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ESTIMATING THE PARAMETERS OF BURST-TYPE SIGNALS

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Abstract: In this paper, we study a model that exhibits burst-type features such as ECG signals. The model was proposed by Sharma and Sircar (2001) and is a generalization of the fixed amplitude sinusoidal model. The least square method is used to estimate the unknown parameters and we show that the least squares estimators are strongly consistent and asymptotically normal. Some numerical results based on simulations are reported for illustrative purposes.

Key words and phrases: Amplitude modulated model, asymptotic distributions, burst-type signals, least squares estimators.

1. Introduction

The estimation of the parameters of a parametric model is a central problem. The present article addresses the estimation of parameters in the model

$$y(t) = \sum_{i=1}^{q} A_i \exp[b_i\{1 - \cos(\alpha t + c_i)\}] \cos(\theta_i t + \phi_i) + e(t), \qquad t = 1, \ldots, N, \quad (1.1)$$

where for $i = 1, \ldots, q$, A_i is the amplitude of the carrier wave, b_i and c_i are the gain part and the phase part of the exponential modulation signal, θ_i is the carrier angular frequency, α is the modulation angular frequency, and ϕ_i is the phase corresponding to the carrier angular frequency. The number of components present in the signal is denoted by q. The error random variables $\{e(t)\}$ are assumed i.i.d. with mean zero and finite variance. The model (1.1) is a sinusoidal model with a time-dependent amplitude $\sum_{i=1}^{q} s_i(t) \cos(\theta_i t + \phi_i) + e(t)$; here $s_i(t)$ takes the particular exponential form $\exp[b_i\{1-\cos(\alpha t+c_i)\}]$ multiplied by a constant A_i. The modulation angular frequency α is assumed to be same through all components, which ensures the presence of burst-like signal.

The model was proposed by Sharma and Sircar (2001) in complex-valued form. As we mostly deal with real-valued observations, it was suggested in Sharma and Sircar (2001) that one estimated the imaginary part of each of them using the Hilbert transform. Then one has complex-valued data (as observed, estimated by Hilbert transform) and the techniques for a complex model can be implemented. Sharma and Sircar (2001) used this in describing a segment of real electrocardiograph (ECG) signal. In an earlier article, Mukhopadhyay and Sircar (1996) proposed a similar kind of model to analyze an ECG signal, actually it was (1.1) with a different representation of parameters. In both papers the authors analyzed ECG data using ad-hoc estimation procedures. Here we consider the particular form (1.1) and our aim is to study the least square estimators (LSEs) of the unknown parameters and to derive their theoretical properties in a systematic manner.

Many data, for example ECG signals, exhibit burst-type features. Similar structures have been observed in the plot of data generated by (1.1) for different sets of values. For an example, see Figure 6 in Section 4. Model (1.1) was proposed to employ one or more amplitude modulated sinusoidal signals with the aim of modelling different features of an ECG output signal separately. Following Sharma and Sircar (2001), we call (1.1) a burst-type signal. For another type of amplitude modulated sinusoidal model, see Sircar and Syali (1996) and Nandi, Kundu and Iyer (2004).

We discuss the parameter estimation in the presence of i.i.d. noise and use the least square method for estimation. It is known that the constant amplitude multiple sinusoidal model does not satisfy the sufficient conditions of Jennrich (1969) or Wu (1981) for the LSEs to be consistent. Model (1.1), being a more complicated general model does not satisfy them. However, the special structure of the model allows us to establish the strong consistency and the large sample distribution of the LSEs of the unknown parameters. To apply the model to data, the main problem is to guess initial estimates, but we do not address any computational problems in this paper. Rather, we study the theoretical properties of the LSEs.

The paper is organized as follows. In Section 2, we state the asymptotic properties of the LSEs for a single burst-type signal ($q = 1$). The results for general q are discussed in Section 3. Numerical results are presented in Section 4. All proofs are in Section S1 of the Supplementary Material.

2. Asymptotic Distribution of LSEs for Single Burst-Type Signal

In this section, we consider the case $q = 1$ and write model (1.1) as

$$y(t) = A\exp[b\{1-\cos(\alpha t + c)\}]\cos(\theta t + \phi) + e(t), \qquad t = 1,\ldots,N. \tag{2.1}$$

It is assumed that $|b| \le J$, so $e^{b\cos(\alpha t)} \le e^{|b|} \le e^J = K$(say) a finite constant, frequencies $\alpha, \theta \in [0,\pi]$, phases $c, \phi \in [-\pi,\pi]$, $A \in \mathcal{R}$ is a finite constant, and the $e(t)$ are i.i.d. with mean zero and finite variance σ^2. We note that A is a linear parameter whereas other parameters are non-linear. The condition $|b| \le J$ is not a serious restriction because A is unbounded. Our problem is to estimate the unknown parameters A, b, α, c, θ, and ϕ from a given sample of size N.

Let $\boldsymbol{\eta} = (A, b, \alpha, c, \theta, \phi)$ with $\boldsymbol{\eta}^0$ as the true value of $\boldsymbol{\eta}$. The LSE $\hat{\boldsymbol{\eta}}$ at (2.1) minimizes the residual sum of squares

$$Q(\boldsymbol{\eta}) = \sum_{t=1}^{N} \Big[y(t) - A \exp[b\{1 - \cos(\alpha t + c)\}] \cos(\theta t + \phi) \Big]^2 \tag{2.2}$$

with respect to $\boldsymbol{\eta}$. Since the model is a partial non-linear regression model, and the parameter space corresponding to the non-linear parameters is compact (stated in Theorem 2.1), following the approach of Lemma 2 of Jennrich (1969), the existence of the LSEs can be established. In this the LSE $\hat{\boldsymbol{\eta}}$ means the local minimum in the neighbourhood of the true parameter value $\boldsymbol{\eta}^0$. First we have consistency.

Theorem 2.1. *Let $\boldsymbol{\eta}^0 = (A^0, b^0, \alpha^0, c^0, \theta^0, \phi^0)$ be an interior point of the parameter space $\{(-\infty, \infty) \times [-\log(K), \log(K)] \times [0, \pi] \times [-\pi, \pi] \times [0, \pi] \times [-\pi, \pi]\}$, where K is a large positive number with $\exp\{|b^0|\} < K$. If the error random variables $e(t)$s are i.i.d., then the LSE $\hat{\boldsymbol{\eta}}$ is a strongly consistent estimator of $\boldsymbol{\eta}^0$.*

For the proof of Theorem 2.1, see the Supplementary Material.

In rest of this section, we develop the joint asymptotic distribution of the LSEs of the unknown parameters for the single component model. We use the usual Taylor series expansion and denote the first derivative vector of $Q(\boldsymbol{\eta})$ as $Q'(\boldsymbol{\eta})$, of order 1×6, and the 6×6 matrix of second order derivatives as $Q''(\boldsymbol{\eta})$. Expanding $Q'(\hat{\boldsymbol{\eta}})$ around $\boldsymbol{\eta}^0$, we have

$$Q'(\hat{\boldsymbol{\eta}}) - Q'(\boldsymbol{\eta}^0) = (\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0) Q''(\bar{\boldsymbol{\eta}}), \tag{2.3}$$

where $\bar{\boldsymbol{\eta}}$ is a point between $\hat{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}^0$. Now define a diagonal matrix of order six as

$$\mathbf{D} = \text{diag}\{N^{-1/2}, N^{-1/2}, N^{-3/2}, N^{-1/2}, N^{-3/2}, N^{-1/2}\}. \tag{2.4}$$

As $Q'(\hat{\boldsymbol{\eta}}) = 0$, (2.3) can be written as

$$(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)\mathbf{D^{-1}} = -\left[\mathbf{Q}'(\boldsymbol{\eta}^{\mathbf{0}})\mathbf{D}\right] \left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\eta}})\mathbf{D}\right]^{\mathbf{-1}}, \tag{2.5}$$

since $\left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\theta}})\mathbf{D}\right]$ is an invertible matrix *a.e.* for large N. From Theorem 2.1, it follows that $\hat{\boldsymbol{\eta}}$ converges *a.s.* to $\boldsymbol{\eta}^0$ and, since each element of $Q''(\boldsymbol{\eta})$ is a continuous function of $\boldsymbol{\theta}$,

$$\lim_{N\to\infty} \left[\mathbf{D}\mathbf{Q}''(\bar{\boldsymbol{\eta}})\mathbf{D}\right] = \lim_{N\to\infty} \left[\mathbf{D}\mathbf{Q}''(\boldsymbol{\theta}^{\mathbf{0}})\mathbf{D}\right] = 2\boldsymbol{\Sigma}(\boldsymbol{\eta}^0) \quad \text{(say)}. \tag{2.6}$$

In obtaining the exact form of the limit matrix $\mathbf{\Sigma}(\boldsymbol{\eta})$, let us write $\boldsymbol{\eta} = (A, \boldsymbol{\xi})$, where $\boldsymbol{\xi} = (b, \alpha, c, \theta, \phi)$. Then $\mathbf{\Sigma}(\boldsymbol{\eta}) = e^{2b^0}\mathbf{\Delta}(\boldsymbol{\eta}^0)$, where

$$\mathbf{\Delta}(\boldsymbol{\eta})=\begin{bmatrix} \delta_1(0) & A\delta_5(0) & Ab\delta_6(1) & Ab\delta_6(0) & A\delta_7(1) & A\delta_7(0) \\ A\delta_5(0) & A^2\delta_2(0) & A^2b\delta_8(1) & A^2b\delta_8(0) & -A^2\delta_9(1) & -A^2\delta_9(0) \\ Ab\delta_6(1) & A^2b\delta_8(1) & A^2b^2\delta_3(2) & A^2b^2\delta_3(1) & -A^2b\delta_{10}(2) & -A^2b\delta_{10}(1) \\ Ab\delta_6(0) & A^2b\delta_8(0) & A^2b^2\delta_3(1) & A^2b^2\delta_3(0) & -A^2b\delta_{10}(1) & -A^2b\delta_{10}(0) \\ Ab\delta_7(1) & -A^2\delta_9(1) & -A^2b\delta_{10}(2) & -A^2b\delta_{10}(1) & A^2\delta_4(2) & A^2\delta_4(1) \\ A\delta_7(0) & -A^2\delta_9(0) & -A^2b\delta_{10}(1) & -A^2b\delta_{10}(0) & A^2\delta_4(1) & A^2\delta_4(0) \end{bmatrix}, \quad (2.7)$$

and $\delta_k(m) = \delta_k(\boldsymbol{\eta}, m)$, $m = 0, 1, 2$, $k = 1, \ldots, 10$ are defined in Section S2 in the Supplementary Material.

$$Q'(\boldsymbol{\eta}^0)\mathbf{D} = \begin{bmatrix} -\frac{2}{\sqrt{N}}\sum_{t=1}^{N} e(t)\exp\{b^0(1-\cos(\alpha^0 t + c^0))\}\cos(\theta^0 t + \phi^0) \\ -\frac{2}{\sqrt{N}}A^0\sum_{t=1}^{N} e(t)\exp\{b^0(1-\cos(\alpha^0 t + c^0))\}(1-\cos(\alpha^0 t + c^0))\cos(\theta^0 t + \phi^0) \\ -\frac{2}{N^{3/2}}A^0 b^0\sum_{t=1}^{N} te(t)\exp\{b^0(1-\cos(\alpha^0 t + c^0))\}\sin(\alpha^0 t + c^0)\cos(\theta^0 t + \phi^0) \\ -\frac{2}{\sqrt{N}}A^0 b^0\sum_{t=1}^{N} e(t)\exp\{b^0(1-\cos(\alpha^0 t + c^0))\}\sin(\alpha^0 t + c^0)\cos(\theta^0 t + \phi^0) \\ \frac{2}{N^{3/2}}A^0\sum_{t=1}^{N} te(t)\exp\{b^0(1-\cos(\alpha^0 t + c^0))\}\sin(\theta^0 t + \phi^0) \\ \frac{2}{\sqrt{N}}A^0\sum_{t=1}^{N} e(t)\exp\{b^0(1-\cos(\alpha^0 t + c^0))\}\sin(\theta^0 t + \phi^0) \end{bmatrix}.$$

All the elements of $Q'(\boldsymbol{\eta}^0)\mathbf{D}$ satisfy the Lindeberg-Feller condition, therefore it converges to a 6-variate normal distribution. Using the limits given in Section S2 of the Supplementary Material, it follows that

$$Q'(\boldsymbol{\eta}^0)\mathbf{D} \to \mathcal{N}_6\Big(\mathbf{0}, 4\sigma^2\mathbf{\Sigma}(\boldsymbol{\eta}^0)\Big). \quad (2.8)$$

Using (2.6) and (2.8) in (2.5), we have

$$(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)\mathbf{D}^{-1} \to \mathcal{N}_{\mathbf{6}}\Big(\mathbf{0}, \sigma^2\mathbf{\Sigma}^{-1}(\boldsymbol{\eta}^0)\Big). \quad (2.9)$$

Now using the inequality (S2.1) of the Supplementary Material, $\delta_k(\boldsymbol{\xi}, p) = 0$ for

$k = 6, \ldots, 10$, and $\delta_5(\boldsymbol{\xi}, p) = \delta_1(\boldsymbol{\xi}, p)$, $p = 0, 1, \ldots$, we can write

$$\begin{aligned} \boldsymbol{\Delta}(\boldsymbol{\eta}) &= \begin{bmatrix} \boldsymbol{\Delta}_1(\boldsymbol{\eta}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_2(\boldsymbol{\eta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_3(\boldsymbol{\eta}) \end{bmatrix}, \\ \boldsymbol{\Delta}_1(\boldsymbol{\eta}) &= \begin{bmatrix} \delta_1(0) & A\delta_1(0) \\ A\delta_1(0) & A^2\delta_2(0) \end{bmatrix}, \ \boldsymbol{\Delta}_2(\boldsymbol{\eta}) = A^2 b^2 \begin{bmatrix} \delta_3(2) & \delta_3(1) \\ \delta_3(1) & \delta_3(0) \end{bmatrix}, \\ \boldsymbol{\Delta}_3(\boldsymbol{\eta}) &= A^2 \begin{bmatrix} \delta_4(2) & \delta_4(1) \\ \delta_4(1) & \delta_4(0) \end{bmatrix}. \end{aligned} \tag{2.10}$$

Thus, the asymptotic variance-covariance matrix of $(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}^0)\mathbf{D}^{-1}$ is

$$\sigma^2 \Sigma^{-1}(\boldsymbol{\eta}^0) = \sigma^2 e^{-2b^0} \boldsymbol{\Delta}^{-1}(\boldsymbol{\eta}^0) = \sigma^2 e^{-2b^0} \begin{bmatrix} \boldsymbol{\Delta}_1^{-1}(\boldsymbol{\eta}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_2^{-1}(\boldsymbol{\eta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_3^{-1}(\boldsymbol{\eta}) \end{bmatrix} \tag{2.11}$$

with

$$\begin{aligned} \boldsymbol{\Delta}_1^{-1}(\boldsymbol{\eta}) &= \frac{1}{\delta_2(0) - \delta_1(0)} \begin{bmatrix} \frac{\delta_2(0)}{\delta_1(0)} & -\frac{1}{A} \\ -\frac{1}{A} & \frac{1}{A^2} \end{bmatrix}, \\ \boldsymbol{\Delta}_2^{-1}(\boldsymbol{\eta}) &= \frac{1}{A^2 b^2 [\delta_3(2)\delta_3(0) - \delta_3(1)^2]} \begin{bmatrix} \delta_3(0) & -\delta_3(1) \\ -\delta_3(1) & \delta_3(2) \end{bmatrix}, \\ \boldsymbol{\Delta}_3^{-1}(\boldsymbol{\eta}) &= \frac{1}{A^2 [\delta_4(2)\delta_4(0) - \delta_4(1)^2]} \begin{bmatrix} \delta_4(0) & -\delta_4(1) \\ -\delta_4(1) & \delta_4(2) \end{bmatrix}. \end{aligned}$$

From (2.11), it implies that the pairs of parameters (A, b), (α, c) and (θ, ϕ) are asymptotically independent of each other, whereas the parameters in each pair are asymptotically dependent.

Remark 1. The rate of convergence of each of A, b, c and ϕ is $O_p(N^{-1/2})$, whereas for the carrier angular frequency θ and the modulating frequency α, the rate of convergence is $O_p(N^{-3/2})$.

3. Theoretical Properties of LSEs for General q

In this section, we provide the asymptotic results for the LSEs for model (1.1). Write $\boldsymbol{\psi}_k = (A_k, b_k, c_k, \theta_k, \phi_k)$, $k = 1, \ldots, q$, and $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_q, \alpha)$ as the parameter vector. Then the LSE of $\boldsymbol{\psi}$ is obtained by minimizing the residual sum of squares defined similarly as at (2.2). Let $\hat{\boldsymbol{\psi}}$ and $\boldsymbol{\psi}^0$ denote the least squares estimator and the true value of $\boldsymbol{\psi}$. The consistency of $\hat{\boldsymbol{\psi}}$ follows similarly as the consistency of $\hat{\boldsymbol{\eta}}$. We state the asymptotic distribution of $\hat{\boldsymbol{\psi}}$; the proof involves routine calculations and the use of multiple Taylor series and a central limit theorem similar to Section 2.

For the asymptotic distribution of $\hat{\boldsymbol{\psi}}$, following the notation used in previous section, we write $\boldsymbol{\xi}_j = (b_j, \alpha, c_j, \theta_j, \phi_j)$, $j = 1, \ldots, q$; $\delta_k(\boldsymbol{\xi}_j, p) = \delta_k^j(p)$, $k = 1(1)4$, $j = 1(1)q$, $p = 0, 1, \ldots$ Let $\mathbf{D}_q$ be the diagonal matrix of order $(5q+1)$ given as

$$\mathbf{D}_q = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & N^{-3/2} \end{bmatrix},$$

where $\mathbf{D}_1 = \mathrm{diag}\{N^{-1/2}, N^{-1/2}, N^{-1/2}, N^{-3/2}, N^{-1/2}\}$. Then as $N \to \infty$,

$$(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}^0)\mathbf{D}_q^{-1} \xrightarrow{d} \mathcal{N}_{5q+1}\Big(\mathbf{0}, \sigma^2 \mathbf{G}_q^{-1}(\boldsymbol{\psi}^0)\Big), \tag{2.12}$$

$$\mathbf{G}_q(\boldsymbol{\psi}) = \begin{bmatrix} e^{2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{w}(\boldsymbol{\psi}_1) \\ \mathbf{0} & e^{2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2) & \cdots & \mathbf{0} & \mathbf{w}(\boldsymbol{\psi}_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & e^{2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q) & \mathbf{w}(\boldsymbol{\psi}_q) \\ \mathbf{w}'(\boldsymbol{\psi}_1) & \mathbf{w}'(\boldsymbol{\psi}_2) & \cdots & \mathbf{w}'(\boldsymbol{\psi}_q) & f^* \end{bmatrix}.$$

Here $f^* = \sum_{j=1}^{q} e^{2b_j} A_j^2 b_j^2 \delta_3^j(2)$ and $\mathbf{w}'(\boldsymbol{\psi}_j) = \Big(0\; 0\; e^{2b_j} A_j^2 b_j^2 \delta_3^j(1)\; 0\; 0\Big)$. The sub-matrix $\boldsymbol{\Gamma}(\boldsymbol{\psi}_j)$ is obtained by deleting third row and third column of $\boldsymbol{\Delta}(\boldsymbol{\eta})$ and replacing (A, b, c, θ, ϕ) by $(A_j, b_j, c_j, \theta_j, \phi_j)$. Thus,

$$\boldsymbol{\Gamma}(\boldsymbol{\psi}_j) = \begin{bmatrix} \delta_1^j(0) & A_j\delta_1^j(0) & 0 & 0 & 0 \\ A_j\delta_1^j(0) & A_j^2\delta_2^j(0) & 0 & 0 & 0 \\ 0 & 0 & A_j^2 b_j^2 \delta_3^j(0) & 0 & 0 \\ 0 & 0 & 0 & A_j^2\delta_4^j(2) & A_j^2\delta_4^j(1) \\ 0 & 0 & 0 & A_j^2\delta_4^j(1) & A_j^2\delta_4^j(0) \end{bmatrix}.$$

The inverse matrix $\mathbf{G}_q^{-1}(\boldsymbol{\psi}^0)$ is

$$\begin{pmatrix} e^{-2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1)^{-1} + \mathbf{F}_{11} & \mathbf{F}_{12} & \cdots & \mathbf{F}_{1q} & e^{-2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1)\mathbf{w}(\boldsymbol{\psi}_1) \\ \mathbf{F}_{21} & e^{-2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2)^{-1} + \mathbf{F}_{22} & \cdots & \mathbf{F}_{2q} & e^{-2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2)\mathbf{w}(\boldsymbol{\psi}_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{F}_{q1} & \mathbf{F}_{q2} & \cdots & e^{-2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q)^{-1} + \mathbf{F}_{qq} & e^{-2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q)\mathbf{w}(\boldsymbol{\psi}_q) \\ e^{-2b_1}\boldsymbol{\Gamma}(\boldsymbol{\psi}_1)\mathbf{w}(\boldsymbol{\psi}_1) & e^{-2b_2}\boldsymbol{\Gamma}(\boldsymbol{\psi}_2)\mathbf{w}(\boldsymbol{\psi}_2) & \cdots & e^{-2b_q}\boldsymbol{\Gamma}(\boldsymbol{\psi}_q)\mathbf{w}(\boldsymbol{\psi}_q) & \frac{1}{d^*} \end{pmatrix},$$

where $d^* = \sum_{j=1}^{q} e^{2b_j} A_j^2 b_j^2 \left[\delta_3^j(2) - \delta_3^j(1)^2/\delta_3^j(0)\right]$ and $\mathbf{F}_{jk}$, $j,k = 1,\ldots,q$, is a 5×5 symmetric matrix whose elements are zero except for the (3,3) element which is $(1/d^*)[\delta_3^j(1)\delta_3^k(1)/\delta_3^j(0)\delta_k^k(0)]$.

4. Numerical Experiments

In this section, we present some numerical results of simulations. We consider the model (1.1) with $q = 4$. Data were generated using the values

$$\begin{gathered} A_1 = 5.70166706 \times 10^{-5},\ A_2 = 3.3049426 \times 10^{-25}, \\ A_3 = 1.002 \times 10^{-3},\qquad A_4 = 3.7575 \times 10^{-4}, \\ b_1 = 4.989495798,\ b_2 = 28.886554622,\ b_3 = 2.62605042,\ b_4 = 2.62605042, \\ c_1 = 0.1904,\quad c_2 = 2.05632,\quad c_3 = 5.9024,\quad c_4 = 3.2368, \\ \theta_1 = 0.07616,\ \theta_2 = 0.26656,\ \theta_3 = 0.03808,\ \theta_4 = 0.03808, \\ \phi_1 = 1.166198163,\ \phi_2 = 18.007071552,\ \phi_3 = 10.928948246,\ \phi_4 = 6.378392654, \\ \alpha = 0.03808. \end{gathered} \tag{2.13}$$

These values were obtained from Mukhopadhyay and Sircar (1996). We consider the case when carrier frequencies are harmonics of the modulation angular frequency i.e., the θ_i's are integer multiples of α. An ECG signal is periodic, θ_i has to be some integer multiple of α. We used the sample size $N = 100$. The error random variables were independent and identically distributed $\mathcal{N}(0, \sigma^2)$. We report results for $\sigma^2 = 0.00001$ and 0.0001. The necessary minimization was carried out by using the downhill simplex method and, for that purpose, routines given in Press, Teukolsky, Vetterling and Flannery (1993) were used. The true parameter values were taken as the initial estimates. Though $q = 4$, the parameter set contains 21 parameters, and the optimization took place in a high dimensional space, the final results are quite satisfactory. We replicated the procedure of data generation and estimation of parameters 1,000 times, then calculated the average estimate (AVEST) and the mean squared error (MSE) of each parameter. We summarize results in Figures 1 and 2. In Figure 1, we present the true values (using circles), and the average estimates (using crosses and triangles when $\sigma^2 = 0.00001$ and 0.0001, respectively). There are 25 subplots in Figure 1 and the jth row corresponds to the parameters of the jth component, $j = 1,\ldots,4$. In Section 3, we developed the asymptotic distribution of LSEs when $q > 1$, and that can be used, in particular, for interval estimation. In particular, we employ the percentile bootstrap (boot-p) method. In Nandi, Iyer and Kundu (2002) and Kundu and Nandi (2008), similar bootstrapping was used for interval estimation in the case of a sinusoidal frequency model and real chirp signal model, respectively. In each replication of our experiment,

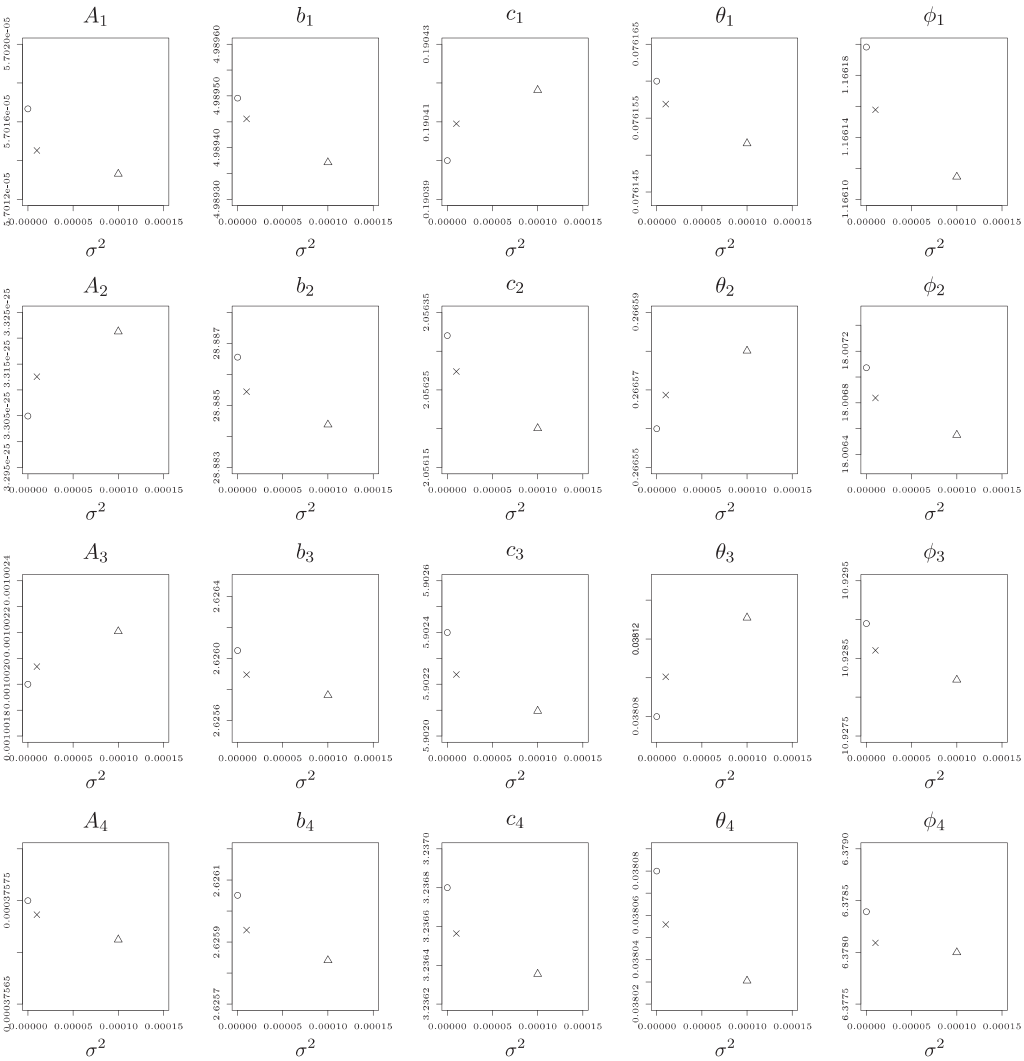

Figure 1. True values (circle) and average estimates when $\sigma^2 = 0.00001$ (cross) and $\sigma^2 = 0.0001$ (triangle), the i^{th} row corresponding to the i^{th} component. The results for α are given in Figure 3.

we generated 1,000 bootstrap resamples using the estimated parameters, and then the bootstrap confidence intervals using the bootstrap quantiles at the 95% nominal level. Thus, from the replicated experiment, we have 1,000 intervals for each parameter. Then we estimated the 95% boot-p coverage probability by calculating the proportion covering the true value of the parameter. All were close to the nominal value, except for c_1 and A_3, and we do not report them here. We also obtained the average lengths of the boot-p confidence intervals; these are

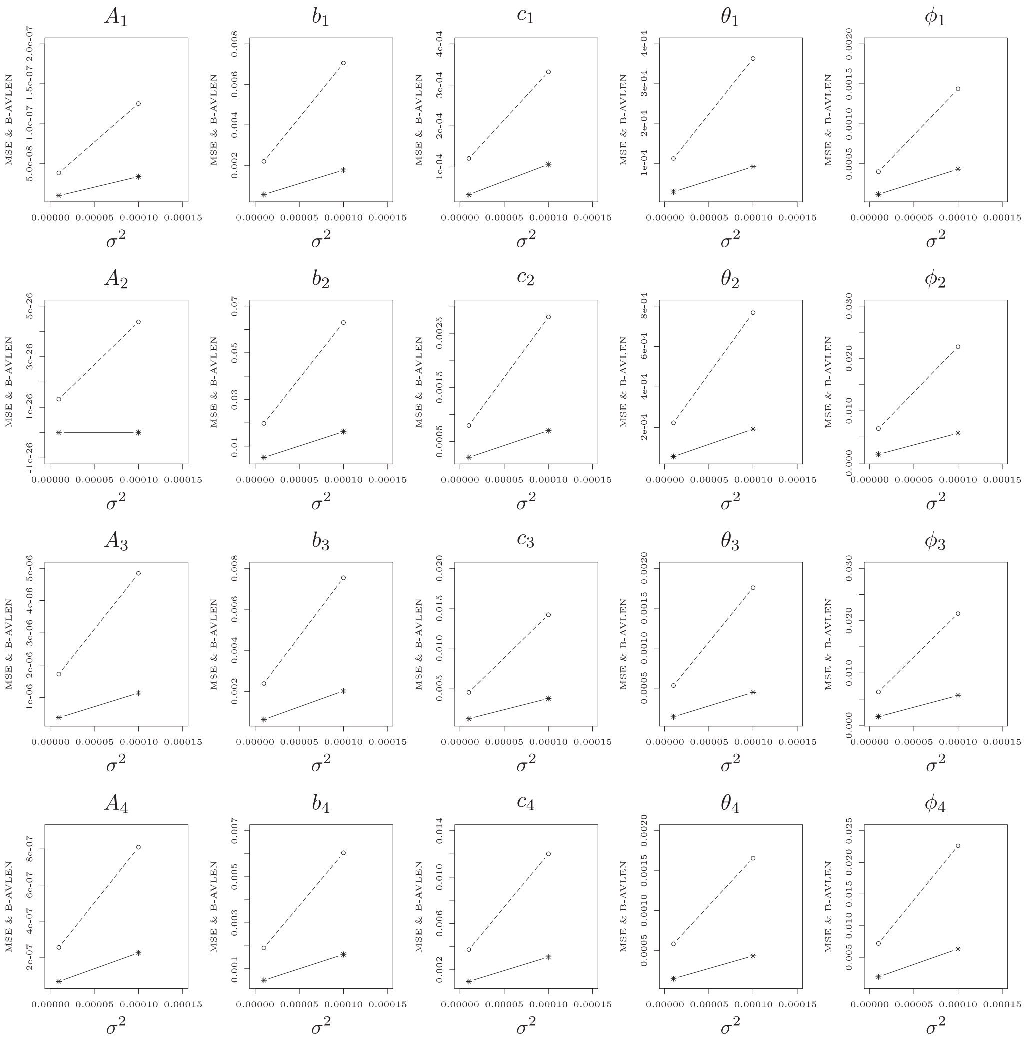

Figure 2. Root mean squared errors of LSEs of parameters when $\sigma^2 = 0.00001$ and $\sigma^2 = 0.0001$ (joined with a continuous line) and average lengths of boot-p confidence intervals (long dashed line). The results corresponding to the first component in (a)-(e), the second in (f)-(j), the third in (k)-(o), and the fourth in (p)-(u) are provided.

reported in Figure 2, along with the root mean squared errors (RMSE). In each sub-figure, for $\sigma^2 = 0.00001$ and 0.0001, we present the RMSEs (circles connected by long dashed lines) and average length of the boot-p confidence intervals ($*$ connected by continuous line). The results for the modulation frequency α are provided in Figure 3. We observe that the average estimates were quite good, which is reflected in Figure 1. The biases were quite small as the estimates (cross

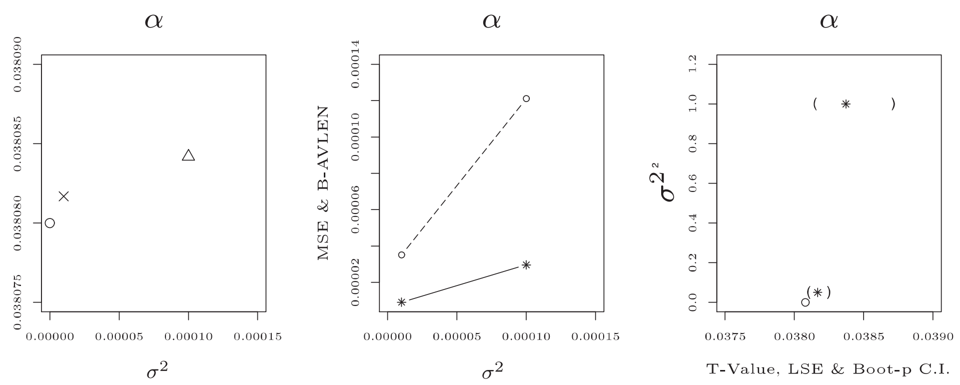

Figure 3. True value and average estimates of α (left), root mean squared errors and average length of boot-p confidence interval for α (middle), and true values, LSEs and 95% confidence intervals.

and triangle) were quite close to the true values (circle). As the error variance increased, the biases increased. The RMSEs and average lengths of the boot-p confidence intervals were reasonably small. Their dependence on the magnitude of the constant amplitude A^0 is quite clear. The asymptotic variances of all parameters, except A, were reciprocally proportional to ${A^0}^2$, which is also visible in RMSEs to some extent. As the error variance increased the RMSEs and average lengths increased. The order of the asymptotic variance is reflected in the RMSE and the length of the interval in each case.

Apart from the replicated experiment, we considered the same model in data analysis format. For that, we generated the data for sample size $N = 500$ and error variance $\sigma^2 = 0.05$. The generated data are plotted when no noise is present, in Figure 5. The corrupted version of the same data set with $\sigma^2 = 0.05$ are presented in Figure 6. We estimated the parameters by minimizing the residual sum of squares and plugging them in the model, to estimate the signal. It is plotted in Figure 7. If the level of noise is increased to $\sigma^2 = 1.0$, the original signal (Figure 5) is totally distorted. We wanted to see whether in this case it is possible to extract the signal. The signal with completely destroyed form is plotted in Figure 8. We obtained the LSEs, and the estimated signal is plotted in Figure 9. In both cases, the LSEs were able to estimate the signal satisfactorily; the estimated graphs (Figs. 7 and 9) match well with the no-noise signal. In simulated experiments, we used the percentile bootstrap method for interval estimation and we used the same method here. Using 1,000 bootstrap resamples, we estimated the 95% confidence intervals for each parameter when $\sigma^2 = 0.05$ and $\sigma^2 = 1.0$. They, as first brackets, are reported in Figure 4, along with their point estimates ($*$) and true values (circle). We see that bias is negligible in most cases when $\sigma^2 = 0.05$. The boot-p confidence intervals in each case included the

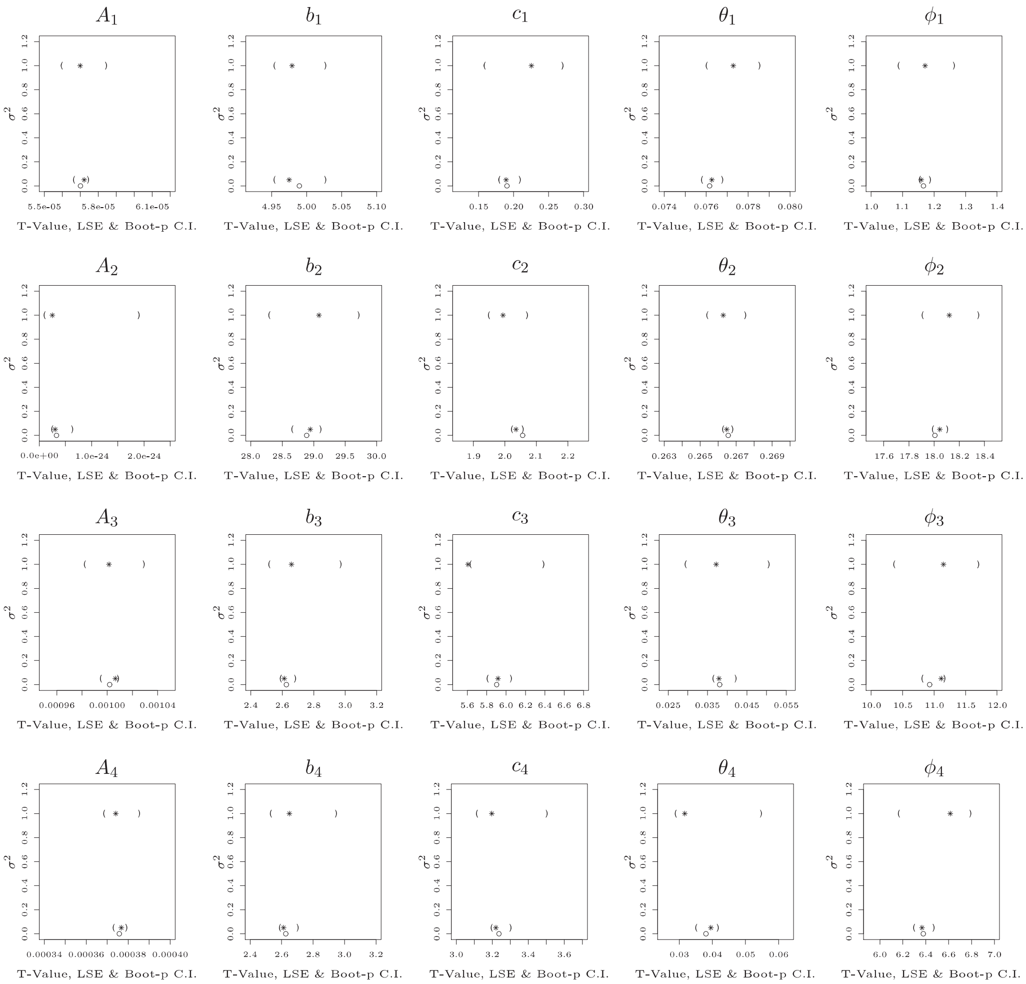

Figure 4. True values (circle), LSEs at $\sigma^2 = 0.05$ and 1.0 ($*$), and 95% percentile bootstrap confidence intervals (brackets) when the sample size $N = 500$.

true parameter value and the order of asymptotic variance is reflected in the length of the interval. We would like to mention that for larger noise levels, biases are large for some parameters, but the overall fit is quite good; the reason may lie in the presence of large absolute values of the amplitude function.

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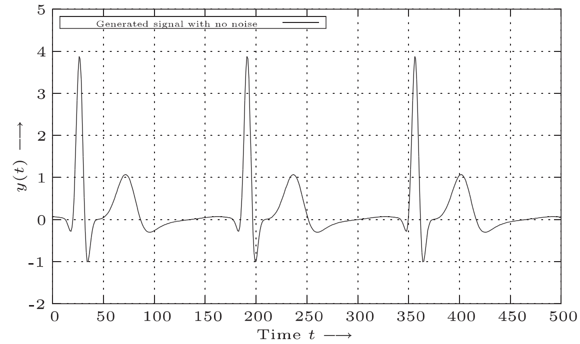

Figure 5. The signal using model (2.13) subjected to no noise.

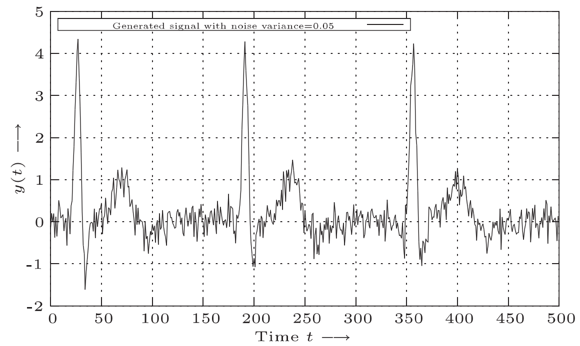

Figure 6. The same signal as in Figure 5 corrupted by noise (variance=0.05).

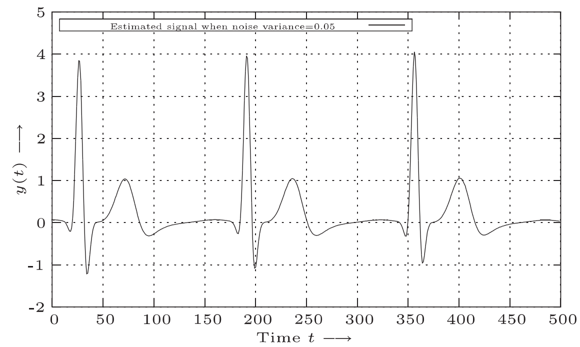

Figure 7. Estimated signal from the signal given in Figure 6.

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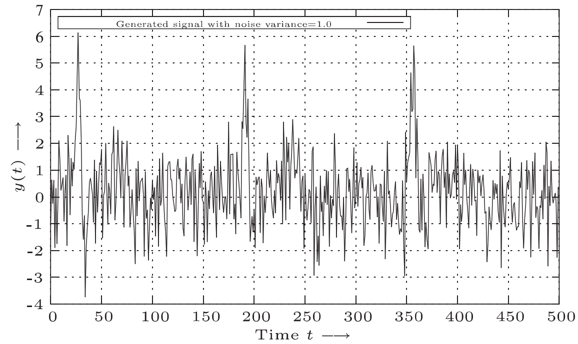

Figure 8. The same signal as in Figure 5 corrupted by noise (Error variance =1.0).

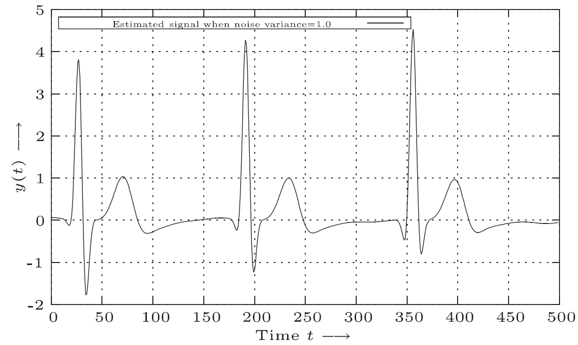

Figure 9. Estimated signal from the signal given in Figure 8.

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