AN ADDITIVE-MULTIPLICATIVE MEAN MODEL FOR MARKER DATA CONTINGENT ON RECURRENT EVENT WITH AN INFORMATIVE TERMINAL EVENT

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Supplementary Material

This supplementary file contains the regularity conditions (C1)-(C6) and the proofs of Theorems 1-3 in Sections 3 and 4 of the paper.

S1 Proofs of Theorems

In order to study the asymptotic properties of the proposed estimators, we need the following regularity conditions:

(C1) $\{m_i(\cdot), N_i(\cdot), T_i, \delta_i, X_i, W_i\}, i = 1, \dots, n$, are independent identically distributed.
(C2) $N(\tau)$, X and W are bounded almost surely, and $P(T \geq \tau) > 0$.
(C3) The link functions $g_\gamma(\cdot)$, $g_\beta(\cdot)$ and $g_\zeta(\cdot)$ are twice continuously differentiable with $g_\gamma(\cdot) > 0$
(C4) The weight functions $W(t)$ and $Q(t)$ have bounded variation and converge to deterministic functions $w(t)$ and $q(t)$, respectively, in probability, uniformly in $t \in [0, \tau]$.
(C5) Ω is nonsingular, where

$$\Omega = E\left[\int_0^\tau \{Z_i(t) - \bar{z}^D(t)\}^{\otimes 2} dN_i^D(t)\right]$$

and $\bar{z}^D(t)$ is the limit of $\bar{Z}^D(t; \eta_0)$.
(C6) A is nonsingular, where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

$$A_{11} = E\left[\int_0^\tau w(t)\{Z_i - \bar{z}(t, Z_i)\} Y_i(t) \times \begin{pmatrix} \{X_i \dot{g}_\beta(\beta_0' X_i) g_\gamma(\gamma_0' Z_i) - g_\beta(\beta_0' X_i) g_\gamma(\gamma_0' Z_i) \bar{x}(t, Z_i)\} dH(t, \log\Lambda_0(t) + \eta_0' Z_i) \\ W_i \dot{g}_\zeta(\zeta_0' W_i) dN_i(t) - g_\beta(\beta_0' X_i) g_\gamma(\gamma_0' Z_i) \bar{w}(t, Z_i) \end{pmatrix}'\right],$$

$$A_{12} = E\bigg[\int_0^\tau w(t)\{Z_i - \bar{z}(t,Z_i)\}Y_i(t)$$
$$\times\{Z_i g_\beta(\beta_0'X_i)\dot{g}_\gamma(\gamma_0'Z_i) - g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\bar{z}^\dagger(t,Z_i)\}'dH(t,\log\Lambda_0(t)+\eta_0'Z_i)\bigg],$$
$$A_{22} = E\bigg[\int_0^\tau q(t)\{Z_i - \bar{z}^N(t,Z_i)\}Y_i(t)\{Z_i\dot{g}_\gamma(\gamma_0'Z_i) - \bar{z}^*(t,Z_i)g_\gamma(\gamma_0'Z_i)\}'d\mu_0(t;v_2)\bigg],$$

and $\bar{z}(t,Z_i)$, $\bar{x}(t,Z_i)$, $\bar{w}(t,Z_i)$, $\bar{z}^\dagger(t,Z_i)$ and $\bar{z}^N(t,Z_i)$ are the limits of $\bar{Z}_i(t;\beta_0,\gamma_0)$, $\bar{X}_i(t;\beta_0,\gamma_0)$, $\bar{W}_i(t;\theta_0,\gamma_0)$, $\bar{Z}_i^\dagger(t;\beta_0,\gamma_0)$, and $\bar{Z}_i^N(t;\gamma_0)$ conditional on Z_i, respectively.

Proof of Theorem 1. Define

$$\Psi_i(t,Z;\eta,\Lambda) = I\{\log\Lambda(T_i)+\eta'Z_i \ge \log\Lambda(t)+\eta'Z \ge \log\Lambda(t)+\eta'Z_i\},$$
$$d\bar{N}(t,Z;\eta,\Lambda) = \frac{\sum_{j=1}^n dN_j(t)\Psi_j(t,Z;\eta,\Lambda)}{\sum_{j=1}^n g_\gamma(\gamma_0'Z_j)\Psi_j(t,Z;\eta,\Lambda)},$$

and

$$d\bar{N}_0(t,Z;\eta,\Lambda) = \frac{E[dN_j(t)\Psi_j(t,Z;\eta,\Lambda)|Z]}{E[g_\gamma(\gamma_0'Z_j)\Psi_j(t,Z;\eta,\Lambda)|Z]}.$$

Let $\Psi_i(t,Z) = \Psi_i(t,Z;\eta_0,\Lambda_0)$. It then follows from the functional delta method (van der Vaart and Wellner, 1996, Theorem 3.9.4, p.374) that

$$n^{-1/2}\sum_{i=1}^n\int_0^\tau q(t)\{Z_i - \bar{z}^N(t,Z_i)\}Y_i(t)g_\gamma(\gamma_0'Z_i)\{d\bar{N}(t,Z_i;\hat{\eta},\hat{\Lambda}_0) - d\bar{N}_0(t,Z_i;\hat{\eta},\hat{\Lambda}_0)\}$$
$$= n^{-1/2}\sum_{i=1}^n\int_0^\tau\int q(t)\{z-\bar{z}^N(t,z)\}I(y\ge t)\frac{g_\gamma(\gamma_0'z)\Psi_i(t,z)}{E[g_\gamma(\gamma_0'Z_i)\Psi_i(t,z)]}dP(z,y)dN_i(t)$$
$$-n^{-1/2}\sum_{i=1}^n\int\bigg[\int_0^\tau q(t)\{z-\bar{z}^N(t,z)\}I(y\ge t)g_\gamma(\gamma_0'Z_i)\Psi_i(t,z)$$
$$\times\frac{E[\Psi_i(t,z)dN_i(t)]}{(E[g_\gamma(\gamma_0'Z_i)\Psi_i(t,z)])^2}\bigg]g_\gamma(\gamma_0'z)dP(z,y)+o_p(1), \qquad (S.1)$$

where $P(z,y)$ is the joint probability measure of (Z_i,T_i). In addition, according to Fleming and Harrington (1991, p.299), we have

$$\hat{\eta}-\eta_0 = \Omega^{-1}n^{-1}\sum_{i=1}^n\int_0^\tau\{Z_i-\bar{z}^D(t)\}dM_i^D(t)+o_p(n^{-1/2}),$$

and

$$\hat{\Lambda}_0(t)-\Lambda_0(t) = n^{-1}\sum_{i=1}^n\int_0^t\frac{dM_i^D(u)}{s^{(0)}(u;\eta_0)} - \int_0^t\bar{z}^D(u)'d\Lambda_0(u)(\hat{\eta}-\eta_0)+o_p(n^{-1/2}),$$

where

$$M_i^D(t) = N_i^D(t) - \int_0^t Y_i(u)\exp(\eta_0'Z_i)d\Lambda_0(u),$$

and $s^{(0)}(t;\eta_0)$ is the limit of $S^{(0)}(t;\eta_0)$. Let $dR_\eta(t,Z)$ and $dR_\Lambda(t,Z)$ be the derivative and the Hadamard derivative of $d\bar{N}_0(t,Z;\eta_0,\Lambda_0)$ with respect to η and Λ, respectively. Then by the functional delta method, we obtain

$$n^{-1/2}\sum_{i=1}^{n}\int_0^\tau q(t)\{Z_i - z^N(t,Z_i)\}Y_i(t)g_\gamma(\gamma_0'Z_i)\{d\bar{N}_0(t,Z_i;\hat{\eta},\hat{\Lambda}_0) - d\bar{N}_0(t,Z_i;\eta_0,\Lambda_0)\}$$

$$= n^{-1/2}\sum_{i=1}^{n}\int_0^\tau \Big[B_1\Omega^{-1}\{Z_i - \bar{z}^D(t)\} + \frac{C_1(t)}{s^{(0)}(t;\eta_0)}\Big]dM_i^D(t) + o_p(1), \qquad (S.2)$$

where

$$C_1(t) = E\bigg[\int_t^\tau q(u)\{Z_i - \bar{z}^N(u,Z_i)\}Y_i(u)g_\gamma(\gamma_0'Z_i)dR_\Lambda(u,Z_i)\bigg],$$

and

$$B_1 = E\bigg[\int_0^\tau q(t)\{Z_i - \bar{z}^N(t,Z_i)\}Y_i(t)g_\gamma(\gamma_0'Z_i)\Big\{dR_\eta(t,Z_i) - \Big(\int_0^t \bar{z}^D(u)'d\Lambda_0(u)\Big)dR_\Lambda(t,Z_i)\Big\}\bigg].$$

Note that

$$n^{-1/2}U_\gamma(\gamma_0) = \sum_{i=1}^{n}\int_0^\tau Q(t)\{Z_i - \bar{Z}_i^N(t;\gamma_0)\}Y_i(t)[dN_i(t) - g_\gamma(\gamma_0'Z_i)d\bar{N}_0(t,Z_i;\eta_0,\Lambda_0)$$

$$-g_\gamma(\gamma_0'Z_i)\{d\bar{N}(t,Z_i;\hat{\eta},\hat{\Lambda}_0) - d\bar{N}_0(t,Z_i;\hat{\eta},\hat{\Lambda}_0)\}$$

$$-g_\gamma(\gamma_0'Z_i)\{d\bar{N}_0(t,Z_i;\hat{\eta},\hat{\Lambda}_0) - d\bar{N}_0(t,Z_i;\eta_0,\Lambda_0)\}] + o_p(1). \qquad (S.3)$$

Since $\sup_{i,t}|\bar{Z}_i^N(t;\gamma_0) - z^N(t,Z_i)| \to 0$ in probability, it follows from (S.1), (S.2) and (S.3) that

$$n^{-1/2}U_\gamma(\gamma_0) = n^{-1/2}\sum_{i=1}^{n}\varphi_i + o_p(1), \qquad (S.4)$$

where

$$\varphi_i = \int_0^\tau q(t)\{Z_i - \bar{z}^N(t,Z_i)\}Y_i(t)[dN_i(t) - g_\gamma(\gamma_0'Z_i)d\bar{N}_0(t,Z_i;\eta_0,\Lambda_0)]$$

$$-\int_0^\tau\int q(t)\{z - \bar{z}^N(t,z)\}I(y\geq t)\frac{g_\gamma(\gamma_0'z)\Psi_i(t,z)}{E[g_\gamma(\gamma_0'Z_i)\Psi_i(t,z)]}dP(z,y)dN_i(t)$$

$$+\int\bigg[\int_0^\tau q(t)\{z - \bar{z}^N(t,z)\}I(y\geq t)g_\gamma(\gamma_0'Z_i)\Psi_i(t,z)$$

$$\times\frac{E[\Psi_i(t,z)dN_i(t)]}{(E[g_\gamma(\gamma_0'Z_i)\Psi_i(t,z)])^2}\bigg]g_\gamma(\gamma_0'z)dP(z,y)$$

$$-\int_0^\tau\Big[B_1\Omega^{-1}\{Z_i - \bar{z}^D(t)\} + \frac{C_1(t)}{s^{(0)}(t;\eta_0)}\Big]dM_i^D(t).$$

Define $dM_i(t) = \{m_i(t) - g_\zeta(\zeta_0'W_i)\}dN_i(t)$,

$$d\bar{M}(t,Z;\eta,\Lambda) = \frac{\sum_{j=1}^n dM_j(t)\Psi_j(t,Z;\eta,\Lambda)}{\sum_{j=1}^n g_\beta(\beta_0'X_j)g_\gamma(\gamma_0'Z_j)\Psi_j(t,Z;\eta,\Lambda)},$$

and

$$d\bar{M}_0(t,Z;\eta,\Lambda)=\frac{E[dM_j(t)\Psi_j(t,Z;\eta,\Lambda)|Z]}{E[g_\beta(\beta_0'X_j)g_\gamma(\gamma_0'Z_j)\Psi_j(t,Z;\eta,\Lambda)|Z]}.$$

Similarly, it follows from the functional delta method that

$$\begin{aligned}
&n^{-1/2}\sum_{i=1}^n\int_0^\tau w(t)\{Z_i-\bar{z}(t,Z_i)\}Y_i(t)g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\{d\bar{M}(t,Z_i;\hat\eta,\hat\Lambda_0)-d\bar{M}_0(t,Z_i;\hat\eta,\hat\Lambda_0)\}\\
&=n^{-1/2}\sum_{i=1}^n\int_0^\tau\int w(t)\{z-\bar{z}(t,z)\}I(y\geq t)\frac{g_\beta(\beta_0'x)g_\gamma(\gamma_0'z)\Psi_i(t,z)}{E[g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\Psi_i(t,z)]}dF(z,x,y)dM_i(t)\\
&-n^{-1/2}\sum_{i=1}^n\int\bigg[\int_0^\tau w(t)\{z-\bar{z}(t,z)\}I(y\geq t)g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\Psi_i(t,z)\\
&\times\frac{E[\Psi_i(t,z)dM_i(t)]}{(E[g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\Psi_i(t,z)])^2}\bigg]g_\beta(\beta_0'x)g_\gamma(\gamma_0'z)dF(z,x,y)+o_p(1), \qquad (S.5)
\end{aligned}$$

where $F(z,x,y)$ is the joint probability measure of (Z_i,X_i,T_i). Let $dR_\eta^*(t,Z)$ and $dR_\Lambda^*(t,Z)$ be the derivative and the Hadamard derivative of $d\bar{M}_0(t,Z;\eta_0,\Lambda_0)$ with respect to η and Λ, respectively. Then in a similar manner, we get

$$\begin{aligned}
&n^{-1/2}\sum_{i=1}^n\int_0^\tau w(t)\{Z_i-\bar{z}(t,Z_i)\}Y_i(t)g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\{d\bar{M}_0(t,Z_i;\hat\eta,\hat\Lambda_0)-d\bar{M}_0(t,Z_i;\eta_0,\Lambda_0)\}\\
&\qquad=n^{-1/2}\sum_{i=1}^n\int_0^\tau\Big[B_2\Omega^{-1}\{Z_i-\bar{z}^D(t)\}+\frac{C_2(t)}{s^{(0)}(t;\eta_0)}\Big]dM_i^D(t)+o_p(1), \qquad (S.6)
\end{aligned}$$

where

$$C_2(t)=E\bigg[\int_t^\tau q(u)\{Z_i-\bar{z}(u,Z_i)\}Y_i(u)g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)dR_\Lambda^*(u,Z_i)\bigg],$$

and

$$\begin{aligned}
B_2=E\bigg[&\int_0^\tau q(t)\{Z_i-\bar{z}(t,Z_i)\}Y_i(t)g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\\
&\times\Big\{dR_\eta^*(t,Z_i)-\Big(\int_0^t\bar{z}^D(u)'d\Lambda_0(u)\Big)dR_\Lambda^*(t,Z_i)\Big\}\bigg].
\end{aligned}$$

Note that

$$\begin{aligned}
U_\theta(\theta_0,\gamma_0)=\sum_{i=1}^n\int_0^\tau &W(t)\{Z_i-\bar{Z}_i(t;\beta_0,\gamma_0)\}Y_i(t)\Big[dM_i(t)-g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)d\bar{M}_0(t,Z_i;\eta_0,\Lambda_0)\\
&-g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\{d\bar{M}(t,Z_i;\hat\eta,\hat\Lambda_0)-d\bar{M}_0(t,Z_i;\hat\eta,\hat\Lambda_0)\}\\
&-g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\{d\bar{M}_0(t,Z_i;\hat\eta,\hat\Lambda_0)-d\bar{M}_0(t,Z_i;\eta_0,\Lambda_0)\}\Big]+o_p(1), \qquad (S.7)
\end{aligned}$$

and $\sup_{i,t}|\bar{Z}_i(t;\beta_0,\gamma_0)-\bar{z}(t,Z_i)|\to 0$ in probability. Thus, it follow from (S.5), (S.6) and (S.7) that

$$n^{-1/2}U_\theta(\theta_0,\gamma_0)=n^{-1/2}\sum_{i=1}^n\xi_i+o_p(1), \qquad (S.8)$$

where

$$\xi_i = \int_0^\tau w(t)\{Z_i - \bar{z}(t, Z_i)\}Y_i(t)[dM_i(t) - g_\beta(\beta_0' X_i)g_\gamma(\gamma_0' Z_i)d\bar{M}_0(t, Z_i; \eta_0, \Lambda_0)]$$
$$- \int_0^\tau \int w(t)\{z - \bar{z}(t, z)\}I(y \geq t)\frac{g_\beta(\beta_0' x)g_\gamma(\gamma_0' z)\Psi_i(t, z)}{E[g_\beta(\beta_0' X_i)g_\gamma(\gamma_0' Z_i)\Psi_i(t, z)]}dF(z, x, y)dM_i(t)$$
$$+ \int \Big[\int_0^\tau w(t)\{z - \bar{z}(t, z)\}I(y \geq t)g_\beta(\beta_0' X_i)g_\gamma(\gamma_0' Z_i)\Psi_i(t; z)$$
$$\times \frac{E[\Psi_i(t, z)dM_i(t)]}{(E[g_\beta(\beta_0' X_i)g_\gamma(\gamma_0' Z_i)\Psi_i(t, z)])^2}\Big]g_\beta(\beta_0' x)g_\gamma(\gamma_0' z)dF(z, x, y)$$
$$- \int_0^\tau \Big[B_2\Omega^{-1}\{Z_i - \bar{z}^D(t)\} + \frac{C_2(t)}{s^{(0)}(t; \eta_0)}\Big]dM_i^D(t).$$

Note that $-n^{-1}\partial U_\theta(\theta_0; \gamma_0)/\partial\theta$, $-n^{-1}\partial U_\theta(\theta_0; \gamma_0)/\partial\gamma$ and $-n^{-1}\partial U_\gamma(\gamma_0)/\partial\gamma$ converge in probability to A_{11}, A_{12} and A_{22}, respectively. It then follows from (S.4), (S.8) and the Taylor expansion that

$$n^{1/2}\begin{pmatrix} \hat{\theta} - \theta_0 \\ \hat{\gamma} - \gamma_0 \end{pmatrix} = A^{-1}n^{-1/2}\begin{pmatrix} U_\theta(\theta_0; \gamma_0) \\ U_\gamma(\gamma_0) \end{pmatrix} + o_p(1),$$
$$= A^{-1}n^{-1/2}\sum_{i=1}^n \begin{pmatrix} \xi_i \\ \varphi_i \end{pmatrix} + o_p(1),$$

which implies that $n^{1/2}(\hat{\theta} - \theta_0)$ and $n^{1/2}(\hat{\gamma} - \gamma_0)$ have asymptotically a joint normal distribution with mean zero and covariance matrix $A^{-1}\Sigma(A')^{-1}$, where $\Sigma = E[(\xi_i', \varphi_i')'^{\otimes 2}]$.

Proof of Theorem 2. In view of the consistency of $\hat{\theta}$, $\hat{\gamma}$, $\hat{\eta}$ and $\hat{\Lambda}_0(t)$, using the Theorem 3.6.13 of van der Vaart and Wellner (1996), we obtain that conditional on the observed data,

$$\Phi_1^* = \sum_{i=1}^n G_i \int_0^\tau w(t)\{Z_i - \bar{z}(t, Z_i)\}Y_i(t)[dM_i(t)$$
$$-g_\beta(\beta_0' X_i)g_\gamma(\gamma_0' Z_i)d\bar{M}_0(t, Z_i; \eta_0, \Lambda_0)] + o_p(n^{1/2}). \quad (S.9)$$

Likewise, it can be shown that conditional on the observed data,

$$\Phi_2^* = -\sum_{i=1}^n G_i \int_0^\tau \int w(t)\{z - \bar{z}(t, z)\}I(y \geq t)\frac{g_\beta(\beta_0' x)g_\gamma(\gamma_0' z)\Psi_i(t, z)}{E[g_\beta(\beta_0' X_i)g_\gamma(\gamma_0' Z_i)\Psi_i(t, z)]}dF(z, x, y)dM_i(t)$$
$$+ \sum_{i=1}^n G_i \int \Big[\int_0^\tau w(t)\{z - \bar{z}(t, z)\}I(y \geq t)g_\beta(\beta_0' X_i)g_\gamma(\gamma_0' Z_i)\Psi_i(t, z)$$
$$\times \frac{E[\Psi_i(t, z)dM_i(t)]}{(E[g_\beta(\beta_0' X_i)g_\gamma(\gamma_0' Z_i)\Psi_i(t, z)])^2}\Big]g_\beta(\beta_0' x)g_\gamma(\gamma_0' z)dF(z, x, y) + o_p(n^{1/2}). \quad (S.10)$$

Let $d\hat{M}_i(t) = \{m_i(t) - g_\varsigma(\hat{\zeta}' W_i)\}dN_i(t)$, and

$$d\hat{M}(t, Z; \eta, \Lambda) = \frac{\sum_{j=1}^n d\hat{M}_j(t)\Psi_j(t, Z; \eta, \Lambda)}{\sum_{j=1}^n g_\beta(\hat{\beta}' X_j)g_\gamma(\hat{\gamma}' Z_j)\Psi_j(t, Z; \eta, \Lambda)}.$$

Thus,

$$\Phi_3^* = \sum_{i=1}^n \int_0^\tau W(t)\{Z_i - \bar{Z}_i(t;\hat\beta,\hat\gamma)\}Y_i(t)g_\beta(\hat\beta' X_i)g_\gamma(\hat\gamma' Z_i)\{d\hat M(t,Z_i;\hat\eta,\hat\Lambda_0) - d\hat M(t,Z_i;\hat\eta^*,\hat\Lambda_0^*)\}.$$

It can be checked that

$$d\hat M(t,Z_i;\hat\eta,\hat\Lambda_0) - d\hat M(t,Z_i;\hat\eta^*,\hat\Lambda_0^*) = \{d\hat M(t,Z_i;\hat\eta,\hat\Lambda_0) - d\bar M_0(t,Z_i;\hat\eta,\hat\Lambda_0)\}$$
$$-\{d\hat M(t,Z_i;\hat\eta^*,\hat\Lambda_0^*) - d\bar M_0(t,Z_i;\hat\eta^*,\hat\Lambda_0^*)\} + \{d\bar M_0(t,Z_i;\hat\eta,\hat\Lambda_0) - d\bar M_0(t,Z_i;\hat\eta^*,\hat\Lambda_0^*)\}.$$

Similarly to (S.1), we obtain

$$n^{-1/2}\sum_{i=1}^n \int_0^\tau W(t)\{Z_i - \bar{Z}_i(t;\hat\beta,\hat\gamma)\}Y_i(t)g_\beta(\hat\beta' X_i)g_\gamma(\hat\gamma' Z_i)\Big[\{d\hat M(t,Z_i;\hat\eta,\hat\Lambda_0)$$
$$-d\bar M_0(t,Z_i;\hat\eta,\hat\Lambda_0)\} - \{d\hat M(t,Z_i;\hat\eta^*,\hat\Lambda_0^*) - d\bar M_0(t,Z_i;\hat\eta^*,\hat\Lambda_0^*)\}\Big] = o_p(1).$$

From an argument similar to that in the proof of (S.6), we have that conditional on the observed data,

$$n^{-1/2}\sum_{i=1}^n \int_0^\tau W(t)\{Z_i - \bar{Z}_i(t;\hat\beta,\hat\gamma)\}Y_i(t)g_\beta(\hat\beta' X_i)g_\gamma(\hat\gamma' Z_i)[d\bar M_0(t,Z_i;\hat\eta^*,\hat\Lambda_0^*) - d\bar M_0(t,Z_i;\hat\eta,\hat\Lambda_0)]$$
$$= n^{-1/2}G_i\int_0^\tau \Big[B_2\Omega^{-1}\{Z_i - \bar z^D(t)\} + \frac{C_2(t)}{s^{(0)}(t;\eta_0)}\Big]dM_i^D(t) + o_p(1). \qquad (S.11)$$

It follows from (S.9), (S.10) and (S.11) that conditional on the observed data,

$$\hat U_1 = n^{-1/2}(\Phi_1^* + \Phi_2^* + \Phi_3^*) = n^{-1/2}\sum_{i=1}^n G_i\xi_i + o_p(1).$$

In a similar manner,

$$\hat U_2 = n^{-1/2}(\Phi_4^* + \Phi_5^* + \Phi_6^*) = n^{-1/2}\sum_{i=1}^n G_i\varphi_i + o_p(1).$$

Thus, by using theorem 3.6.13 of van der Vaart and Wellner (1996), $E_G[\hat U^{\otimes 2}]$ converges in probability to Σ.

Proof of Theorem 3. Note that

$$\mathcal{F}(z,t) = n^{-1/2}\sum_{i=1}^n \int_0^t I(Z_i \le z)Y_i(u)\Big[\{m_i(u) - g_\zeta(\zeta_0' W_i)\}dN_i(t)$$
$$-g_\beta(\beta_0' X_i)g_\gamma(\gamma_0' Z_i)\{d\bar m_i(u;\beta_0,\gamma_0) - d\bar g_i(u;\theta_0,\gamma_0)\}\Big]$$
$$-\Gamma_1(z,t)' n^{1/2}(\hat\theta - \theta_0) - \Gamma_2(z,t)' n^{1/2}(\hat\gamma - \gamma_0) + o_p(1), \qquad (S.12)$$

where $\Gamma_1(z,t)$ and $\Gamma_2(z,t)$ are the limit of $\hat{\Gamma}_1(z,t)$ and $\hat{\Gamma}_2(z,t)$, respectively. By following similar arguments as in the proof of Theorem 1, the first term on the right-hand side of (S.12) equals

$$n^{-1/2}\sum_{i=1}^{n}\phi_i(z,t)+o_p(1), \qquad (S.13)$$

where

$$\begin{aligned}
\phi_i(z,t) = &\int_0^t I(Z_i\le z)Y_i(u)[dM_i(u) - g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)d\bar{M}_0(u,Z_i;\eta_0,\Lambda_0)]\\
&-\int_0^t\int I(Z_i\le z)I(y\ge u)\frac{g_\beta(\beta_0'x)g_\gamma(\gamma_0's)\Psi_i(u,s)}{E[g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\Psi_i(u,s)]}dF(s,x,y)dM_i(u)\\
&+\int\Big[\int_0^t I(Z_i\le z)I(y\ge u)g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\Psi_i(u,s)\\
&\times\frac{E[\Psi_i(u,s)dM_i(u)]}{(E[g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\Psi_i(u,s)])^2}\Big]g_\beta(\beta_0'x)g_\gamma(\gamma_0's)dF(s,x,y)\\
&-B_3(t,z)\Omega^{-1}\int_0^\tau\{Z_i-\bar{z}^D(u)\}dM_i^D(u)-\int_0^t\frac{C_3(t,u,z)}{s^{(0)}(u;\eta_0)}dM_i^D(u),\\
B_3(t,z) = &E\Big[\int_0^t I(Z_i\le z)Y_i(u)g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)\Big\{dR_\eta^*(u,Z_i)\\
&-\Big(\int_0^u\bar{z}^D(s)'d\Lambda_0(s)\Big)dR_\Lambda^*(u,Z_i)\Big\}\Big],
\end{aligned}$$

and

$$C_3(t,u,z)=E\Big[\int_u^t I(Z_i\le z)Y_i(s)g_\beta(\beta_0'X_i)g_\gamma(\gamma_0'Z_i)dR_\Lambda^*(s,Z_i)\Big].$$

Thus, it follows from (S.12), (S.13) and Theorem 1 that

$$\mathcal{F}(z,t)=n^{-1/2}\sum_{i=1}^{n}\Big[\phi_i(z,t)-\Gamma(z,t)'A^{-1}(\xi_i',\varphi_i')'\Big]+o_p(1), \qquad (S.14)$$

where $\Gamma(z,t)=(\Gamma_1(z,t)',\Gamma_2(z,t)')'$. By the multivariate central limit theorem, $\hat{\mathcal{F}}(z,t)$ converges in finite-dimensional distribution to a zero-mean Gaussian process. The first term of (S.14) is tight because any function of bounded variation can be written as the difference of two increasing functions. Note that $\Gamma(z,t)$ is a deterministic function, and ξ_i and φ_i do not involve t. Thus the second term is also tight. Hence $\mathcal{F}(z,t)$ is tight and converges weakly to a zero-mean Gaussian process. By the arguments analogous to those in the proof of Theorem 2, we obtain that $\mathcal{F}(z,t)$ can be approximated by the zero-mean Gaussian process $\hat{\mathcal{F}}(z,t)$ given by (7).

References

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