SEMI-PARAMETRIC INFERENCE FOR COPULA MODELS FOR TRUNCATED DATA

SUPPLEMENTARY MATERIALS

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***Abstract*:** In the main article, we propose semiparametric inference methods for estimating the association parameter for the dependently truncated data. Here we provide supplementary information for the results. The regularity conditions and proofs for the asymptotic analyses are given in Appendix A.1 and Appendix A.2 respectively. Appendix A.3 discusses variance estimation of the proposed estimators based on the asymptotic linear expression derived in Appendix A.2. In Appendix A.4, the analytical variance estimator under the Clayton model is derived and compared with the jackknife alternative by simulations. In Appendix B we derive the relationship between the proposed estimating function for the association parameter and that proposed by Chaieb, Rivest and Abdous (2006). In Appendix C, explicit formulas of the proposed estimating functions for selected examples of Archimedean copula models are given.

***Key words**:* Archimedean copula model, Conditional likelihood, Functional delta method, Product-limit estimator, Kendall's tau, Truncation data.

Appendix

Appendix A1: *Regularity Conditions*

(A-I) A parameter space $\Theta \subset R^2$ for (α, c) is compact;

(A-II) Deterministic functions $\phi_\alpha(v)$, $\phi'_\alpha(v) \equiv \partial\phi_\alpha(v)/\partial v$, $\phi''_\alpha(v) \equiv \partial^2\phi_\alpha(v)/\partial v^2$, $\phi^{-1}{}_\alpha(v)$, $\dot{\phi}^{-1}_\alpha(v) = \partial\phi^{-1}_\alpha(v)/\partial\alpha$, $\theta_\alpha(v)$, $\theta'_\alpha(v) \equiv \partial\theta_\alpha(v)/\partial v$, $\theta''_\alpha(v) \equiv \partial^2\theta_\alpha(v)/\partial v^2$, $\dot{\theta}_\alpha(v) \equiv \partial\theta_\alpha(v)/\partial\alpha$, $\tilde{w}_\alpha(v)$, and $\tilde{w}'_\alpha(v) \equiv \partial\tilde{w}_\alpha(v)/\partial v$ are twice continuously differentiable and bounded function of (α, v);

(A-III) There exists a function $\tilde{w}_\alpha\{\cdot\}: R \to R$ such that

$$\sup_{x,y}\left|\tilde{w}_{\alpha,c}(x,y) - \tilde{w}_\alpha\{c\hat{\pi}(x,y)\}\right| = o_p(n^{-1/2});$$

(A-IV) There exists two positive numbers $y_0 < x_0$ such that

$$F_X(y_0) > 0,\ \ S_Y(y_0) = 1,\ \ F_X(x_0) = 1 \ \text{ and } \ S_Y(x_0) > 0.$$

(A-V) The (2×2) matrix $A = E[\dot{U}_{\alpha_0,c_0}(X,Y)]$ is non-singular, which is given later.

These regularity assumptions guarantee sufficient conditions for the use of the functional delta method and empirical process theory (Van Der Vaart, 1998) in the proofs. Application of the functional delta method requires somewhat stringent differentiability assumption (A-II) together with the compactness of the parameter space (A-I). Condition (A-III) demands that the weight function is approximately expressed as, or exactly equal to, a smooth function of an empirical process. Note that (A-IV) is a condition for the identifiability of $(F_X(\cdot), S_Y(\cdot))$, which has been routinely used in theoretical analysis of truncation data. For example, the upper limit x_0 plays the same role as the notation T^* in Wang, Jewell and Tsai (1986). Non-singularity of (A-V) appears to be difficult to verify, but it is a standard regularity assumption imposed for consistency of Z-estimators.

Appendix A2: *Asymptotic Analysis*

To simplify the notations, we define $g_\alpha(v) \equiv 1/\{1+\theta_\alpha(v)\}$, $\pi(\infty, y) = \Pr(Y > y \mid X \le Y)$ and $\pi(x,0) = \Pr(X \le x \mid X \le Y)$. Also, let $\{D[0,\infty)\}^2$ be a space consisting of right-continuous function $(f_1(t), f_2(t))^T$ with left-side limits, where $f_k(t):[0,\infty) \mapsto R$ for $k = 1, 2$. The metric is defined as $d(f, g) = \max\{\sup_{0 \le t < \infty} | f_k(t) - g_k(t) |; k = 1, 2\}$ for $f, g \in \{D[0,\infty)\}^2$. Similarly, the space $D\{[0,\infty)^2\}$ consists of right-continuous function $f(s,t)$ with left-side limits, where $f(s,t):[0,\infty)^2 \mapsto R$, equipped with the usual sup-norm. Let $\Theta \subset R^2$ be the parameter space for (α, c), and $(\alpha_0, c_0) \in \Theta$ is denoted as the true parameter value. Hereafter, expectation symbols represent the conditional expectation given $X \le Y$.

All the estimating functions and estimators in this paper can be approximately expressed as a Hadamard differentiable function of the empirical process $\hat{\pi}(x, y) = R(x, y)/n$. The functional delta method is applied based on the weak convergence result of $n^{1/2}(\hat{\pi}(x, y) - \pi(x, y))$ to a Gaussian process $W(x, y)$ on $D\{[0,\infty)^2\}$ with the covariance structure given by:

$$\operatorname{cov}\{W(x_1, y_1), W(x_2, y_2)\} = \pi(x_1 \wedge x_2, y_1 \vee y_2) - \pi(x_1, y_1)\pi(x_2, y_2)$$

for $x_1 \le y_1$ and $x_2 \le y_2$.

The first step is to obtain the asymptotic linear expression for $\tilde{U}_w(\alpha,c)$ and $U_c(\alpha,c)$. By applying the functional delta method, the estimating function $\tilde{U}_w(\alpha,c)$ can be expressed as:

$$\binom{n}{2}^{-1} \tilde{U}_w(\alpha,c) = \Phi(\pi;\alpha,c) + \frac{1}{n}\sum_{i=1}^{n} \Phi'_\pi(\hat{\pi}_{(X_i,Y_i)} - \pi;\alpha,c) + o_p(n^{-1/2}) \qquad \text{(A.1)}$$

where $\pi \in D\{[0,\infty)^2\}$, $\hat{\pi}_{(X_i,Y_i)}(x,y) = I(X_i \le x, Y_i \ge y)$,

$$\Phi(\pi;\alpha,c) = \iint\iint_{x\vee x^*\le y\wedge y^*} \tilde{w}_\alpha\{c\pi(x\vee x^*, y\wedge y^*)\}$$
$$\times[I((x-x^*)(y-y^*)>0) - g_\alpha\{c\pi(x\vee x^*, y\wedge y^*)\}]\,d\pi(x,y)\,d\pi(x^*,y^*),$$

and

$$\Phi'_\pi(h;\alpha,c) = c\iint\iint_{x\vee x^*\le y\wedge y^*} h(x\vee x^*, y\wedge y^*)$$
$$\times\Big(\tilde{w}'_\alpha\{c\pi(x\vee x^*, y\wedge y^*)\}[I\{(x-x^*)(y-y^*)>0\} - g_\alpha\{c\pi(x\vee x^*, y\wedge y^*)\}]$$
$$-\tilde{w}_\alpha\{c\pi(x\vee x^*, y\wedge y^*)\}g'_\alpha\{c\pi(x\vee x^*, y\wedge y^*)\}\Big)d\pi(x,y)\,d\pi(x^*,y^*)$$
$$+2\iint\iint_{x\vee x^*\le y\wedge y^*} \tilde{w}_\alpha\{c\pi(x\vee x^*, y\wedge y^*)\}$$
$$\times[I\{(x-x^*)(y-y^*)>0\} - g_\alpha\{c\pi(x\vee x^*, y\wedge y^*)\}]\,dh(x,y)\,d\pi(x^*,y^*).$$

Similarly $U_c(\alpha,c)$ can be expressed as

$$U_c(\alpha,c) = \Psi(\pi;\alpha,c) + \frac{1}{n}\sum_{i=1}^{n}\Psi'_\pi(\hat{\pi}_{(X_i,Y_i)} - \pi;\alpha,c) + o_p(n^{-1/2}), \tag{A.2}$$

where

$$\Psi(\pi;\alpha,c) = c\int_{y_L}^{x_U}\phi'_\alpha\{c\pi(u,u)\}d\pi(u,0) + \phi_\alpha\{c\pi(y_L,y_L)\}$$

and

$$\Psi'_\pi(\pi_{(X_i,Y_i)} - \pi;\alpha,c) = c^2\int_{y_L}^{x_U}\phi''_\alpha\{c\pi(u,u)\}\{I(X_i \le u, Y_i \ge u) - \pi(u,u)\}d\pi(u,0)$$
$$-c\int_{y_L}^{x_U}\{I(X_i < u) - \pi(u,0)\}d\phi'_\alpha(c\pi(u,u)).$$

Both terms of $\Phi'_\pi(\pi_{(X_i,Y_i)} - \pi;\alpha,c)$ and $\Psi'_\pi(\pi_{(X_i,Y_i)} - \pi;\alpha,c)$ have zero-means for any value of (α,c). Using the following notations:

$$U_{\alpha,c}(X_i,Y_i) = \begin{bmatrix}\Phi'_\pi(\hat{\pi}_{(X_i,Y_i)} - \pi;\alpha,c) + \Phi(\pi;\alpha,c)\\ \Psi'_\pi(\hat{\pi}_{(X_i,Y_i)} - \pi;\alpha,c) + \Psi(\pi;\alpha,c)\end{bmatrix}, \tag{A.3}$$

we obtain the asymptotic expression

$$\begin{bmatrix} \binom{n}{2}^{-1} \tilde{U}_w(\alpha, c) \\ U_c(\alpha, c) \end{bmatrix} = \frac{1}{n}\sum_{i=1}^{n} U_{\alpha,c}(X_i, Y_i) + o_p(n^{-1/2}) . \qquad \text{(A.4)}$$

Based on equation (A.4), now we prove the consistency of $(\hat{\alpha}, \hat{c})$ based on the empirical process techniques. From (A.1) and (A.2), we have $\sum_{i=1}^{n} U_{\hat{\alpha},\hat{c}}(X_i, Y_i)/n = o_p(n^{-1/2})$. This formula implies that $(\hat{\alpha}, \hat{c})$ is an approximate Z-estimator (Van Der Vaart, 1998, p.46) for the criterion function $(\alpha, c) \mapsto U_{\alpha,c}(x, y)$. The consistency of $(\hat{\alpha}, \hat{c})$ follows by checking the two conditions: (i) The point (α_0, c_0) is the unique zero for $E[U_{\alpha,c}(X, Y)] = 0$ in its neighborhood and (ii) the set of functions $\Im = \{U_{\alpha,c}(\cdot,\cdot); (\alpha, c) \in \Theta\}$ is Glivenko-Cantelli (Van Der Vaart, 1998). For (i) we first need to check $\Phi(\pi; \alpha_0, c_0) = 0$ and $\Psi(\pi; \alpha_0, c_0) = 0$. The first equation follows from the fact that the conditional expectation of Δ_{ij} given $(\tilde{X}_{ij}, \tilde{Y}_{ij})$ is $g_{\alpha_0}\{c_0 \pi(\tilde{X}_{ij}, \tilde{Y}_{ij})\}$. The second equation can be directly shown from the identity:

$$\phi_{\alpha_0}\{c_0\pi(y_0, y_0)\} = \phi_{\alpha_0}\{F_X(y_0)\} = \int_{y_0}^{\infty} c_0\pi(u,u)\phi'_\alpha\{c_0\pi(u,u)\}\frac{d\pi(u,0)}{\pi(u,u)} .$$

Let $\dot{U}_{\alpha,c}(X_i, Y_i) = \partial U_{\alpha,c}(X_i, Y_i)/\partial(\alpha, c)$ and $A = E[\dot{U}_{\alpha_0,c_0}(X, Y)]$. Then, the non-singularity of $A = E[\dot{U}_{\alpha_0,c_0}(X, Y)]$ is sufficient to show the uniqueness of the zero-point of (α_0, c_0) in its neighborhood. To prove (ii), we note that sufficient conditions for Glivenko-Cantelli are that $(\alpha, c) \mapsto U_{\alpha,c}(x, y)$ is continuous in (α, c) for any fixed point (x, y) and that the function $(\alpha, c, x, y) \mapsto U_{\alpha,c}(x, y)$ is bounded (Van Der Vaart, 1999). These requirements are satisfied under the continuity condition (A-II) along with the compactness of the parameter space (A-I). Thus, the conditions for the consistency of $(\hat{\alpha}, \hat{c})$ are justified.

Finally, based on the Taylor expansion of equation (A.4), one can show that

$$\sqrt{n}\begin{bmatrix} \hat{\alpha}-\alpha_0 \\ \hat{c}-c_0 \end{bmatrix} = \frac{1}{n^{1/2}}\sum_{i=1}^{n} A^{-1}U_{\alpha_0,c_0}(X_i,Y_i) + o_p(1) .$$

Thus, the statement of Theorem 2 holds by letting $B = E[U_{\alpha_0,c_0}(X,Y)U_{\alpha_0,c_0}(X,Y)^T]$.

For the marginal estimators, we derive the following asymptotic linear expression:

$$n^{1/2}\begin{bmatrix} \phi_{\hat{\alpha}}\{\hat{S}_Y(t)\} - \phi_{\alpha_0}\{S_Y(t)\} \\ \phi_{\hat{\alpha}}\{\hat{F}_X(t)\} - \phi_{\alpha_0}\{F_X(t)\} \end{bmatrix} = \frac{1}{n^{1/2}}\sum_{i=1}^{n}\left\{\begin{bmatrix} L^Y{}_{\alpha_0,c_0}(X_i,Y_i;t) \\ L^X{}_{\alpha_0,c_0}(X_i,Y_i;t) \end{bmatrix} + \begin{bmatrix} h_Y(t)^T \\ h_X(t)^T \end{bmatrix} A^{-1}U_{\alpha_0,c_0}(X_i,Y_i)\right\} + o_p(1),$$

where

$$H^Y{}_{\alpha,c}(\pi;t) = \left[\int_{y_0}^{t}\frac{\partial}{\partial\alpha}\psi\{\alpha,c;\pi(u,u)\}d\pi(\infty,u) \quad \int_{y_0}^{t}\frac{\partial}{\partial c}\psi\{\alpha,c;\pi(u,u)\}d\pi(\infty,u)\right]^T ,$$

$$H^X{}_{\alpha,c}(\pi;t) = -\left[\int_{t}^{x_0}\frac{\partial}{\partial\alpha}\psi\{\alpha,c;\pi(u,u)\}d\pi(u,0) \quad \int_{t}^{x_0}\frac{\partial}{\partial c}\psi\{\alpha,c;\pi(u,u)\}d\pi(u,0)\right]^T$$

and $L^Y{}_{\alpha,c}(X_i,Y_i;t)$ and $L^X{}_{\alpha,c}(X_i,Y_i;t)$ equal

$$c^2\int_{y_0}^{t}\phi''_{\alpha}\{c\pi(u,u)\}\{I(X_i \le u, Y_i \ge u) - \pi(u,u)\}d\pi(\infty,u) + c\int_{y_0}^{t}\phi'_{\alpha}\{c\pi(u,u)\}d\{I(Y_i \ge u) - \pi(\infty,u)\},$$

$$-c^2\int_{t}^{x_0}\phi''_{\alpha}\{c\pi(u,u)\}\{I(X_i \le u, Y_i \ge u) - \pi(u,u)\}d\pi(u,0) - c\int_{t}^{x_0}\phi'_{\alpha}\{c\pi(u,u)\}d\{I(X_i \le u) - \pi(u,0)\}$$

respectively. The terms in the summation are i.i.d. mean-zero stochastic processes and the summation is a tight process. The notation $o_p(1)$ holds uniformly for $t \in [0,\infty)$. Then it can be shown that

$$n^{1/2}\begin{bmatrix} \hat{S}_Y(t) - S_Y(t) \\ \hat{F}_X(t) - F_X(t) \end{bmatrix} = \frac{1}{n^{1/2}}\sum_{i=1}^{n}\begin{bmatrix} V^Y{}_{\alpha_0,c_0}(X_i,Y_i;t) \\ V^X{}_{\alpha_0,c_0}(X_i,Y_i;t) \end{bmatrix} + o_p(1), \qquad \text{(A.5)}$$

where

$$V^Y{}_{\alpha_0,c_0}(X_i,Y_i;t)=\frac{L^Y{}_{\alpha_0,c_0}(X_i,Y_i;t)}{\phi'_{\alpha_0}\{S_Y(t)\}}+\left\{\frac{h_Y(t)}{\phi'_{\alpha_0}\{S_Y(t)\}}+\begin{bmatrix}\dot{\phi}_\alpha^{-1}\{\phi_{\alpha_0}S_Y(t)\}\\0\end{bmatrix}\right\}^T A^{-1}U_{\alpha_0,c_0}(X_i,Y_i)$$

and

$$V^X{}_{\alpha_0,c_0}(X_i,Y_i;t)=\frac{L^X{}_{\alpha_0,c_0}(X_i,Y_i;t)}{\phi'_{\alpha_0}\{F_X(t)\}}+\left\{\frac{h_X(t)}{\phi'_{\alpha_0}\{F_X(t)\}}+\begin{bmatrix}\dot{\phi}_\alpha^{-1}\{\phi_{\alpha_0}F_X(t)\}\\0\end{bmatrix}\right\}^T A^{-1}U_{\alpha_0,c_0}(X_i,Y_i)$$

are mean-zero i.i.d. stochastic processes and their summations are tight processes. Let $V_n(t)=n^{1/2}(\hat{S}_Y(t)-S_Y(t),\hat{F}_X(t)-F_X(t))^T$. Also, let $G(t)=(G_X(t),G_Y(t))^T$ be a zero-mean Gaussian random field, the covariance function being specified as $E[G_Y(s)G_Y(t)]=E[V^Y_{\alpha_0,c_0}(X,Y;s)V^Y_{\alpha_0,c_0}(X,Y;t)]$, $E[G_Y(s)G_X(t)]=E[V^Y_{\alpha_0,c_0}(X,Y;s)V^Y_{\alpha_0,c_0}(X,Y;t)]$ and $E[G_X(s)G_X(t)]=E[V^X_{\alpha_0,c_0}(X,Y;s)V^X_{\alpha_0,c_0}(X,Y;t)]$ for $0\le s,t<\infty$. Based on the central limit theorem, the finite dimensional distribution of $V_n(t)$ converges weakly to that of $G(t)$ and the tightness property of $V_n(t)$, we can prove *Theorem 3.*

Appendix A3: *Estimation of Variance under AC models*

The asymptotic analysis in Appendix A.2 has proven that $\sqrt{n}(\hat{\alpha}-\alpha_0,\hat{c}-c_0)$ has an asymptotic variance $A^{-1}B(A^{-1})^T$. Plug-in estimators can be used to obtain the estimators of the asymptotic variance by $\hat{A}_{\hat{\alpha},\hat{c}}$ where

$$\hat{A}_{\alpha,c}=\begin{bmatrix}\partial\Phi(\hat{\pi};\alpha,c)/\partial(\alpha,c)\\\partial\Psi(\hat{\pi};\alpha,c)/\partial(\alpha,c)\end{bmatrix}, \tag{A.6-a}$$

$$\begin{aligned}\frac{\partial\Phi(\hat{\pi};\alpha,c)}{\partial(\alpha,c)}&=\frac{1}{n^2}\sum_{k,l}I\{A_{kl}\}\frac{\partial\tilde{w}_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}}{\partial(\alpha,c)}[\Delta_{kl}-g_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}]\\&\quad-\frac{1}{n^2}\sum_{k,l}I\{A_{kl}\}\tilde{w}_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}\frac{\partial g_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}}{\partial(\alpha,c)},\end{aligned}$$

$$\frac{\partial\Psi(\hat{\pi};\alpha,c)}{\partial(\alpha,c)}=\frac{1}{n}\sum_{j;y_L\le X_j}\frac{\partial c\phi_\alpha'\{c\hat{\pi}(X_j,X_j)\}}{\partial(\alpha,c)}+\frac{\partial\phi_\alpha\{c\hat{\pi}(y_L,y_L)\}}{\partial(\alpha,c)},$$

and $\hat{B}_{\hat{\alpha},\hat{c}}=\frac{1}{n}\sum_j \hat{U}_{\hat{\alpha},\hat{c}}(X_j,Y_j)\hat{U}_{\hat{\alpha},\hat{c}}(X_j,Y_j)^T$, where

$$\hat{U}_{\hat{\alpha},\hat{c}}(X_i,Y_i)=\begin{bmatrix}\Phi_{\hat{\pi}}'(\hat{\pi}_{(X_i,Y_i)}-\hat{\pi};\hat{\alpha},\hat{c})\\ \Psi_{\hat{\pi}}'(\hat{\pi}_{(X_i,Y_i)}-\hat{\pi};\hat{\alpha},\hat{c})\end{bmatrix}, \tag{A.6-b}$$

$$\begin{aligned}
&\Phi_{\hat{\pi}}'(\hat{\pi}_{(X_i,Y_i)}-\hat{\pi};\alpha,c)\\
&=\frac{c}{n^2}\sum_{k,l}I\{A_{kl}\}\{I(X_i\le\tilde{X}_{kl},Y_i\ge\tilde{Y}_{kl})-\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}\\
&\qquad\times\left(\tilde{w}_\alpha'\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}[\Delta_{kl}-g_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}]-\tilde{w}_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}g_\alpha'\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}\right)\\
&+\frac{2}{n}\sum_l I\{A_{il}\}\tilde{w}_\alpha\{c\hat{\pi}(\tilde{X}_{il},\tilde{Y}_{il})\}[\Delta_{il}-g_\alpha\{c\hat{\pi}(\tilde{X}_{il},\tilde{Y}_{il})\}]\\
&-\frac{2}{n^2}\sum_{k,l}I\{A_{kl}\}\tilde{w}_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}[\Delta_{kl}-g_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}],
\end{aligned}$$

$$\begin{aligned}
&\Psi_{\hat{\pi}}'(\hat{\pi}_{(X_i,Y_i)}-\hat{\pi};\alpha,c)\\
&=\frac{c^2}{n}\sum_k I(y_L\le X_k)\phi_\alpha''\{c\hat{\pi}(X_k,X_k)\}\{I(X_i\le X_k,Y_i\ge X_k)-\hat{\pi}(X_k,X_k)\}\\
&-c\sum_k I(y_L\le X_k)\{I(X_i\le X_k)-\hat{\pi}(X_k,X_k)\}[\phi_\alpha'\{c\hat{\pi}(X_k,X_k)\}-\phi_\alpha'\{c\hat{\pi}(X_k-,X_k-)\}].
\end{aligned}$$

In the above expression, the asymptotic variances of the proposed likelihood estimator and the estimator of Chaieb et al. (2006) can be obtained by setting $\tilde{w}_\alpha(v)=\dot{\theta}_\alpha(v)\{1+1/\theta_\alpha(v)\}/v$ and $\tilde{w}_\alpha(v)=1$ respectively. Expressions (A.6-a) and (A.6-b) are applicable to the whole AC families by setting an arbitrary generating function $\phi_\alpha(v)$. Although the asymptotic variance estimators from (A.6-a) and (A.6-b) are explicit, the formulae are somewhat complicated. In the next section, we explain how to apply these formulae to the Clayton model.

Appendix A4: *Estimation of Variance under Clayton Model*

Expressions (A.6-a) and (A.6-b) become relatively easier to calculate for the Clayton AC model of the form $\phi_\alpha(t) = (t^{-(\alpha-1)} - 1)/(\alpha - 1)$. Applying $g_\alpha(v) = (1+\alpha)^{-1}$, $g'_\alpha(v) = 0$, $\phi'_\alpha(t) = -(\alpha-1)t^{-\alpha}$ and $\phi''_\alpha(t) = \alpha(\alpha-1)t^{-\alpha-1}$, one obtains

$$\frac{\partial \Phi(\hat{\pi};\alpha,c)}{\partial \alpha}$$

$$= \frac{1}{n^2}\sum_{k,l} I\{A_{kl}\}\frac{\partial \tilde{w}_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}}{\partial \alpha}[\Delta_{kl} - 1/(1+\alpha)] + \frac{1}{n^2(1+\alpha)^2}\sum_{k,l} I\{A_{kl}\}\tilde{w}_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\},$$

$$\Phi'_{\hat{\pi}}(\hat{\pi}_{(X_i,Y_i)} - \hat{\pi};\alpha,c)$$

$$= \frac{c}{n^2}\sum_{k,l} I\{A_{kl}\}\{I(X_i \le \tilde{X}_{kl}, Y_i \ge \tilde{Y}_{kl}) - \hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}\tilde{w}'_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}[\Delta_{kl} - 1/(1+\alpha)]$$

$$+ \frac{2}{n}\sum_{l} I\{A_{il}\}\tilde{w}_\alpha\{c\hat{\pi}(\tilde{X}_{il},\tilde{Y}_{il})\}[\Delta_{il} - 1/(1+\alpha)] - \frac{2}{n^2}\sum_{k,l} I\{A_{kl}\}\tilde{w}_\alpha\{c\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}[\Delta_{kl} - 1/(1+\alpha)],$$

It turns out that the above formulae do not depend on c when one chooses the weight $\tilde{w}_\alpha(v) = 1$ and $\tilde{w}_\alpha(v) = c(1+\alpha)/(v\alpha)$ corresponding to the estimator of Chaieb, Rivest and Abdous (2006) and the proposed likelihood estimator respectively. The above expression can be used for approximating the asymptotic variance of $\hat{\alpha}$ by

$$Var(\sqrt{n}\hat{\alpha}) \approx \hat{A}_{\hat{\alpha}}^{-2}(n)\hat{B}_{\hat{\alpha}}(n) \tag{A.7}$$

where $\hat{A}_{\hat{\alpha}}(n)$ and $\hat{B}_{\hat{\alpha}}(n)$ are defined separately as follows:

1) Estimating equation for Chaieb, Rivest and Abdous (2006): $\tilde{w}_\alpha(v) = 1$

$$\hat{A}_\alpha(n) = \frac{1}{(1+\alpha)^2 n^2}\sum_{k,l} I\{A_{kl}\},$$

$$\hat{B}_\alpha(n) = \frac{1}{n}\sum_i\left(\frac{2}{n}\sum_l I\{A_{il}\}[\Delta_{il} - 1/(1+\alpha)] - \frac{2}{n^2}\sum_{k,l} I\{A_{kl}\}[\Delta_{kl} - 1/(1+\alpha)]\right)^2.$$

2) The proposed likelihood equation: $\tilde{w}_\alpha(v) = c(1+\alpha)/(v\alpha)$

$$\hat{A}_\alpha(n) = -\frac{1}{\alpha^2 n^2}\sum_{k,l} I\{A_{kl}\}\frac{\Delta_{kl} - 1/(1+\alpha)}{\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})} + \frac{1}{\alpha(1+\alpha)n^2}\sum_{k,l}\frac{I\{A_{kl}\}}{\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})},$$

$$\hat{B}_\alpha(n) = \frac{1}{n}\sum_i \Bigg(-\frac{1+\alpha}{\alpha n^2}\sum_{k,l} I\{A_{kl}\}\frac{[\Delta_{kl} - 1/(1+\alpha)]}{\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})^2}\{I(X_i \le \tilde{X}_{kl}, Y_i \ge \tilde{Y}_{kl}) - \hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})\}$$

$$+\frac{2(1+\alpha)}{\alpha n}\sum_l I\{A_{il}\}\frac{[\Delta_{il} - 1/(1+\alpha)]}{\hat{\pi}(\tilde{X}_{il},\tilde{Y}_{il})} - \frac{2(1+\alpha)}{\alpha n^2}\sum_{k,l} I\{A_{kl}\}\frac{[\Delta_{kl} - 1/(1+\alpha)]}{\hat{\pi}(\tilde{X}_{kl},\tilde{Y}_{kl})}\Bigg)^2 .$$

We have performed Monte Carlo simulations to assess the performance of the above variance estimators and to compare them with the jackknife estimator. Data $\{(X_j, Y_j)\ (j = 1,\ldots,n)\}$ were generated by the Clayton semi-survival AC model with exponential marginal distributions subject to the truncating condition $X_j \le Y_j$. For each run, the estimator of the asymptotic variance $\hat{\sigma}^2_{Analytic} = \hat{A}^{-2}_{\hat{\alpha}}(n)\hat{B}_{\hat{\alpha}}(n)$ and the jackknife estimator

$$\hat{\sigma}^2_{Jack} = (n-1)\sum_j (\hat{\alpha}^{(-j)} - \overline{\alpha}^{(\cdot)})(\hat{\alpha}^{(-j)} - \overline{\alpha}^{(\cdot)})',$$

where $\hat{\alpha}^{(-j)}$ is the estimator ignoring j th observation $\overline{\alpha}^{(\cdot)} = \sum_j \hat{\alpha}^{(-j)}/n$, are calculated. The mean squared error (MSE) with respect to the sample variance of $\hat{\alpha}$ is compared based on 200 runs. Also, their empirical coverage probability for the confidence limits $\hat{\alpha} \pm 1.96\hat{\sigma}_{Analytic}/\sqrt{n}$ and $\hat{\alpha} \pm 1.96\hat{\sigma}_{Jack}/\sqrt{n}$ is compared.

Table 1 and 2 summarize the simulation results. Note that results from Table 1 use the weight $\tilde{w}_\alpha(v) = 1$ while those of Table 2 use $\tilde{w}_\alpha(v) = c(1+\alpha)/(v\alpha)$ for variance estimation in (A.7). From the results in both Table 1 and 2, the estimators (A.7) are fairly close to the true variances. The mean squared error (MSE) of the estimators reduces as the sample size gets large in all cases. The analytic estimator has smaller MSE than the jackknife estimator in all cases, but the difference is not obvious. The coverage rates of the 95% confidence intervals using the two variance formulas are similar and both are close to the nominal 95% level in all cases. In summary, the estimation of the asymptotic variance using the analytic and jackknife estimator is appropriate under the Clayton model.

Appendix B: *Equivalence of Different Estimating Functions*

For a pair (X_i, Z_i, δ_i) and (X_j, Z_j, δ_j), we define

$$B_{ij} = \{I(X_{ij} \le Z_{ij}) = 1\} \cap$$

$$\cap[\{\delta_i = \delta_j = 1\} \cup \{\delta_i = 1, \delta_j = 0, Z_j > Z_i\} \cup \{\delta_i = 0, \delta_j = 1, Z_i > Z_j\}]$$

as the event that the pair (i, j) is orderable and comparable (Martin and Betensky, 2005). We aim to establish the following identity:

$$I = \iint_{(x,y)\in\varphi} w_{\alpha,c^*}(x,y)\left[\Delta(x,y) - \frac{\theta_\alpha\{c * \hat{v}(x,y)\}}{R(x,y) - 1 + \theta_\alpha\{c * \hat{v}(x,y)\}}\right]$$

$$= -\sum_{i<j} I\{B_{ij}\} \frac{w_{\alpha,c^*}(\breve{X}_{ij}, \tilde{Z}_{ij})[1 + \theta_\alpha\{c * \hat{v}(\breve{X}_{ij}, \tilde{Z}_{ij})\}]}{R(\breve{X}_{ij}, \tilde{Z}_{ij}) - 1 + \theta_\alpha\{c * \hat{v}(\breve{X}_{ij}, \tilde{Z}_{ij})\}}$$

$$\times\left[\Delta_{ij} - \frac{1}{1 + \theta_\alpha\{c * \hat{v}(\breve{X}_{ij}, \tilde{Z}_{ij})\}}\right].$$

As a special case with $C_i = \infty$, the above identity yields equation (9a).

The following proof is for the general situation that permits external censoring. Let $\hat{\theta}(x,y) = \theta_\alpha\{c * \hat{v}(x,y)\}$ and $w(x,y) = w_{\alpha,c^*}(x,y)$. Writing the integral via the finite sum, we obtain

$$I = \sum_{i=1}^{n} \sum_{j: X_i < Z_j \le Z_i, X_j < X_i} \delta_j w(X_i, Z_j)\left[N_{11}(dX_i, dZ_j) - \frac{\hat{\theta}(X_i, Z_j)}{R(X_i, Z_j) - 1 + \hat{\theta}(X_i, Z_j)}\right]$$

$$= \sum_{i=1}^{n} \delta_i w(X_i, Z_i)\left[1 - \frac{\hat{\theta}(X_i, Z_i)}{R(X_i, Z_i) - 1 + \hat{\theta}(X_i, Z_i)}\right] - \sum_{i=1}^{n} \sum_{j: X_i < Z_j < Z_i, X_j < X_i} \frac{\delta_j w(X_i, Z_j)\hat{\theta}(X_i, Z_j)}{R(X_i, Z_j) - 1 + \hat{\theta}(X_i, Z_j)}$$

$$\equiv I_1 + I_2.$$

The first term I_1 can be written as

$$\sum_{i=1}^{n} \frac{\delta_i w(X_i, Z_i)\{R(X_i, Z_i) - 1\}}{R(X_i, Z_i) - 1 + \hat{\theta}(X_i, Z_i)} = \sum_{i=1}^{n} \sum_{j: X_i < Z_j, X_j < X_i} \frac{\delta_i (1 - \Delta_{ij}) w(\breve{X}_{ij}, \tilde{Z}_{ij})}{R(\breve{X}_{ij}, \tilde{Z}_{ij}) - 1 + \hat{\theta}(\breve{X}_{ij}, \tilde{Z}_{ij})}.$$

The above equation follows by noting that the number of j satisfying $Z_j > Z_i, X_j < X_i$ is $R(X_i, Z_j) - 1$ and using the notation $\breve{X}_{ij}$ and $\tilde{Z}_{ij}$. It is easy to see that

$$I_2 = -\sum_{i=1}^{n} \sum_{j: X_i < Z_j < Z_i, X_j < X_i} \frac{\delta_j w(\breve{X}_{ij}, \tilde{Z}_{ij}) \hat{\theta}(\breve{X}_{ij}, \tilde{Z}_{ij})}{R(\breve{X}_{ij}, \tilde{Z}_{ij}) - 1 + \hat{\theta}(\breve{X}_{ij}, \tilde{Z}_{ij})}.$$

$$= -\sum_{i=1}^{n} \sum_{j: X_i < Z_j, X_j < X_i} \frac{\delta_j \Delta_{ij} w(\breve{X}_{ij}, \tilde{Z}_{ij}) \hat{\theta}(\breve{X}_{ij}, \tilde{Z}_{ij})}{R(\breve{X}_{ij}, \tilde{Z}_{ij}) - 1 + \hat{\theta}(\breve{X}_{ij}, \tilde{Z}_{ij})}.$$

By combining these terms, we have

$$I = \sum_{i=1}^{n} \sum_{j: X_i < Z_j, X_j < X_i} w(\breve{X}_{ij}, \tilde{Z}_{ij}) \frac{\delta_i (1 - \Delta_{ij}) - \delta_j \Delta_{ij} \hat{\theta}(\breve{X}_{ij}, \tilde{Z}_{ij})}{R(\breve{X}_{ij}, \tilde{Z}_{ij}) - 1 + \hat{\theta}(\breve{X}_{ij}, \tilde{Z}_{ij})}.$$

$$= -\sum_{i<j} I\{B_{ij}\} \frac{w(\breve{X}_{ij}, \tilde{Z}_{ij})\{-1 + \Delta_{ij} + \Delta_{ij} \hat{\theta}(\breve{X}_{ij}, \tilde{Z}_{ij})\}}{R(\breve{X}_{ij}, \tilde{Z}_{ij}) - 1 + \hat{\theta}(\breve{X}_{ij}, \tilde{Z}_{ij})}$$

$$= -\sum_{i<j} I(B_{ij}) \frac{w_{\alpha, c^*}(\breve{X}_{ij}, \tilde{Z}_{ij})[1 + \theta_\alpha\{c^* \hat{v}(\breve{X}_{ij}, \tilde{Z}_{ij})\}]}{R(\breve{X}_{ij}, \tilde{Z}_{ij}) - 1 + \theta_\alpha\{c^* \hat{v}(\breve{X}_{ij}, \tilde{Z}_{ij})\}} \left[\Delta_{ij} - \frac{1}{1 + \theta_\alpha\{c^* \hat{v}(\breve{X}_{ij}, \tilde{Z}_{ij})\}} \right].$$

Appendix C: *Examples*

For illustration, we derive explicit formulas for the Clayton and Frank models.

C1: *Clayton model (Clayton, 1978)*

The Clayton copula is indexed by $\phi_\alpha(t) = t^{-(\alpha-1)} - 1$ ($\alpha > 0$) and, by equation (1), $\theta_\alpha(v) = \alpha$. The semi-survival Clayton model follows that

$$\Pr(X \le x, Y > y \mid X \le Y) = (1/c)[\max\{F_X(x)^{-(\alpha-1)} + S_Y(y)^{-(\alpha-1)} - 1, 0\}]^{-\frac{1}{\alpha-1}}.$$

Note that the above expression also accommodates the case of $0 < \alpha < 1$, where $\phi_\alpha(0) < \infty$ (Nelsen, 1999, p.92). By equations (2) or (5), $\theta^*(x, y) = \alpha$ but its interpretation is the reciprocal of the usual odds ratio. Hence, when $0<\alpha<1$, we have $\theta^*(x,y)=\alpha<1$ which implies positive association between X and Y.

The proposed estimating function is given by

$$U_L(\alpha)=\iint_{(x,y)\in\varphi}\frac{1}{\alpha}\left[\Delta(x,y)-\frac{\alpha}{R(x,y)-1+\alpha}\right]=0.$$

By solving $U_L(\alpha)=0$, $\hat{\alpha}$ can be obtained without estimating c^* or c. The second estimating function $U_c(\alpha,c^*)=0$ reduces to the explicit formula

$$c^*=\left(\left(\frac{1}{n}\right)^{1-\alpha}+\sum_{j;x_{(1)}<x_j}\left[\left\{\frac{\tilde{R}(x_j)}{n\hat{S}_C(x_j)}\right\}^{1-\alpha}-\left\{\frac{\tilde{R}(x_j)-1}{n\hat{S}_C(x_j)}\right\}^{1-\alpha}\right]\right)^{-\frac{1}{1-\alpha}}.$$

Plugging $\hat{\alpha}$ in the above equation, we obtain $\hat{c}^*$. The recursive algorithm yields the following marginal estimators:

$$\hat{S}_Y(t)=\left(1-\sum_{j;z_j\le t,\delta_j=1}\left[\left\{\hat{c}^*\frac{\tilde{R}(z_j)}{n\hat{S}_C(z_j)}\right\}^{1-\hat{\alpha}}-\left\{\hat{c}^*\frac{\tilde{R}(z_j)-1}{n\hat{S}_C(z_j)}\right\}^{1-\hat{\alpha}}\right]\right)^{\frac{1}{1-\hat{\alpha}}},$$

$$\hat{F}_X(t)=\left(\left(\frac{\hat{c}^*}{n}\right)^{1-\hat{\alpha}}-\sum_{j;x_{(1)}<x_j\le t}\left[\left\{\hat{c}^*\frac{\tilde{R}(x_j)}{n\hat{S}_C(x_j)}\right\}^{1-\hat{\alpha}}-\left\{\hat{c}^*\frac{\tilde{R}(x_j)-1}{n\hat{S}_C(x_j)}\right\}^{1-\hat{\alpha}}\right]\right)^{\frac{1}{1-\hat{\alpha}}}.$$

C2: *Frank model (Genest, 1987)*

The Frank copula is indexed by $\phi_\alpha(t)=\log\{(1-\alpha^{-1})/(1-\alpha^{-t})\}$, where $1>\alpha>0$ corresponds to the positive association and $1<\alpha$ corresponds to the negative association. It can be shown that $\theta_\alpha(v)=v\log(\alpha)/(1-\alpha^{-v})$. The semi-survival model of the Frank model can be written as

$$\Pr(X\le x,Y>y\mid X<Y)=(1/c)\log_{\alpha^{-1}}[1-(1-\alpha^{-F_X(x)})(1-\alpha^{-S_Y(y)})/(1-\alpha^{-1})].$$

Consider the transformation $\gamma = -c^* \log(\alpha)$. The likelihood estimating function can be expressed in terms of γ, and the proposed estimating function of γ is given by

$$U_L(\gamma) \propto \iint_{(x,y)\in\varphi} \hat{w}_\gamma(x,y)\left[\Delta(x,y) - \frac{\gamma\,\hat{v}(x,y)}{\{e^{\gamma\hat{v}(x,y)}-1\}\{R(x,y)-1\}+\gamma\,\hat{v}(x,y)}\right],$$

where $\hat{w}_\gamma(x,y) = 1-\gamma\,\hat{v}(x,y)e^{\gamma\hat{v}(x,y)}/(1-e^{\gamma\hat{v}(x,y)})$. Let $\hat{\gamma}$ be the solution to $U_L(\gamma)=0$. The association parameter α can be estimated by

$$\hat{\alpha}^{-1} = 1 + \left(e^{\hat{\gamma}/n}-1\right)\prod_{j;x_{(1)}<x_j}\left[\frac{e^{\hat{\gamma}\tilde{R}(x_j)/\{n\hat{S}_C(x_j)\}}-1}{e^{\hat{\gamma}\{\tilde{R}(x_j)-1\}/\{n\hat{S}_C(x_j)\}}-1}\right]$$

and hence $\hat{c}^* = -\hat{\gamma}/\log(\hat{\alpha})$. Explicit formula for the marginal estimators are given by

$$\hat{S}_Y(t) = \log_{\hat{\alpha}^{-1}}\left(1+(\hat{\alpha}^{-1}-1)\prod_{j;z_j\le t,\delta_j=1}\left[\frac{\alpha^{-\hat{c}^*\{\tilde{R}(z_j)-1\}/\{n\hat{S}_C(z_j)\}}-1}{\alpha^{-\hat{c}^*\tilde{R}(z_j)/\{n\hat{S}_C(z_j)\}}-1}\right]\right),$$

$$\hat{F}_X(t) = \log_{\hat{\alpha}^{-1}}\left(1+(\hat{\alpha}^{-\hat{c}^*/n}-1)\prod_{j;x_{(1)}<x_j\le t}\left[\frac{\hat{\alpha}^{-\hat{c}^*\tilde{R}(x_j)/\{n\hat{S}_C(x_j)\}}-1}{\hat{\alpha}^{-\hat{c}^*\{\tilde{R}(x_j)-1\}/\{n\hat{S}_C(x_j)\}}-1}\right]\right).$$

References

Chaieb, L. L., Rivest, L.-P. and Abdous, B. (2006). Estimating survival under a dependent truncation. *Biometrika*, **93**, 655-69.

Clayton, D. G. (1978). A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika* **65**, 141-51.

Genest, C. (1987). Frank’s family of bivariate distributions. *Biometrika* **74**, 549-55

Martin, E. C. and Betensky, R. A. (2005). Testing quasi-independence of failure and truncation via Conditional Kendall’s Tau. *J. Am. Statist. Assoc.* **100**, 484-92.

Nelsen, R. B. (1999). *An Introduction to copulas.* New York: Springer-Verlag.

Van Der Vaart. A. W. (1998). *Asymptotic statistics.* Cambridge Series in Statistics and Probabilistic Mathematics. Cambridge: Cambridge University Press.

Wang, M. C., Jewell, N. P. and Tsai, W. Y. (1986). Asymptotic properties of the product-limit estimate and right censored data. *Ann. Statist*. **13**, 1597-605.

Table 1. Comparison of two (analytic vs. jackknife) variance estimators of $\sqrt{n}\hat{\alpha}_\tau$ when (X, Y) are generated from the Clayton AC model.

α	c	n	$Var(\sqrt{n}\hat{\alpha}_\tau)$	Average (MSE) of the variance estimators		95% coverage	
				Analytic	Jackknife	Analytic	Jackknife
5/3	0.67	150	6.050	5.849 (2.402)	6.047 (2.551)	0.970	0.970
		250	5.822	5.642 (1.117)	5.754 (1.139)	0.950	0.955
	0.50	150	5.986	5.864 (2.497)	6.064 (2.690)	0.930	0.935
		250	5.763	5.624 (1.230)	5.736 (1.264)	0.940	0.940
	0.33	150	5.980	6.063 (2.751)	6.271 (3.055)	0.950	0.955
		250	5.779	5.750 (1.463)	5.864 (1.536)	0.960	0.960
3	0.67	150	20.957	20.389 (30.365)	21.226 (33.248)	0.925	0.925
		250	19.339	20.119 (17.272)	20.598 (19.263)	0.945	0.945
	0.50	150	20.459	20.762 (29.898)	21.621 (34.222)	0.955	0.960
		250	20.545	20.268 (16.005)	20.753 (16.883)	0.955	0.955
	0.33	150	20.239	20.626 (29.501)	21.459 (33.791)	0.930	0.940
		250	20.911	20.560 (19.164)	21.052 (20.119)	0.925	0.925

NOTE: $Var(\sqrt{n}\hat{\alpha}_\tau)$ is obtained based on 10,000 Monte Carlo simulations. The MSE of the analytic (jackknife) variance estimator is calculated based on 200 Monte Carlo simulations by $\text{MSE} = \sum_{r=1}^{200}\{\hat{\sigma}_r^2 - Var(\sqrt{n}\hat{\alpha}_\tau)\}^2 / 200$, where $\hat{\sigma}_r^2$ denotes the estimates of analytic (jackknife) variance estimates.

Table 2. Comparison of two (analytic vs. jackknife) variance estimators of $\sqrt{n}\hat{\alpha}_L$ when (X, Y) are generated from the Clayton AC model.

α	c	n	$Var(\sqrt{n}\hat{\alpha}_L)$	Average (MSE) of the variance estimators		95% coverage	
				Analytic	Jackknife	Analytic	Jackknife
5/3	0.67	150	5.236	4.499 (2.467)	5.374 (2.509)	0.920	0.950
		250	5.026	4.419 (1.391)	4.992 (1.253)	0.965	0.970
	0.50	150	5.231	4.433 (3.142)	5.356 (3.358)	0.925	0.950
		250	4.892	4.435 (1.565)	5.008 (1.624)	0.925	0.930
	0.33	150	5.174	4.603 (2.425)	5.505 (2.909)	0.945	0.975
		250	4.969	4.573 (1.491)	5.1463 (1.624)	0.945	0.960
3	0.67	150	17.905	16.486 (26.487)	18.374 (29.440)	0.945	0.955
		250	16.634	16.581 (11.499)	17.715 (14.121)	0.960	0.960
	0.50	150	17.376	16.638 (23.376)	18.583 (28.912)	0.960	0.975
		250	17.233	16.599 (11.402)	17.725 (12.719)	0.940	0.950
	0.33	150	17.380	17.045 (26.364)	18.950 (33.545)	0.930	0.940
		250	17.340	16.770 (13.885)	17.903 (15.351)	0.915	0.925

NOTE: $Var(\sqrt{n}\hat{\alpha}_\tau)$ is obtained based on 10,000 Monte Carlo simulations. The MSE of the analytic (jackknife) variance estimator is calculated based on 200 Monte Carlo simulations by $\text{MSE} = \sum_{r=1}^{200}\{\hat{\sigma}_r^2 - Var(\sqrt{n}\hat{\alpha}_\tau)\}^2 / 200$, where $\hat{\sigma}_r^2$ denotes the estimates of analytic (jackknife) variance estimates.