^2\tau^2} \right) + \left(\frac{\nu\alpha_0^2}{\varepsilon_0^2\omega^2} - \frac{\nu\alpha_1^2}{\varepsilon_1^2\omega^2} \right) + O\left(\omega^{-4}\tau^{-4}\right) \right| \qquad \text{(S1.2)}$$

as $\omega, \tau \to \infty$. Assume $\lim_{\omega,\tau\to\infty} (\omega/\tau) = k < \infty$, there exists a constant $k^* > 0$ such that $k^*(\omega^2 + \tau^2) \leq \min(\varepsilon_0^2\varepsilon_1^2\omega^2\tau^2, \varepsilon_0^2\varepsilon_1^2\omega^2, \varepsilon_0^2\varepsilon_1^2\tau^2)$ as $\omega, \tau \to \infty$. Then, from (S1.2),

$$\begin{aligned}
& \left| \left(\frac{\nu\alpha_0^2\beta_0^2}{\varepsilon_0^2\omega^2\tau^2} - \frac{\nu\alpha_1^2\beta_1^2}{\varepsilon_1^2\omega^2\tau^2} \right) + \left(\frac{\nu\beta_0^2}{\varepsilon_0^2\tau^2} - \frac{\nu\beta_1^2}{\varepsilon_1^2\tau^2} \right) + \left(\frac{\nu\alpha_0^2}{\varepsilon_0^2\omega^2} - \frac{\nu\alpha_1^2}{\varepsilon_1^2\omega^2} \right) + O\left(\omega^{-4}\tau^{-4}\right) \right| \\
\leq & \frac{\left|\nu\left(\varepsilon_1^2\alpha_0^2\beta_0^2 - \varepsilon_0^2\alpha_1^2\beta_1^2\right)\right|}{\varepsilon_0^2\varepsilon_1^2\omega^2\tau^2,} + \frac{\left|\nu\left(\varepsilon_1^2\beta_0^2 - \varepsilon_0^2\beta_1^2\right)\right|}{\varepsilon_0^2\varepsilon_1^2\tau^2} + \frac{\left|\nu\left(\varepsilon_1^2\alpha_0^2 - \varepsilon_0^2\alpha_1^2\right)\right|}{\varepsilon_0^2\varepsilon_1^2\omega^2} + \left|O\left(\omega^{-4}\tau^{-4}\right)\right| \\
\leq & \frac{\nu\left(\left|\varepsilon_1^2\alpha_0^2\beta_0^2 - \varepsilon_0^2\alpha_1^2\beta_1^2\right| + \left|\varepsilon_1^2\beta_0^2 - \varepsilon_0^2\beta_1^2\right| + \left|\varepsilon_1^2\alpha_0^2 - \varepsilon_0^2\alpha_1^2\right|\right)}{k^*\left(\omega^2 + \tau^2\right)} + O\left(\omega^{-4}\tau^{-4}\right).
\end{aligned}$$

Therefore, by Lemma 1,

$$\begin{aligned}
& \iint_{\|\boldsymbol{\omega},\tau\|>c} \omega^{d-1}\left\{\frac{f_1(\omega,\tau)-f_0(\omega,\tau)}{f_0(\omega,\tau)}\right\}^2 \mathrm{d}\tau\mathrm{d}\omega \\
\leq\ & \iint_{\mathcal{A}} \omega^{d-1}\left\{\frac{f_1(\omega,\tau)-f_0(\omega,\tau)}{f_0(\omega,\tau)}\right\}^2 \mathrm{d}\tau\mathrm{d}\omega \\
\leq\ & \sum_{j=1}^{5}\iint_{\mathcal{A}_j} \omega^{d-1}\left\{\frac{f_1(\omega,\tau)-f_0(\omega,\tau)}{f_0(\omega,\tau)}\right\}^2 \mathrm{d}\tau\mathrm{d}\omega \\
\leq\ & \left[\frac{\nu\left(|\varepsilon_1^2\alpha_0^2\beta_0^2-\varepsilon_0^2\alpha_1^2\beta_1^2|+|\varepsilon_1^2\beta_0^2-\varepsilon_0^2\beta_1^2|+|\varepsilon_1^2\alpha_0^2-\varepsilon_0^2\alpha_1^2|\right)}{k^*}\right]^2 \times \sum_{j=1}^{5}\iint_{\mathcal{A}_j}\frac{\omega^{d-1}}{\left(\omega^2+\tau^2\right)^2}\mathrm{d}\tau\mathrm{d}\omega.
\end{aligned} \tag{S1.3}$$

Direct integration shows that all $\iint_{\mathcal{A}_j}\omega^{d-1}\left(\omega^2+\tau^2\right)^{-2}\mathrm{d}\tau\mathrm{d}\omega, j=1,\ldots,5$ are finite for $d\leq 2$. Hence, (S1.3) is finite for $d\leq 2$.

For the necessity part of (a), if $\sigma_0^2\alpha_0^{2\nu-d}\beta_0^{2\nu-1}\neq\sigma_1^2\alpha_1^{2\nu-d}\beta_1^{2\nu-1}$, let

$$\sigma_2^2=\sigma_1^2\left(\frac{\alpha_1^{2\nu-d}\beta_1^{2\nu-1}}{\alpha_0^{2\nu-d}\beta_0^{2\nu-1}}\right).$$

Then, we have $\sigma_2^2\alpha_0^{2\nu-d}\beta_0^{2\nu-1}=\sigma_1^2\alpha_1^{2\nu-d}\beta_1^{2\nu-1}$ and therefore, $\sigma_2^2M_\nu^1(\alpha_0,\beta_0)$ and $\sigma_1^2M_\nu^1(\alpha_1,\beta_1)$ define two equivalent measures. It remains to show that $\sigma_2^2M_\nu^1(\alpha_0,\beta_0)$ and $\sigma_0^2M_\nu^1(\alpha_0,\beta_0)$ define two orthogonal measures since Gaussian measures are either equivalent or orthogonal (Ibragimov and Rozanov, 1978, Thm 1, p. 77). This can be seen following the lines in the proof of Theorem 1 in Zhang (2004, p. 260). This completes the proof of parts (a) and (b).

For part (c), when $\varepsilon=0$, by direct calculation,

$$\gamma=\frac{\sigma^2\alpha^{2\nu-d}\beta^{2\nu-1}\Gamma(\nu)}{\Gamma\left(\nu-\frac{d+1}{2}\right)}.$$

For the sufficiency part, with the techniques used in proving part (a), assume $\beta_0/\alpha_0=$

$\beta_1/\alpha_1 = r$ and $\sigma_0^2\alpha_0^{-d}\beta_0^{2\nu-1} = \sigma_1^2\alpha_1^{-d}\beta_1^{2\nu-1}$, it can be shown that

$$\begin{aligned}
\left|\frac{f_1(\omega,\tau)-f_0(\omega,\tau)}{f_0(\omega,\tau)}\right| &= \left|\frac{\sigma_1^2\alpha_1^{2\nu-d}\beta_1^{2\nu-1}\left(\alpha_0^2\beta_0^2+\beta_0^2\omega^2+\alpha_0^2\tau^2\right)^\nu}{\sigma_0^2\alpha_0^{2\nu-d}\beta_0^{2\nu-1}\left(\alpha_1^2\beta_1^2+\beta_1^2\omega^2+\alpha_1^2\tau^2\right)^\nu}-1\right| \\
&= \left|\frac{\sigma_1^2\alpha_1^{2\nu-d}\beta_1^{2\nu-1}/\alpha_1^{2\nu}\left(\beta_0^2+\frac{\beta_0^2}{\alpha_0^2}\omega^2+\tau^2\right)^\nu}{\sigma_0^2\alpha_0^{2\nu-d}\beta_0^{2\nu-1}/\alpha_0^{2\nu}\left(\beta_1^2+\frac{\beta_1^2}{\alpha_1^2}\omega^2+\tau^2\right)^\nu}-1\right| \\
&= \left|\frac{\left(\beta_0^2+r^2\omega^2+\tau^2\right)^\nu}{\left(\beta_1^2+r^2\omega^2+\tau^2\right)^\nu}-1\right| \\
&\le \frac{\left|\left(\beta_0^2+r^2\omega^2+\tau^2\right)^\nu-\left(\beta_1^2+r^2\omega^2+\tau^2\right)^\nu\right|}{\left(r^2\omega^2+\tau^2\right)^\nu} \\
&= \left|\left(\frac{\beta_0^2}{r^2\omega^2+\tau^2}+1\right)^\nu-\left(\frac{\beta_1^2}{r^2\omega^2+\tau^2}+1\right)^\nu\right| \\
&= \left|\frac{\nu\left(\beta_0^2-\beta_1^2\right)}{r^2\omega^2+\tau^2}+O\left(\left(\omega^2+\tau^2\right)^{-2}\right)\right| \quad \text{as } \omega,\tau\to\infty \\
&\le \frac{\nu\left|\beta_0^2-\beta_1^2\right|}{r^2\omega^2+\tau^2}+O\left(\left(\omega^2+\tau^2\right)^{-2}\right).
\end{aligned}$$

Hence, it can be checked that the integral in (2.3) is finite for $d \le 2$ using similar techniques as in the proof of Theorem 1.

For the necessity part, we first assume $r_1 = r_0 = r$ but $\sigma_1^2\alpha_1^{-d}\beta_1^{2\nu-1} \ne \sigma_0^2\alpha_0^{-d}\beta_0^{2\nu-1}$, i.e., $\sigma_1^2\beta_1^{2\nu-d-1} \ne \sigma_0^2\beta_0^{2\nu-d-1}$, then the covariograms under P_i is $\sigma_i^2 M_\nu(y|\beta_i)$ for $y = \sqrt{r^2u^2+h^2}$. Hence, by Theorem 3 of Zhang (2004), P_0 and P_1 are orthogonal since $\sigma_1^2\beta_1^{2\nu-d-1} \ne \sigma_0^2\beta_0^{2\nu-d-1}$.

Next, we assume $\sigma_1^2\alpha_1^{-d}\beta_1^{2\nu-1} = \sigma_0^2\alpha_0^{-d}\beta_0^{2\nu-1}$ but $r_1 \ne r_0$. It is intended to show that P_1 which is defined by $\sigma_1^2 M_\nu^0(\alpha_1,\beta_1)$ and P_0 which is defined by $\sigma_0^2 M_\nu^0(\alpha_0,\beta_0)$ are orthogonal. Let P_2 be the Gaussian measure defined by $\sigma_2^2 M_\nu^0(\alpha_0, r_1\alpha_0)$ such that $r_2 = r_1$ and $\sigma_2^2\alpha_0^{-d}(r_1\alpha_0)^{2\nu-1} = \sigma_1^2\alpha_1^{-d}\beta_1^{2\nu-1} = \sigma_0^2\alpha_0^{-d}\beta_0^{2\nu-1}$. From the sufficiency part, we can immediately conclude that $P_2 \equiv P_1$. Hence, it suffices to show that P_2 and P_0 are orthogonal. Note that the covariograms under P_0 and P_2 are $\sigma_0^2 M_\nu^0(\alpha_0,\beta_0)$ and $\sigma_0^2(r_0/r_1)^{2\nu-d-1} M_\nu^0(\alpha_0, r_1\alpha_0)$

respectively. By Lemma 1 of Ibragimov and Rozanov (1978, p. 72), two probability measures are equivalent if and only if they are also equivalent on any linear combinations of the random variables generated. Define

$$\psi_i = \sum_{j,k=1}^{n_i} a_{ijk} X\left(\boldsymbol{s}_{ij}, t_{ik}\right) \tag{S1.4}$$

with $a_{ijk} = 0$ if $t_{ik} \neq t_0$ for some $t_0 \in \mathcal{D}_T \subset \mathbb{R}$ and $n_i > 0$. Note that under this construction, $u_{i,i'} = |t_{ik} - t_{i'k}| = 0$ and it can be seen that $\mathbb{E}_2\left(\psi_i\psi_{i'}\right) = \left(r_0/r_1\right)^{2\nu-d-1}\mathbb{E}_0\left(\psi_i\psi_{i'}\right)$ where $\mathbb{E}_i$ denotes the expectation under probability measure i. Therefore

$$\sum_{i,i'=1}^{\infty}\left[\mathbb{E}_2\left(\psi_i\psi_{i'}\right) - \mathbb{E}_0\left(\psi_i\psi_{i'}\right)\right]^2 = \infty.$$

By Theorem 7 of Stein (1999, p. 129), P_2 and P_0 are orthogonal implying that P_0 and P_1 are orthogonal.

Finally, if $r_1 \neq r_0$ and $\sigma_1^2\alpha_1^{-d}\beta_1^{2\nu-1} \neq \sigma_0^2\alpha_0^{-d}\beta_0^{2\nu-1}$, we can find a measure P_2 defined by $\sigma_2^2 M_\nu^0\left(\alpha_1, r_0\alpha_1\right)$ such that $\sigma_2^2\alpha_1^{-d}\left(r_0\alpha_1\right)^{2\nu-1} = \sigma_0^2\alpha_0^{-d}\left(r_0\alpha_0\right)^{2\nu-1}$, i.e., $\sigma_2^2\alpha_1^{2\nu-d-1} = \sigma_0^2\alpha_0^{2\nu-d-1}$, then $P_2 \equiv P_0$. Because $\sigma_2^2\alpha_1^{-d}\left(r_0\alpha_1\right)^{2\nu-1} \neq \sigma_1^2\alpha_1^{-d}\beta_1^{2\nu-1}$, there exists a positive finite constant $c \neq 1$ such that $\sigma_2^2\alpha_1^{-d}\left(r_0\alpha_1\right)^{2\nu-1} = c\sigma_1^2\alpha_1^{-d}\beta_1^{2\nu-1}$ or $\sigma_2^2 = c\left(r_1/r_0\right)^{2\nu-1}\sigma_1^2$.

Defining ψ_i as in (S1.4), it can be seen that $\mathbb{E}_2\left(\psi_i\psi_{i'}\right) = c\left(r_1/r_0\right)^{2\nu-1}\mathbb{E}_1\left(\psi_i\psi_{i'}\right)$ and hence

$$\sum_{i,i'=1}^{\infty}\left[\mathbb{E}_2\left(\psi_i\psi_{i'}\right) - \mathbb{E}_1\left(\psi_i\psi_{i'}\right)\right]^2 = \infty$$

implying that P_2 and P_1 are orthogonal (Stein, 1999, Thm. 7, p. 129). Therefore, P_0 and P_1 are orthogonal. □

Proof of Corollary 1

Based on Theorem 1, i.e., the existence of equivalent Gaussian measures, Corollary 1 follows immediately from Corollary 1 of Zhang (2004). □

Proof of Theorem 2

Define $\phi = \sigma^{-2}$, since α and β are fixed, ignoring the terms without ϕ, write the log-likelihood (2.4) as

$$\ell_n(\phi) = -\frac{1}{2}\left(KT\log\frac{1}{\phi} + \phi \boldsymbol{X}_n^\top \boldsymbol{\Gamma}_{\alpha_1,\beta_1}^{-1}\boldsymbol{X}_n\right)$$

and it is readily seen that the second derivative of $\ell_n(\phi)$ with respect to ϕ is $\ell_n''(\phi) = -KT/(2\phi^2) < 0$. The concavity of $\ell_n(\phi)$ together with Theorem 1 guarantee the result $\hat{\sigma}_n^2\alpha_1^{2\nu-d}\beta_1^{2\nu-1} \to \sigma_0^2\alpha_0^{2\nu-d}\beta_0^{2\nu-1}$ holds almost surely as $K, T \to \infty$ under P_0, following Theorem 3 of Zhang (2004). □

Lemma 2. *Under the conditions in Theorem 1 hold and* $\varepsilon = 1$, *if* $0 < \alpha_a \le \alpha_b < \infty$ *and* $0 < \beta_a \le \beta_b < \infty$, *then* $\zeta^1_{\alpha_b,\beta_b} \ge \zeta^1_{\alpha_a,\beta_a}$.

Proof. The difference

$$\zeta^1_{\alpha_b,\beta_b} - \zeta^1_{\alpha_a,\beta_a} = \frac{1}{KT}\boldsymbol{X}_n^\top\left\{\alpha_b^{2\nu-d}\beta_b^{2\nu-1}\boldsymbol{\Gamma}^{-1}_{\alpha_b,\beta_b} - \alpha_a^{2\nu-d}\beta_a^{2\nu-1}\boldsymbol{\Gamma}^{-1}_{\alpha_a,\beta_a}\right\}\boldsymbol{X}_n$$

is non-negative if $\alpha_b^{2\nu-d}\beta_b^{2\nu-1}\boldsymbol{\Gamma}^{-1}_{\alpha_b,\beta_b} - \alpha_a^{2\nu-d}\beta_a^{2\nu-1}\boldsymbol{\Gamma}^{-1}_{\alpha_a,\beta_a}$ is non-negative definite, which is the case if

$$\alpha_a^{d-2\nu}\beta_a^{1-2\nu}\boldsymbol{\Gamma}_{\alpha_a,\beta_a} - \alpha_b^{d-2\nu}\beta_b^{1-2\nu}\boldsymbol{\Gamma}_{\alpha_b,\beta_b} \tag{S1.5}$$

is non-negative definite (Horn and Johnson, 1990, Chapter 7). The latter can be proven through Fourier transform. Indeed, it remains to show that the matrix formed by the differences

$$\eta(h,u|\alpha_a,\alpha_b,\beta_a,\beta_b) = \alpha_a^{d-2\nu}\beta_a^{1-2\nu}M_\nu^1(h,u|\alpha_a,\beta_a) - \alpha_b^{d-2\nu}\beta_b^{1-2\nu}M_\nu^1(h,u|\alpha_b,\beta_b)$$

is a non-negative function. By direct calculation, the Fourier transform of η is

$$\begin{aligned}
&\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{-i\left(\omega^\top h+\tau u\right)}\eta\left(h,u|\alpha_a,\alpha_b,\beta_a,\beta_b\right)\mathrm{d}\boldsymbol{h}\mathrm{d}u \\
=\ &\frac{\Gamma\left(\nu-\frac{d}{2}\right)\Gamma\left(\nu-\frac{1}{2}\right)}{\pi^{\frac{d+1}{2}}}\left\{\left(\alpha_a^2+\omega^2\right)^{-\nu}\left(\beta_a^2+\tau^2\right)^{-\nu}-\left(\alpha_b^2+\omega^2\right)^{-\nu}\left(\beta_b^2+\tau^2\right)^{-\nu}\right\}
\end{aligned}$$

which is always non-negative since $\alpha_a \leq \alpha_b$ and $\beta_a \leq \beta_b$. □

Proof of Theorem 3

From the setting, it is guaranteed that $\alpha_L \leq \hat{\alpha} \leq \alpha_U$ and $\beta_L \leq \hat{\beta} \leq \beta_U$. By Lemma 2, we have $\zeta^1_{\alpha_L,\beta_L} \leq \zeta^1_{\hat{\alpha},\hat{\beta}} \leq \zeta^1_{\alpha_U,\beta_U}$. From Theorem 2, the left hand side and right hand side of the inequality both converge to $\sigma_0^2\alpha_0^{2\nu-d}\beta_0^{2\nu-1}$ almost surely as $K,T\to\infty$. Therefore, $\zeta^1_{\hat{\alpha},\hat{\beta}} \to \sigma_0^2\alpha_0^{2\nu-d}\beta_0^{2\nu-1}$ almost surely. □

Proof of Theorem 4

Since $\beta_1/\alpha_1 = \beta_0/\alpha_0 = r$, the covariogram (1.5) reduces to

$$\frac{\sigma_i^2 2^{1-\nu+\frac{d+1}{2}}}{\Gamma\left(\nu-\frac{d+1}{2}\right)}\left(\alpha_i x\right)^{\nu-\frac{d+1}{2}}\mathcal{K}_{\nu-\frac{d+1}{2}}\left(\alpha_i x\right), \tag{S1.6}$$

where $i=0,1$ and $x=\sqrt{h^2+r^2u^2}$, which is in the form of (1.1). In light of Theorem 3 of Zhang (2004), if $\hat{\sigma}_n^2$ was obtained from maximizing the likelihood function using (S1.6) as the covariance function, then

$$\begin{aligned}
\hat{\sigma}_n^2\alpha_1^{2\nu-d-1} &\to \sigma_0^2\alpha_0^{2\nu-d-1} \\
\Longrightarrow \hat{\sigma}_n^2\alpha_1^{2\nu-d-1}\left(\frac{\beta_1}{\beta_1}\right)^{2\nu-d-1} &\to \sigma_0^2\alpha_0^{2\nu-d-1}\left(\frac{\beta_0}{\beta_0}\right)^{2\nu-d-1} \\
\Longrightarrow \hat{\sigma}_n^2 r^d\beta_1^{2\nu-d-1} &\to \sigma_0^2 r^d\beta_0^{2\nu-d-1}
\end{aligned}$$

which completes the proof. □

Proof of Theorem 5

Let f_{n,r,σ^2} the probability density function of $\boldsymbol{X}_n$ observed in the region $\mathcal{D}_n$ under the probability measure P_{r,σ^2}, where P_{r,σ^2} is the mean-zero Gaussian measure with the Matérn covariogram $\sigma^2 M_\nu^0(\alpha_2, r\alpha_2)$. Write

$$\sigma_r^2 = \frac{\sigma_0^2 r_0^{2\nu-1}\alpha_0^{2\nu-d-1}}{r^{2\nu-1}\alpha_2^{2\nu-d-1}}.$$

The Radon-Nikodym derivative of P_{r,σ^2} with respect to $P_{r_0,\sigma^2_{r_0}}$ is $f_{n,r,\sigma^2}/f_{n,r_0,\sigma^2_{r_0}}$. Let $p_n(r,\sigma^2)$ be the logarithm of the Radon-Nikodym derivative such that $p_n(r,\sigma^2) = \ell_n(r,\sigma^2) - \ell_n(r_0, \sigma^2_{r_0})$. As shown in Gikhman and Skorokhod (1974, Thm. 1, p. 442) and Zhang (2004), if $P_{r,\sigma^2} \equiv P_{r_0,\sigma^2_{r_0}}$, then p_n converges to a limit, ρ, say with $P_{r_0,\sigma^2_{r_0}}$-probability 1. Otherwise, if P_{r,σ^2} and $P_{r_0,\sigma^2_{r_0}}$ are orthogonal, then $p_n \to -\infty$. By Theorem 4, $P_0 \equiv P_{r_0,\sigma^2_{r_0}}$ where P_0 is the probability measure defined by $\sigma_0^2 M_\nu^0(\alpha_0, r_0\alpha_0)$.

For any r, σ_n^2 is defined by $(KT)^{-1}\boldsymbol{X}_n^\top \boldsymbol{\Gamma}_{\alpha_2, r\alpha_2}\boldsymbol{X}_n$ according to (2.6). From Theorem 4, $\lim_{n\to\infty} p_n(r,\sigma^2) \to \rho > -\infty$ if $r = r_0$. Furthermore, from Theorem 1(c), $\lim_{n\to\infty} p_n(r,\sigma^2) \to -\infty$ if $r \neq r_0$. Hence, r_0 is the unique maximizer of $\ell_n(r, \hat{\sigma}_n^2)$ and therefore $\hat{r}_n \to r_0$ almost surely. Hence, $\zeta^0_{\alpha_2,\hat{r}_n} = \hat{\sigma}_n^2(\hat{r}_n)^d \beta_2^{2\nu-d-1} \to \hat{\sigma}_n^2(r_0)^d\beta_2^{2\nu-d-1}$. Thus, from Theorem 4, $\zeta^0_{\alpha_2,\hat{r}_n} \to \sigma_0^2 r_0^d \beta_0^{2\nu-d-1}$. □

Lemma 3. *Under the conditions in Theorem 1 and* $\varepsilon = 0$, *if* $0 < \alpha_a \le \alpha_b < \infty$ *and* r_a, r_b *are estimated from (2.12), then* $\zeta^0_{\alpha_b,r_b} \ge \zeta^0_{\alpha_a,r_a}$ *for some large enough sample size.*

Proof. Similar to the proof of Lemma 2, it can be verified that

$$
\begin{aligned}
& \zeta^0_{\alpha_b,r_b} - \zeta^0_{\alpha_a,r_a} \\
= \; & \frac{1}{KT}\boldsymbol{X}_n^\top \left\{ r_b^d \left(r_b\alpha_b\right)^{2\nu-d-1} \boldsymbol{\Gamma}^{-1}_{\alpha_b,r_b\alpha_b} - r_a^d \left(r_a\alpha_a\right)^{2\nu-d-1} \boldsymbol{\Gamma}^{-1}_{\alpha_a,r_a\alpha_a} \right\} \boldsymbol{X}_n.
\end{aligned}
$$

Hence, it suffices to show that the matrix

$$
r_a^{-d}\left(r_a\alpha_a\right)^{1-2\nu+d} \boldsymbol{\Gamma}_{\alpha_a,r_a\alpha_a} - r_b^{-d}\left(r_b\alpha_b\right)^{1-2\nu+d} \boldsymbol{\Gamma}_{\alpha_b,r_b\alpha_b} \tag{S1.7}
$$

is non-negative definite. The Fourier transform of (S1.7) is

$$
\begin{aligned}
& \frac{\Gamma\left(\nu - \frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \left[\frac{\alpha_a^{2\nu-d}\left(r_a\alpha_a\right)^{2\nu-1}}{r_a^d\left(r_a\alpha_a\right)^{2\nu-d-1}\left(\alpha_a^2 r_a^2 \alpha_a^2 + r_a^2\alpha_a^2\omega^2 + \alpha_a^2\tau^2\right)^\nu} \right. \\
& \qquad \left. - \frac{\alpha_b^{2\nu-d}\left(r_b\alpha_b\right)^{2\nu-1}}{r_b^d\left(r_b\alpha_b\right)^{2\nu-d-1}\left(\alpha_b^2 r_b^2 \alpha_b^2 + r_b^2\alpha_b^2\omega^2 + \alpha_b^2\tau^2\right)^\nu} \right] \\
= \; & \frac{\Gamma\left(\nu - \frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \left[\frac{\alpha_a^{2\nu}}{\left(\alpha_a^2 r_a^2 \alpha_a^2 + r_a^2\alpha_a^2\omega^2 + \alpha_a^2\tau^2\right)^\nu} - \frac{\alpha_b^{2\nu}}{\left(\alpha_b^2 r_b^2 \alpha_b^2 + r_b^2\alpha_b^2\omega^2 + \alpha_b^2\tau^2\right)^\nu} \right] \\
= \; & \frac{\Gamma\left(\nu - \frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \left[\left(\alpha_a^2 r_a^2 + r_a^2\omega^2 + \tau^2\right)^{-\nu} - \left(\alpha_b^2 r_b^2 + r_b^2\omega^2 + \tau^2\right)^{-\nu} \right].
\end{aligned} \tag{S1.8}
$$

By Theorem 5, since $r_a, r_b \to r_0$ almost surely as $K, T \to \infty$. With $\alpha_a \leq \alpha_b$, (S1.8) converges to

$$
\frac{\Gamma\left(\nu - \frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \left[\left(\alpha_a^2 r_0^2 + r_0^2\omega^2 + \tau^2\right)^{-\nu} - \left(\alpha_b^2 r_0^2 + r_0^2\omega^2 + \tau^2\right)^{-\nu} \right] \geq 0.
$$

□

Proof of Theorem 6

By Lemma 3 and Theorem 5, the result in Theorem 6 can be proven in a similar fashion as the proof of Theorem 3. We omit the proof here.

S2 Detailed Simulation Results

This section is devoted to supplement the simulation results provided in Section 3 of the paper. In particular, summary statistics (quartiles, mean and standard deviation) of the estimated parameters under the 11 scenarios are provided.

Table 1: Simulation results. Summary statistics of estimated α, β, σ^2 and ζ under scenarios 1 to 4 for different values of K (and T). Under these scenarios, $\varepsilon = 1$ (the separable case).

	$K = 25$				$K = 36$				$K = 49$			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}^2$	$\hat{\zeta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}^2$	$\hat{\zeta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}^2$	$\hat{\zeta}$
Scenario 1: $(\alpha, \beta) = (0.3, 3.0)$												
Q_1	—	—	0.964	2.343	—	—	0.975	2.368	—	—	0.979	2.380
Q_2	—	—	1.001	2.434	—	—	1.001	2.433	—	—	0.999	2.428
Q_3	—	—	1.037	2.520	—	—	1.026	2.494	—	—	1.019	2.475
Mean	—	—	1.001	2.433	—	—	1.002	2.434	—	—	0.999	2.427
SD	—	—	0.053	0.130	—	—	0.040	0.097	—	—	0.031	0.076
Scenario 2: $(\alpha, \beta) = (0.4, 2.0)$												
Q_1	—	—	1.732	2.217	—	—	1.795	2.297	—	—	1.838	2.353
Q_2	—	—	1.802	2.307	—	—	1.840	2.355	—	—	1.866	2.389
Q_3	—	—	1.883	2.410	—	—	1.890	2.419	—	—	1.893	2.423
Mean	—	—	1.805	2.310	—	—	1.841	2.357	—	—	1.866	2.388
SD	—	—	0.108	0.138	—	—	0.072	0.092	—	—	0.041	0.052
Scenario 3: $(\alpha, \beta) = (1.0, 1.0)$												
Q_1	—	—	3.307	3.307	—	—	3.125	3.125	—	—	2.903	2.903
Q_2	—	—	3.563	3.563	—	—	3.295	3.295	—	—	2.972	2.972
Q_3	—	—	3.855	3.855	—	—	3.476	3.476	—	—	3.057	3.057
Mean	—	—	3.591	3.591	—	—	3.312	3.312	—	—	2.981	2.981
SD	—	—	0.397	0.397	—	—	0.260	0.260	—	—	0.119	0.119
Scenario 4: $(\hat{\alpha}, \hat{\beta})$ obtained from (2.7) and (2.9)												
Q_1	0.254	2.635	0.217	2.306	0.265	2.629	0.199	2.361	0.250	2.642	0.195	2.359
Q_2	0.349	3.231	0.609	2.606	0.369	3.194	0.589	2.542	0.342	3.224	0.670	2.517
Q_3	0.481	4.188	1.837	3.247	0.501	3.979	1.625	2.921	0.470	4.073	1.923	2.821
Mean	0.393	3.525	2.108	3.082	0.413	3.445	2.003	2.883	0.387	3.442	2.158	2.695
SD	0.198	1.291	4.716	1.423	0.216	1.209	6.014	1.310	0.195	1.132	5.016	0.558

Table 2: Simulation results. Summary statistics of estimated α, β, σ^2 and ζ under scenarios 5 to 7 for different values of K (and T). Under these scenarios, $\varepsilon = 0$ (the non-separable case).

	$K=25$				$K=36$				$K=49$			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}^2$	$\hat{\zeta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}^2$	$\hat{\zeta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}^2$	$\hat{\zeta}$
Scenario 5: $(\alpha, \beta) = (0.3, 3.0)$												
Q_1	—	—	0.966	289.8	—	—	0.977	293.0	—	—	0.980	293.9
Q_2	—	—	1.003	301.0	—	—	1.002	300.5	—	—	0.999	299.6
Q_3	—	—	1.045	313.4	—	—	1.027	308.1	—	—	1.022	306.6
Mean	—	—	1.004	301.2	—	—	1.002	300.7	—	—	1.001	300.3
SD	—	—	0.058	17.41	—	—	0.040	12.06	—	—	0.031	9.336
Scenario 6: $(\alpha, \beta) = (0.1, 1.0)$												
Q_1	—	—	2.849	284.9	—	—	2.917	291.7	—	—	2.946	294.6
Q_2	—	—	2.982	298.2	—	—	2.995	299.5	—	—	2.993	299.3
Q_3	—	—	3.093	309.3	—	—	3.064	306.4	—	—	3.049	304.9
Mean	—	—	2.977	297.7	—	—	2.996	299.6	—	—	2.998	299.8
SD	—	—	0.173	17.25	—	—	0.110	11.00	—	—	0.080	8.011
Scenario 7: $(\alpha, \beta) = (1.0, 1.0)$												
Q_1	—	—	2.223	2.223	—	—	2.377	2.377	—	—	2.501	2.501
Q_2	—	—	2.389	2.389	—	—	2.497	2.497	—	—	2.594	2.594
Q_3	—	—	2.565	2.565	—	—	2.618	2.618	—	—	2.686	2.686
Mean	—	—	2.410	2.410	—	—	2.506	2.506	—	—	2.596	2.596
SD	—	—	0.263	0.263	—	—	0.187	0.187	—	—	0.140	0.140

Table 3: Simulation results. Summary statistics of estimated α, r, σ^2 and ζ under scenarios 8 to 11 for different values of K (and T). Under these scenarios, $\varepsilon = 0$ (the non-separable case).

	$K=25$				$K=36$				$K=49$			
	$\hat{\alpha}$	$\hat{r}$	$\hat{\sigma}^2$	$\hat{\zeta}$	$\hat{\alpha}$	$\hat{r}$	$\hat{\sigma}^2$	$\hat{\zeta}$	$\hat{\alpha}$	$\hat{r}$	$\hat{\sigma}^2$	$\hat{\zeta}$
Scenario 8: $\alpha = 0.3$												
Q_1	—	9.335	0.957	246.0	—	9.646	0.976	271.3	—	9.705	0.979	277.2
Q_2	—	10.01	0.994	300.1	—	10.03	1.000	305.7	—	9.996	1.000	299.0
Q_3	—	10.77	1.037	366.9	—	10.48	1.030	341.2	—	10.30	1.024	324.6
Mean	—	10.07	0.999	312.8	—	10.05	1.003	308.5	—	9.998	1.002	301.7
SD	—	0.993	0.060	88.85	—	0.645	0.040	57.78	—	0.451	0.032	37.38
Scenario 9: $\alpha = 0.5$												
Q_1	—	9.318	0.582	244.4	—	9.490	0.587	259.8	—	9.701	0.590	276.5
Q_2	—	9.909	0.605	301.0	—	9.945	0.603	297.7	—	9.961	0.603	299.3
Q_3	—	10.62	0.632	359.9	—	10.34	0.619	333.6	—	10.26	0.615	322.9
Mean	—	9.992	0.607	310.0	—	9.950	0.603	300.6	—	9.991	0.603	302.1
SD	—	0.981	0.037	90.45	—	0.653	0.024	56.79	—	0.439	0.019	37.78
Scenario 10: $\alpha = 1.0$												
Q_1	—	8.621	0.310	213.7	—	9.154	0.304	242.5	—	9.531	0.302	267.0
Q_2	—	9.312	0.323	261.0	—	9.624	0.312	278.3	—	9.801	0.308	289.5
Q_3	—	10.02	0.339	316.6	—	10.06	0.322	316.1	—	10.10	0.314	317.4
Mean	—	9.369	0.324	273.7	—	9.628	0.313	282.7	—	9.811	0.309	292.9
SD	—	1.051	0.023	86.43	—	0.670	0.014	55.31	—	0.451	0.010	37.61
Scenario 11: $\hat{\alpha}$ obtained from (2.7) and $\hat{r}$ from (2.12)												
Q_1	0.244	9.346	0.574	246.6	0.222	9.553	0.546	265.6	0.239	9.705	0.583	275.6
Q_2	0.356	10.01	0.834	301.7	0.370	10.02	0.825	302.0	0.354	9.975	0.841	298.9
Q_3	0.534	10.63	1.233	362.6	0.553	10.47	1.354	341.4	0.523	10.27	1.268	324.1
Mean	0.418	10.02	1.007	311.4	0.418	10.02	1.063	305.9	0.413	9.992	1.015	301.9
SD	0.251	0.994	0.633	89.06	0.250	0.641	0.762	55.33	0.246	0.434	0.653	37.04

References

Gikhman, I. I. and A. V. Skorokhod (1974). *The Theory of Stochastic Processes I.* Springer, Berlin.

Gradshteyn, I. S. and I. M. Ryzhik (2007). *Table of Integrals, Series, and Products* (7th ed.). Elsevier Inc.

Horn, R. A. and C. R. Johnson (1990). *Matrix Analysis.* Cambridge University Press.

Ibragimov, I. A. and Y. A. Rozanov (1978). *Gaussian Random Processes.* Springer–Verlag, New York.

Prudnikov, A., Y. A. Brychkov, and O. Marichev (1986). *Integrals and Series, Vol. 1.* Gordon and Breach Science Publishers, New York.

Stein, M. L. (1999). *Interpolation of Spatial Data: Some Theory for Kriging.* Springer–Verlag, New York.

Zhang, H. (2004). Inconsistent estimation and asymptotically equal interpolations in model–based geostatistics. *J. Am. Stat. Assoc.* **99**, 250–261.