BAYESIAN MULTIPLE TESTING UNDER SPARSITY FOR POLYNOMIAL-TAILED DISTRIBUTIONS

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Supplementary Material

This file provides detailed proofs of the theoretical results.

S1 Proof of Proposition 1

Proof of Property 1. First observe that

$$\frac{g(x/\theta)}{g(x)} = \frac{d(x/\theta)}{\theta^{\gamma+1}d(x)}.$$

Because of the third condition in the definition of the MPT family, $g(x/\theta)/g(x)$ is a strictly increasing function for $x > 0$ if $\theta > 1$. Also notice that $g(x/\theta)/g(x) \to 1$ as $x \to \infty$. Combining the two, we get $g(x/\theta)/g(x) < 1$. Therefore, for any $x_1 > x_2 > 0$,

$$\frac{g(x_2)}{g(x_1)} = \frac{g(x_1/(x_1/x_2))}{g(x_1)} < 1.$$

□

Proof Property 2. $h'(x) = -d(x)x^{\gamma} + \gamma x^{\gamma-1}\{1 - D(x)\}$. To show h is a strictly increasing function, it is sufficient to show $h'(x) > 0$, or equivalently, $\{1 - D(x)\}/d(x) > x/\gamma$.

$$\frac{1 - D(x)}{d(x)} = \frac{\int_x^\infty d(y)dy}{d(x)} = \int_x^\infty \frac{d(y)y^{\gamma+1}}{d(x)x^{\gamma+1}}\frac{x^{\gamma+1}}{y^{\gamma+1}}dy > \int_x^\infty \frac{x^{\gamma+1}}{y^{\gamma+1}}dy = x/\gamma.$$

□

Proof of Property 3. Let $\psi(x) = \theta^{-1}d(x/\theta)/d(x)$. For $x_1 > x_2 > 0$,

$$\frac{1-D(x_2/\theta)}{1-D(x_2)} = \frac{\int_{x_2}^{x_1} d(x)\psi(x)dx + \int_{x_1}^{\infty} d(x)\psi(x)dx}{\int_{x_2}^{x_1} d(x)dx + \int_{x_1}^{\infty} d(x)dx} < \frac{\int_{x_1}^{\infty} d(x)\psi(x)dx}{\int_{x_1}^{\infty} d(x)dx} = \frac{1-D(x_1/\theta)}{1-D(x_1)},$$

since by the monotonicity of $\psi(x)$,

$$\frac{\int_{x_2}^{x_1} d(x)\psi(x)dx}{\int_{x_2}^{x_1} d(x)dx} < \psi(x_1) \text{ and } \frac{\int_{x_1}^{\infty} d(x)\psi(x)dx}{\int_{x_1}^{\infty} d(x)dx} > \psi(x_1).$$

□

Proof of Property 4.

$$f'(x) = 2d(x)/\gamma + 2xd'(x)/\gamma + 2d(x) = 2d(x)(1+1/\gamma)\left(1 + \frac{xd'(x)}{(1+\gamma)d(x)}\right).$$

By Property 1, $\log(x^{\gamma+1}d(x))$ is a strictly increasing function in x for $x > 0$, so

$$\left\{\log(x^{\gamma+1}d(x))\right\}' = \frac{\gamma+1}{x} + \frac{d'(x)}{d(x)} = \frac{\gamma+1}{x}\left(1 + \frac{xd'(x)}{(1+\gamma)d(x)}\right) > 0.$$

Thus $f'(x) > 0$, for $x > 0$. □

S2 Proof of Proposition 2

Proof. Since the oracle threshold ω_{opt} does not depend on identically distributed $X_1, \ldots, X_m$, t_{1i} and t_{2i} do not vary with i. Hence the Bayes risk of the oracle Bayes rule can be simplified to

$$R_{\text{opt}} = m(1-p)t_1\delta_0 + mpt_2\delta_A = mp\delta_A(vt_1 + t_2),$$

and (2.7) is immediate from (2.6). The asymptotic behavior of ω_{opt}^2 and t_2 in (2.6) follows immediately from (2.5). Also, since $\omega_{\text{opt}} \to \infty$ as $u \to \infty$ and $1 - D(x) \sim (C_d/\gamma)x^{-\gamma}$ as $x \to \infty$, we have

$$\begin{aligned} vt_1 = 2v\{1 - D(\omega_{\text{opt}})\} &\sim 2v(C_d/\gamma)\omega_{\text{opt}}^{-\gamma} \sim 2vu^{-\gamma/2}(C_d/\gamma)C^{-\gamma/2} \\ &\to 2C_0(C_d/\gamma)C^{-\gamma/2} = C_1. \end{aligned}$$

□

S3 Proof of Theorem 1

Proof. The equivalence of (3.1) and (3.2) is straightforward due to (2.6). We will show that (3.2) is necessary and sufficient.

If (3.2) holds, then

$$\omega^2/(1+u) \sim C(v/C_0)^{2/\gamma}u^{-1} \to C, \quad v(\omega^2)^{-\gamma/2} \to C^{-\gamma/2}C_0.$$

Similar to the proof of Proposition 2,

$$2D\left(\omega(1+u)^{-1/2}\right) - 1 \to 2D(\sqrt{C}) - 1, \quad 2v\{1 - D(\omega)\} \to C_1.$$

Therefore, $R/R_{\text{opt}} \to 1$, as $m \to \infty$.

Next, we will show that condition (3.2) is necessary. First, we claim that $\omega^2 \to \infty$ as $m \to \infty$ is a necessary condition of a multiple testing rule to be ABOS. In fact, if the limit of ω^2 is not infinity, there is a subsequence $\tilde{\omega}^2$ of ω^2 such that $\tilde{\omega}^2$ converges to a finite constant. Let the subsequence of risks corresponding to $\tilde{\omega}^2$ be $\tilde{R}$ and the corresponding subsequence of R_{opt} be $\tilde{R}_{\text{opt}}$. Since $\tilde{R}/\tilde{R}_{\text{opt}} \to \infty$, R/R_{opt} will not converge to 1.

Suppose (3.2) does not hold. One of the following statements is true: (a) $\omega^2 v^{-2/\gamma} \to \tilde{C}C_0^{-2/\gamma}$ as $m \to \infty$ where $\tilde{C} \neq C$ is a nonnegative constant; (b) $\omega^2 v^{-2/\gamma} \to \infty$ as $m \to \infty$; (c) The limit of $\omega^2 v^{-2/\gamma}$ does not exist and $\omega^2 v^{-2/\gamma}$ is bounded.

The proof will be complete if we show that the multiple testing procedure is not ABOS in the three cases. If (a) is true, then

$$\frac{R}{R_{\text{opt}}} \to \frac{2D(\sqrt{\tilde{C}}) - 1 + C_1(\tilde{C}/C)^{-\gamma/2}}{2D(\sqrt{C}) - 1 + C_1} = \frac{h_C(\sqrt{\tilde{C}})}{h_C(\sqrt{C})},$$

where $h_C(x) = 2D(x) - 1 + C_1(x/\sqrt{C})^{-\gamma}$. Since h_C reaches its global minimum only at $\sqrt{C}$, so the procedure is not ABOS. If (b) is true,

$$\frac{R}{R_{\text{opt}}} \to \frac{1}{2D(\sqrt{C}) - 1 + C_1} \neq 1.$$

If (c) is true, there exists a subsequence of $\omega^2 v^{-2/\gamma}$ such that its limit is not $CC_0^{-2/\gamma}$. Then this situation goes back to case (a). □

S4 Proof of Theorem 2

Proof. To simplify notations, for any measurable set B with respect to the σ-algebra of the probability space of the random vector $(X_1, \ldots, X_m)$, we let

$$P_{0i}(B) = P\left(B \,|\, s_i = 0\right), \quad P_{Ai}(B) = P\left(B \,|\, s_i = 1\right), \quad i = 1, \ldots, m.$$

Let Q denote the event $\left\{|\hat{\omega} - \omega_{\text{opt}}| > \epsilon v^{1/\gamma}\right\}$. It is easy to see that

$$P(Q) = (1-p)P_{0i}(Q) + pP_{Ai}(Q), \quad i = 1, \ldots, m,$$

so

$$P(Q) = (1-p)\bar{P}_{0\cdot}(Q) + p\bar{P}_{A\cdot}(Q),$$

where $\bar{P}_{0\cdot}(Q) = \sum P_{0i}(Q)/m$ and $\bar{P}_{A\cdot}(Q) = \sum P_{Ai}(Q)/m$. If we assume that δ does not converge to 0, (3.5) implies (3.3) and (3.4), namely,

$$\bar{P}_{0\cdot}(Q) = o(v^{-1}), \quad \bar{P}_{A\cdot}(Q) = o(1).$$

We only need to show the sufficiency of (3.3) and (3.4).

For any random thresholding procedure,

$$t_{1i} = P_{0i}(\{|X_i/\sigma| \geq \hat{\omega}\} \cap Q) + P_{0i}(\{|X_i/\sigma| \geq \hat{\omega}\} \cap Q^C),$$

which implies

$$P_{0i}(|X_i/\sigma| \geq \omega_{\text{opt}} + \epsilon v^{1/\gamma}) - P_{0i}(Q) \leq t_{1i} \leq P_{0i}(Q) + P_{0i}(|X_i/\sigma| \geq \omega_{\text{opt}} - \epsilon v^{1/\gamma}).$$

Then

$$\frac{P_{01}(|X_1/\sigma| \geq \omega_{\text{opt}} + \epsilon v^{1/\gamma}) - \bar{P}_{0\cdot}(Q)}{P_{01}(|X_1/\sigma| \geq \omega_{\text{opt}})} \leq \frac{\bar{t}_{1\cdot}}{t_1} \leq \frac{\bar{P}_{0\cdot}(Q) + P_{01}(|X_1/\sigma| \geq \omega_{\text{opt}} - \epsilon v^{1/\gamma})}{P_{01}(|X_1/\sigma| \geq \omega_{\text{opt}})},$$

where $\bar{t}_{j\cdot} = \sum t_{ji}/m$ $(j = 1, 2)$. Because of (3.3) and (3.4), both sides of the above inequality converge to 1 as $m \to \infty$ and $\epsilon \to 0$. Hence, we have $\bar{t}_{1\cdot}/t_1 \to 1$. Similarly, $\bar{t}_{2\cdot}/t_2 \to 1$. Therefore,

$$\frac{R}{R_{\text{opt}}} = \frac{vt_1}{vt_1 + t_2}\frac{\bar{t}_{1\cdot}}{t_1} + \frac{t_2}{vt_1 + t_2}\frac{\bar{t}_{2\cdot}}{t_2} \to 1.$$

□

S5 Proof of Proposition 3

Proof. By Theorem 1, in order to prove that (4.4) is both necessary and sufficient for a procedure controlling BFDR to be ABOS, we need only show that it is equivalent to

$$\omega_B^2/(1+u) \to C, \quad v\{1-D(\omega_B)\} \to C_1/2. \tag{S5.1}$$

It is straightforward to see (S5.1) implies (4.4) via (4.2). Notice that (4.4) is equivalent to

$$\frac{v\{1-D(\omega_B)\}}{1-D(\omega_B(1+u)^{-1/2})} \to \frac{C_1}{1-C_2}.$$

If we can show that (4.4) implies $\omega_B^2/(1+u) \to C$, then it also implies $v\{1-D(\omega_B)\} \to C_1/2$. To see this, we first observe that (4.4) implies that $\omega_B \to \infty$; otherwise there exists a subsequence of δr_α whose limit is infinite which is in conflict with (4.4). Suppose $\omega_B \to \infty$ but $\omega_B^2/(1+u) \not\to C$. We want to show that (4.4) does not hold.

One of the following three statements must be true if $\omega_B^2/(1+u) \not\to C$: (a) $\omega_B^2/(1+u) \to \tilde{C}$ where $\tilde{C} \neq C$ is a nonnegative constant; (b) $\omega_B^2/(1+u) \to \infty$; (c) $\omega_B^2/(1+u)$ is bounded but its limit does not exist.

If (a) holds, then

$$\delta r_\alpha = \frac{v\{1-D(\omega_B)\}}{1-D(\omega_B(1+u)^{-1/2})} \sim \frac{vC_d(\tilde{C}u)^{-\gamma/2}/\gamma}{1-D(\sqrt{\tilde{C}})} \to \frac{C_1}{2(1-D(\sqrt{C}))}\frac{g(\sqrt{\tilde{C}})}{g(\sqrt{C})},$$

where $g(x) = x^{-\gamma}\{1-D(x)\}^{-1}$. Since g is a strictly decreasing function, $g(\sqrt{\tilde{C}})/g(\sqrt{C}) \neq 1$. If (b) holds, then

$$\delta r_\alpha = \frac{v\{1-D(\omega_B)\}}{1-D(\omega_B(1+u)^{-1/2})} \sim \frac{v\omega_B^{-\gamma}C_d/\gamma}{(\omega_B u^{-1/2})^{-\gamma}C_d/\gamma} \to C_0 \neq \frac{C_1}{2(1-D\sqrt{C})}.$$

If (c) holds, there exists a subsequence of $\omega_B^2/(1+u)$ such that its limit exists but is not C, and we are back to case (a).

By our assumptions,

$$1-D(\omega_B) \sim \omega_B^{-\gamma}C_d/\gamma. \tag{S5.2}$$

By (4.2),

$$1-D(\omega_B) = \frac{r_\alpha}{f}\{1-D(\sqrt{C})\}(1+o(1)). \tag{S5.3}$$

(S5.2) and (S5.3) imply (4.5). □

S6 Proof of Proposition 4

Proof. Equation (4.8) is equivalent to

$$\frac{1 - D(\omega_{\mathrm{GW}})}{1 - D(\omega_{\mathrm{GW}}(1+u)^{-1/2})} = \frac{pr_\alpha}{1 + pr_\alpha} = \frac{r_{\alpha'}}{f}, \tag{S6.1}$$

where $\alpha' = \alpha(1-p)$. Comparing (S6.1) with (4.2), the result is an immediate consequence of Theorem 1 and the fact that $r_{\alpha'}/r_\alpha \to 1$. □

S7 Proof of Theorem 3

The proof of Theorem 3 is based on Theorem 2 and the following lemmas. The proofs of these lemmas are very similar to Lemmas 11.2 and 11.4 in the supplementary material of Bogdan, Chakrabarti, Frommlet, and Ghosh (2011), but we include the proofs for the sake of completeness.

Lemma 1. *Assume there exists a constant $\rho > 0$ such that $mp^{1+\rho} \to \infty$. Let ω_1 be the GW threshold at level $\alpha_1 = \alpha(1-\xi)$, where $\xi^2 = v^{1+\rho}/m$. If (4.4) and (4.10) hold, then*

$$P(\omega_{BH} \geq \omega_1) = o(v^{-1}).$$

Also, $\omega_1 = \omega_{GW} + o(v^{1/\gamma})$.

Proof. Since $mp^{1+\rho} \to \infty$ and $\delta \to \delta_\infty$,

$$\xi^2 = \frac{v^{1+\rho}}{m} \sim \frac{{\delta_\infty}^{1+\rho}}{mp^{1+\rho}} \to 0,$$

which implies that $r_{\alpha_1}/r_\alpha \to 1$. If (4.4) holds, then according to Proposition 4, both the multiple testing rule corresponding to ω_{GW} and the rule corresponding to ω_1 are ABOS. Therefore $\omega_1 = \omega_{\mathrm{GW}} + o(v^{1/\gamma})$. The other part of the theorem also holds since

$$\begin{aligned}
P(\omega_{\mathrm{BH}} \geq \omega_1) &\leq P\left(\frac{2\{1-D(\omega_1)\}}{1-\hat{F}(\omega_1)} > \alpha\right) \\
&= P\left(\frac{1-\hat{F}(\omega_1)}{1-F(\omega_1)} < \frac{2\{1-D(\omega_1)\}}{\alpha\{1-F(\omega_1)\}}\right) \\
&= P\left(\frac{1-\hat{F}(\omega_1)}{1-F(\omega_1)} < 1-\xi\right) \qquad \text{(By (4.8))} \\
&\leq \exp\left(-\frac{1}{4}m\xi^2\{1-F(\omega_1)\}\right) \qquad \text{(By Bernstein's inequality)} \\
&= \exp\left(-\frac{1}{2}m\xi^2\{1-D(\sqrt{C})\}\frac{r_\alpha+1}{f}(1+o(1))\right) \\
&\leq \exp\left(-\frac{C_1}{4}v^\rho(1+o(1))\right) = o(v^{-1}).
\end{aligned}$$

□

Lemma 2. *Let ω_2 be the GW threshold at level $\alpha_2 = \alpha(1+\xi)$ where $\xi = 1/\log(m)$. Under the assumptions (4.4), (4.9) and (4.10),*

$$P(\omega_{BH} < \omega_2) = o(v^{-1}).$$

Also, $\omega_2 = \omega_{GW} + o(v^{1/\gamma})$.

Proof. Since r_{α_2}/r_α converges to 1 and (4.4) holds, according to Theorem 4, both the multiple testing rule corresponding to ω_{GW} and the rule corresponding to ω_2 are ABOS. Therefore $\omega_2 = \omega_{\mathrm{GW}} + o(v^{1/\gamma})$. Since $\{1-D(x)\}/\{1-F(x)\}$ is an decreasing function in x,

$$\omega_{\mathrm{BH}} < \omega_2 \Rightarrow \frac{2\{1-D(\omega_{\mathrm{BH}})\}}{1-F(\omega_{\mathrm{BH}})} > \frac{2\{1-D(\omega_2)\}}{1-F(\omega_2)} = \alpha(1+\xi).$$

□

By the definition of ω_{BH},

$$\frac{2\{1-D(\omega_{\mathrm{BH}})\}}{1-\hat{F}(\omega_{\mathrm{BH}})} \leq \alpha.$$

Combining the above two inequalities, we have

$$\omega_{\rm BH} < \omega_2 \Rightarrow \frac{1-\hat{F}(\omega_{\rm BH})}{1-F(\omega_{\rm BH})} > 1+\xi.$$

Therefore,

$$P(\omega_{\rm BH} < \omega_2) \le P\left(\sup_{c\in[0,\omega_2)} \frac{1-\hat{F}(c)}{1-F(c)} > 1+\xi\right) = P\left(\sup_{t\in[0,z_1)} \frac{1-\hat{G}(t)}{1-F(t)} > 1+\xi\right),$$

where $z_1 = F(\omega_2) = 1 - C_3 p(1+o(1))$, $C_3 = 2\{1-D(\sqrt{C})\}/(1-\alpha_\infty)$, and $\hat{G}$ is the empirical cdf of $U_i = F\left(|X_i|/\sigma_0\right)$ $(i=1,\ldots,m)$. Let $u_i = i/m$, $k_1 = \lceil m(1-C_3p)\rceil$. Notice that for $t \in [u_i, u_i+1/m)$,

$$1-\hat{G}(t) \le 1-\hat{G}(u_i) \text{ and } 1-t > 1-u_i-1/m.$$

Then for sufficiently large m,

$$\begin{aligned} P(\omega_{\rm BH} < \omega_2) &\le P\left(\max_{i\in\{0,1,\ldots,k_1\}} \frac{1-\hat{G}(u_i)}{1-u_i-1/m} > 1+\xi\right) \\ &\le \sum_{i=0}^{k_1} P\left(1-\hat{G}(u_i) > (1+\xi)(1-u_i-1/m)\right). \end{aligned}$$

If $i=0$,

$$P\left(1-\hat{G}(u_i) > (1+\xi)(1-u_i-1/m)\right) = P(1 > (1-1/m)(1+\xi)) = 0$$

for sufficiently large m. For $i = 1,\ldots,k_1$,

$$1-u_i-\frac{1}{m} = (1-u_i)(1-\tau_i),$$

where $\tau_i = m^{-1}(1-u_i)^{-1}$. Since

$$1-u_i \ge 1-k_1/m \ge C_3p - 1/m,$$

we have

$$\tau_i \le (C_3mp-1)^{-1} = o(\xi).$$

By Bernstein's inequality,

$$\begin{aligned} P\left(1-\hat{G}(u_i) > (1-u_i-1/m)(1+\xi)\right) &\leq \exp\left[-\frac{m(1-u_i)\{(1+\xi)(1-\tau)-1\}^2}{4(1+\xi)(1-\tau)}\right] \\ &= \exp\left\{-\frac{1}{4}m\xi^2(1-u_i)(1+o(1))\right\} \\ &\leq \exp\left\{-\frac{1}{4}C_3mp\xi^2(1+o(1))\right\}. \end{aligned}$$

Therefore, for sufficiently large m

$$P(\omega_{\mathrm{BH}} < \omega_2) \leq m\exp\left\{-\frac{1}{4}C_3mp\xi^2(1-u_i)(1+o(1))\right\} = o(v^{-1}),$$

if $p \propto 1/\log(m)$ or $p \propto m^{-\kappa}$.

Proof of Theorem 3. By Theorems 2 and 4, it is sufficient to show that for any $\epsilon > 0$,

$$P(|\omega_{\mathrm{BH}} - \omega_{\mathrm{GW}}| > \epsilon v^{1/\gamma}) = o(v^{-1}).$$

By Lemmas 1 and 2,

$$P(|\omega_{\mathrm{BH}} - \omega_{\mathrm{GW}}| > \epsilon v^{1/\gamma}) \leq P(\omega_{\mathrm{BH}} > \omega_1) + P(\omega_{\mathrm{BH}} < \omega_2) = o(v^{-1}).$$

□

References

Bogdan, M., Chakrabarti, A., Frommlet, F., and Ghosh, J. K. (2011). Asymptotic Bayes-optimality under sparsity of some multiple testing procedures. *The Annals of Statistics* **39**, 1551–1579.