Statistica Sinica, **20**(2010), Supplement, 000∼000

ESTIMATING THE PARAMETERS OF BURST-TYPE SIGNALS

Swagata Nandi[1] and Debasis Kundu[2]

Indian Statistical Institute and Indian Institute of Technology, Kanpur

Supplementary Material

This note contains the proofs of the results stated in Section 2 and definition of some limits which has been used in obtaining the asymptotic distribution of the least squares estimators. Lemmas 1 and 2 are first stated and proved. They are required to prove theorem 2.1. Then Theorem 2.1 is proved. Several limits are defined afterwards.

S1. Proof of Consistency

The technique used to prove Theorem 2.1, is that of Wu (1981). Lemma 2 gives a sufficient condition for strong consistency of the LSEs and Lemma 1 is required to verify the condition given in Lemma 2 under the condition that the error random variables are i.i.d. The methodology adopted in the following might be applicable to the case of undamped periodic signal models.

Lemma S1.1. Let $X(1), X(2), \ldots$ be i.i.d. random variables with mean zero and finite second moment, and let b be a real number such that $e^{|b|} \leq K$. Let $\Pi = (0,\pi)\times(0,\pi) \in \mathcal{R}^2$. Then

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N}\sum_{t=1}^{N} X(t)\exp\{b\cos(\alpha t)\}\cos(\theta t) \stackrel{a.s.}{\rightarrow} 0, \quad \text{as} \quad N\rightarrow\infty.$$

Proof of Lemma S1.1. If $Z(t) = X(t)$ when $|X(t)| \leq t^{\frac{1}{2}}$ and is 0 otherwise, then

$$\begin{aligned}
\sum_{t=1}^{\infty} P[Z(t)\neq X(t)] &= \sum_{t=1}^{\infty} P[|X(t)| > t^{\frac{1}{2}}] \\
&= \sum_{t=1}^{\infty}\sum_{2^{t-1}\leq n\leq 2^t} P[|X(1)| > n^{\frac{1}{2}}] \\
&\leq \sum_{t=1}^{\infty} 2^t P[|X(1)| > 2^{\frac{t-1}{2}}] \\
&\leq \sum_{t=1}^{\infty} 2^t \sum_{j=t}^{\infty} P[2^{\frac{j-1}{2}} \leq |X(1)| < 2^{\frac{j}{2}}] \\
&\leq \sum_{j=1}^{\infty} P[2^{\frac{j-1}{2}} \leq |X(1)| < 2^{\frac{j}{2}}] \sum_{t=1}^{j} 2^t
\end{aligned}$$

$$\leq \quad c\sum_{j=1}^{\infty} 2^{j-1} P[2^{\frac{j-1}{2}} \leq |X(1)| < 2^{\frac{j}{2}}] \quad \leq \quad cE|X(1)|^2 < \infty.$$

So, $P[Z(t) \neq X(t) \text{ i.o.}] = 0$ and $Z(t)$ and $X(t)$ are equivalent random variables. Thus

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N}\sum_{t=1}^{N} X(t)\exp\{b\cos(\alpha t)\}\cos(\theta t) \stackrel{a.s.}{\to} 0$$
$$\Leftrightarrow \quad \sup_{(\alpha,\theta)\in\Pi} \frac{1}{N}\sum_{t=1}^{N} Z(t)\exp\{b\cos(\alpha t)\}\cos(\theta t) \stackrel{a.s.}{\to} 0.$$

Let $U_t = Z(t) - E(Z(t))$. Then

$$\sup_{(\alpha,\theta)\in\Pi} \left| \frac{1}{N}\sum_{t=1}^{N} Z(t)\exp\{b\cos(\alpha t)\}\cos(\theta t)\right| \leq e^{|b|}\frac{1}{N}\sum_{t=1}^{N}|Z(t)| \to 0.$$

Thus, it is enough to show that

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N}\sum_{t=1}^{N} U_t\exp\{b\cos(\alpha t)\}\cos(\theta t) \stackrel{a.s.}{\to} 0.$$

For any fixed $\epsilon > 0$, assume that $0 \leq h \leq \frac{1}{2N^{1/2}K}$. Then $|hU_t\cos(\theta t)e^{b\cos(\alpha t)}| \leq \frac{1}{2}$. Now, using $e^{|x|} \leq 2e^x$ and $e^x \leq 1 + x + 2x^2$ for $|x| \leq \frac{1}{2}$, we have

$$\begin{aligned}
P\left[\left|\frac{1}{N}\sum_{t=1}^{N} U_t\cos(\theta t)e^b\cos(\alpha t)\right| \geq \epsilon\right] \quad &\leq \quad e^{-hN\epsilon}\prod_{t=1}^{N} E\left(\exp\{|hU_t\cos(\theta t)e^{b\cos(\alpha t)}|\}\right) \\
&\leq \quad 2e^{-hN\epsilon}\prod_{t=1}^{N} E\left(\exp\{hU_t\cos(\theta t)e^{b\cos(\alpha t)}\}\right) \\
&\leq \quad 2e^{-hN\epsilon}\prod_{t=1}^{N}(1 + 2h^2\sigma^2K^2) \\
&\leq \quad 2e^{-hN\epsilon + 2Nh^2\sigma^2K^2}.
\end{aligned}$$

If $h = \frac{1}{2N^{1/2}K}$ in the above inequality,

$$P\left[\left|\frac{1}{N}\sum_{t=1}^{N} U_t\cos(\theta t)e^b\cos(\alpha t)\right| \geq \epsilon\right] \leq 2e^{-\frac{1}{2}N^{\frac{1}{2}}K^{-1}\epsilon + \frac{1}{2}\sigma^2} \leq ce^{-\frac{1}{2}N^{\frac{1}{2}}K^{-1}\epsilon}.$$

Let $L = N^2$. Choose $(\alpha_1, \theta_1), \ldots, (\alpha_L, \theta_L)$ such that for each $(\alpha, \theta) \in \Pi$, we have a (α_j, θ_j) satisfying $|\alpha_j - \alpha| \leq \frac{\pi}{N^2}$ and $|\theta_j - \theta| \leq \frac{\pi}{N^2}$. From

$$\left|\cos(\theta t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha_j t)}\right|$$

$$\begin{aligned} &= \left|\cos(\theta t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha t)} + \cos(\theta_j t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha_j t)}\right| \\ &\leq |e^{b\cos(\alpha t)}|\,|\cos(\theta t) - \cos(\theta_j t)| + |\cos(\theta_j t)|\left|e^{b\cos(\alpha t)} - e^{b\cos(\alpha_j t)}\right| \\ &\leq Kt|\theta_j - \theta| + Kt|b||\alpha_j - \alpha|, \end{aligned}$$

$$\begin{aligned} \left|\frac{1}{N}\sum_{t=1}^{N} U_t \left(\cos(\theta t)e^{b\cos(\alpha t)} - \cos(\theta_j t)e^{b\cos(\alpha_j t)}\right)\right| &\leq \frac{2}{N}\sum_{t=1}^{N} t^{\frac{1}{2}} Kt(|\theta_j - \theta| + |b||\alpha_j - \alpha|) \\ &\leq \frac{2}{N}\sum_{t=1}^{N} t^{\frac{1}{2}} Kt\frac{\pi}{N^2}(1 + |b|) \\ &\leq 2K(1 + |b|)\frac{\pi}{\sqrt{N}} \;\to\; 0, \text{ as } N \to \infty. \end{aligned}$$

Therefore, for large N,

$$\begin{aligned} P\left[\sup_{\alpha,\theta}\left|\frac{1}{N}\sum_{t=1}^{N} U_t \cos(\theta t)e^{b\cos(\alpha t)}\right| \geq 2\epsilon\right] &\leq P\left[\max_{j\leq N^2}\left|\frac{1}{N}\sum_{t=1}^{N} U_t\cos(\theta_j t)e^{b\cos(\alpha_j t)}\right| \geq \epsilon\right] \\ &\leq cN^2 e^{-\frac{1}{2}N^{\frac{1}{2}}K^{-1}\epsilon}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} n^2 e^{-\frac{1}{2}n^{\frac{1}{2}}K^{-1}\epsilon} < \infty$, we have

$$\sup_{(\alpha,\theta)\in\Pi} \frac{1}{N}\sum_{t=1}^{N} X(t)\exp\{b\cos(\alpha t)\}\cos(\theta t) \overset{a.s.}{\to} 0,$$

as $N \to \infty$, using the Borel Cantelli Lemma. ∎

Lemma S1.2. Let $S_{\epsilon,M} = \left\{\boldsymbol{\eta} : |\boldsymbol{\eta} - \boldsymbol{\eta}^0| > 6\epsilon, |A| \leq M\right\}$. If for any $\epsilon > 0$ and for some $M < \infty$, $\liminf_{N\to\infty} \inf_{\boldsymbol{\eta}\in S_{\epsilon,M}} \frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right] > 0$ *a.s.* then $\hat{\boldsymbol{\eta}}$ is a strongly consistent estimator of $\boldsymbol{\eta}^0$.

Proof of Lemma S1.2. It is simple and can be proved by contradiction along the same lines as Wu (1981), so it is not provided here. ∎

Proof of Theorem 2.1: In this proof, we denote $\hat{\boldsymbol{\eta}}$ by $\hat{\boldsymbol{\eta}}_N = (A_N, b_N, \alpha_N, c_N, \theta_N, \phi_N)$ to emphasize the dependence on N. Assume that $\hat{\boldsymbol{\eta}}_N$ is not a consistent estimator for $\boldsymbol{\eta}^0$ and consider two cases.

CASE I: For all sub sequences $\{N_k\}$ of $\{N\}$, $|\hat{A}_{N_k}| \to \infty$. This implies $\frac{1}{N_k}\left[Q(\hat{\boldsymbol{\eta}}_{N_k}) - Q(\boldsymbol{\eta}^0)\right]$ $\to \infty$. But as $\hat{\boldsymbol{\eta}}_{N_k}$ is the LSE of $\boldsymbol{\eta}^0$ with sample size N_k, we have $Q(\hat{\boldsymbol{\eta}}_{N_k}) - Q(\boldsymbol{\eta}^0) < 0$, which leads to a contradiction. So $\hat{\boldsymbol{\eta}}_N$ is a strongly consistent estimator of $\boldsymbol{\eta}^0$.

CASE II: For at least one sub sequence $\{N_k\}$ of $\{N\}$, $\hat{\boldsymbol{\eta}}_{N_k} \in S_{\epsilon,M}$ for some $\epsilon > 0$ and some $0 < M < \infty$. Now we write $\frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right] = f(\boldsymbol{\eta}) + g(\boldsymbol{\eta})$, where

$$\begin{aligned} f(\boldsymbol{\eta}) &= \frac{1}{N}\sum_{t=1}^{N}\Big[A^0\exp\{b^0(1-\cos(\alpha^0 t + c^0))\}\cos(\theta^0 t+\phi^0) \\ &\quad -A\exp\{b(1-\cos(\alpha t + c))\}\cos(\theta t+\phi)\Big]^2, \\ g(\boldsymbol{\eta}) &= \frac{2}{N}\sum_{t=1}^{N}e(t)\Big[A^0\exp\{b^0(1-\cos(\alpha^0 t + c^0))\}\cos(\theta^0 t+\phi^0) \\ &\quad -A\exp\{b(1-\cos(\alpha t + c))\}\cos(\theta t+\phi)\Big]. \end{aligned}$$

Using Lemma 1, we have $\lim_{N\to\infty}\sup_{\eta\in S_{\epsilon,M}} g(\boldsymbol{\eta}) = 0$, a.s. Define sets $S^i_{\epsilon,M}$, $i = 1,\ldots,6$, as $S^i_{\epsilon,M} = \left\{\boldsymbol{\eta} : |\eta_i - \eta_i^0| > \epsilon, |A| \le M\right\}$, where η_i, $i = 1,\ldots,6$ stands for the elements of $\boldsymbol{\eta}$, that is, A, b, α, c, θ and ϕ. Note that $S_{\epsilon,M} \subset \cup_{i=1}^6 S^i_{\epsilon,M} = S$ (say). Therefore,

$$\liminf_{N\to\infty}\inf_{S_{\epsilon,M}}\frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right] \ge \liminf_{N\to\infty}\inf_{S}\frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right].$$

Our aim is to show that $\liminf_{N\to\infty}\inf_{S^i_{\epsilon,M}}\frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right] = \liminf_{N\to\infty}\inf_{S^i_{\epsilon,M}} f(\boldsymbol{\eta}) > 0$, a.s. for $i = 1,\ldots,6$ which would imply $\liminf_{N\to\infty}\inf_{S_{\epsilon,M}}\frac{1}{N}\left[Q(\boldsymbol{\eta}) - Q(\boldsymbol{\eta}^0)\right] > 0$ a.s. So, for $i = 1$,

$$\begin{aligned} &\liminf_{N\to\infty}\inf_{S^1_{\epsilon,M}} f(\boldsymbol{\eta}) \\ =\ &\liminf_{N\to\infty}\inf_{|A-A^0|>\epsilon}\frac{1}{N}\sum_{t=1}^{N}\Big[A^0\exp\{b^0(1-\cos(\alpha^0 t + c^0))\}\cos(\theta^0 t+\phi^0) \\ &-A\exp\{b(1-\cos(\alpha t + c))\}\cos(\theta t+\phi)\Big]^2 \\ =\ &\lim_{N\to\infty}\inf_{|A-A^0|>\epsilon}\frac{1}{N}\sum_{t=1}^{N}(A-A^0)^2\exp\{2b^0(1-\cos(\alpha^0 t + c^0))\}\cos^2(\theta^0 t+\phi^0) \\ \ge\ & e^{2b^0}\epsilon^2\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\exp\{-2b^0\cos(\alpha^0 t + c^0))\}\cos^2(\theta^0 t+\phi^0) \\ \ge\ & e^{2b^0}e^{-|2b^0|}\epsilon^2\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\cos^2(\theta^0 t+\phi^0) \quad = \quad \frac{c_{b^0}\epsilon^2}{2} \quad > \quad 0 \quad \text{a.s.} \end{aligned}$$

where $c_b = 1$, if $b > 0$ and $c_b = e^{-|4b^0|}$. Using a similar technique, the inequality holds for other i as well and the theorem is proved.

S2. Limits Used in Asymptotic Distribution

For $p = 0, 1, 2, \ldots$, the following limits have been used to obtain the asymptotic distribution of the LSE $\hat{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}^0$:

$$\lim_{N\to\infty} \frac{1}{N^p+1} \sum_{t=1}^{N} t^p e^{-2b\cos(\alpha t+c)} \cos^2(\theta t+\phi) = \delta_1(\boldsymbol{\xi}, p);$$

$$\lim_{N\to\infty} \frac{1}{N^p+1} \sum_{t=1}^{N} t^p e^{-2b\cos(\alpha t+c)} (1-\cos(\alpha t+c))^2 \cos^2(\theta t+\phi) = \delta_2(\boldsymbol{\xi}, p);$$

$$\lim_{N\to\infty} \frac{1}{N^p+1} \sum_{t=1}^{N} t^p e^{-2b\cos(\alpha t+c)} \sin^2(\alpha t+c)\cos^2(\theta t+\phi) = \delta_3(\boldsymbol{\xi}, p);$$

$$\lim_{N\to\infty} \frac{1}{N^p+1} \sum_{t=1}^{N} t^p e^{-2b\cos(\alpha t+c)} \sin^2(\theta t+\phi) = \delta_4(\boldsymbol{\xi}, p);$$

$$\lim_{N\to\infty} \frac{1}{N^p+1} \sum_{t=1}^{N} t^p e^{-2b\cos(\alpha t+c)} (1-\cos(\alpha t+c)) \cos^2(\theta t+\phi) = \delta_5(\boldsymbol{\xi}, p);$$

$$\lim_{N\to\infty} \frac{1}{N^p+1} \sum_{t=1}^{N} t^p e^{-2b\cos(\alpha t+c)} \sin(\alpha t+c)\cos^2(\theta t+\phi) = \delta_6(\boldsymbol{\xi}, p);$$

$$\lim_{N\to\infty} \frac{1}{N^p+1} \sum_{t=1}^{N} t^p e^{-2b\cos(\alpha t+c)} \sin(\theta t+\phi)\cos(\theta t+\phi) = \delta_7(\boldsymbol{\xi}, p);$$

$$\lim_{N\to\infty} \frac{1}{N^p+1} \sum_{t=1}^{N} t^p e^{-2b\cos(\alpha t+c)} \sin(\alpha t+c)\}(1-\cos(\alpha t+c)) \cos^2(\theta t+\phi) = \delta_8(\boldsymbol{\xi}, p);$$

$$\lim_{N\to\infty} \frac{1}{N^p+1} \sum_{t=1}^{N} t^p e^{-2b\cos(\alpha t+c)} (1-\cos(\alpha t+c)) \sin(\theta t+\phi)\cos(\theta t+\phi) = \delta_9(\boldsymbol{\xi}, p);$$

$$\lim_{N\to\infty} \frac{1}{N^p+1} \sum_{t=1}^{N} t^p e^{-2b\cos(\alpha t+c)} \sin(\alpha t+c)\sin(\theta t+\phi)\cos(\theta t+\phi) = \delta_{10}(\boldsymbol{\xi}, p).$$

Note that

$$\exp\{-2|b|\} \leq \exp\{-2b\cos(\alpha t+c)\} \leq \exp\{2|b|\}. \tag{S2.1}$$

Using it in the first sequence listed above, with $p = 0$, we have

$$e^{-2|b|}\frac{1}{N}\sum_{t=1}^{N}\cos^2(\theta t+\phi) \leq \frac{1}{N}\sum_{t=1}^{N} e^{-2b\cos(\alpha t+c)}\cos^2(\theta t+\phi) \leq e^{2|b|}\frac{1}{N}\sum_{t=1}^{N}\cos^2(\theta t+\phi).$$

Now taking limit as $N \to \infty$, we get $\dfrac{e^{-2|b|}}{2} \leq \delta_1(\psi, 0) \leq \dfrac{e^{2|b|}}{2}$. For notational simplicity, $\delta_k(\psi, p) = \delta_k(p)$, $k = 1, \ldots, 10$, has been used in obtaining the asymptotic distribution of the LSEs.

Using the inequality given in (S2.1), in $\delta_6(\boldsymbol{\xi}, p)$, we have

$$\begin{aligned}\delta_6(\boldsymbol{\xi}, p) &\leq \left\{\geq \quad \right\} e^{|2b|} \left\{e^{-|2b|}\right\} \lim_{N\to\infty} \frac{1}{N^{p+1}} \sum_{t=1}^{N} t^p \sin(\alpha t + c)\cos^2(\theta t + \phi) \\ &\to \quad e^{|2b|}\left\{e^{-|2b|}\right\} \times 0.\end{aligned}$$

This implies that $0 \leq \delta_6(\boldsymbol{\xi}, p) \leq 0 \Rightarrow \delta_6(\boldsymbol{\xi}, p) \to 0$, for all p and $\boldsymbol{\xi}$. In a similar way, we find that $\delta_k(\boldsymbol{\xi}, p) \to 0$ for all p and $\boldsymbol{\xi}$ for $k = 7, \ldots, 10$ and $\delta_5(\boldsymbol{\xi}, p) = \delta_1(\boldsymbol{\xi}, p)$.