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A GENERAL CLASS FOR QUASI-INDEPENDENCE TESTS FOR LEFT-TRUNCATED RIGHT-CENSORED DATA

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Abstract: In survival studies, classical inferences for left-truncated data require quasi-independence, a property that the joint density of truncation time and failure time is factorizable into their marginal densities in the observable region. The quasi-independence hypothesis is testable; many authors have developed tests for left-truncated data with or without right-censoring. In this paper, we propose a class of test statistics for testing the quasi-independence that unifies the existing methods and generates new useful statistics such as conditional Spearman's rank correlation coefficient. Asymptotic normality of the proposed class of statistics is given. We show that a new set of tests can be powerful under certain alternatives by theoretical and empirical power comparison.

Key words and phrases: Left-truncation, quasi-independence testing, survival data.

1. Introduction

Left truncation is a common type of incompleteness in survival analysis along with right-censoring. A typical prevalent cohort study produces left-truncated data. A famous example is the study of the residents of the Channing House retirement community in Palo Alto, California. The subjects who died before becoming eligible for the retirement community cannot be recruited and thus are truncated. Left truncation also induces length-biased sampling (Asgharian, M'Lan and Wolfson (2002); Tsai (2009)). Analysis of left-truncated data with or without right censoring has been studied (Lynden-Bell (1971); Hyde (1977); Wang, Jewell and Tsai (1986)). To accommodate left truncation, a key is to redefine the risk set for truncated data to apply the Kaplan-Meier estimator and Cox proportional hazard models (Andersen et al. (1993)). While these methods assume an independence between failure time and truncation time, Tsai (1990) pointed out that we need a weaker assumption than independence, quasi-independence, which is independence between failure and truncation times only in the observed region.

Unlike independence between failure time and censoring time, quasi-independence is testable. Tsai (1990) proposed a statistic based on the Kandall's tau conditioning on a set of observations that can be interpreted as "comparable" sets of data points. Jones and Crowley (1992) presented a score-type test from the Cox proportional hazard model. Chen, Tsai and Chao (1996) suggested a test statistic for quasi-independence based on the conditional product-moment correlation among comparable data points if failure and truncation times follow truncated bivariate normal distribution. Martin and Betensky (2005) extended the conditional Kendall's tau test to more complex truncation schemes. Both Tsai (1990) and Chen, Tsai and Chao (1996) adapted usual association measures for complete bivariate data. While other measures can be adjusted for left truncation and right censoring questions remain: (1) which association measure for complete bivariate data can be tailored to survival data with left truncation?; (2) if they can, how is the given measure adjusted?

We propose a general class of test statistics that unifies the existing tests and induces some new statistics. The class reveals that any association measure with a form U-statistic comprising skew-symmetric transformations can induce a version for left truncated data, restricting the measure onto the mean among "comparable pairs". The class is broad enough to embed tests by Tsai (1990), Jones and Crowley (1992) and Chen, Tsai and Chao (1996), and also Spearman's rank correlation. We derive asymptotic properties of the proposed class of statistics. We further specify sufficient conditions to guarantee that the proposed class has a valid asymptotic null distribution under right censoring. We compare various statistics through simulation studies and theoretical efficacies under a sequence of contiguous alternatives.

The remainder of this paper is organized as follows. In Section 2, we introduce notation and review existing quasi-independence tests. In Section 3 we propose the general class of test statistics and derive asymptotic properties. We also introduce useful statistics derived from the proposed class. In Section 4, we compare asymptotic relative efficiencies of some cases of our class to Kendall's tau under a special case of contiguous excess and relative risk models. In Section 5, we compare the empirical power of the different tests within the class under various alternatives. Section 6 illustrates application of the proposed tests to the Channing House data. Concluding remarks follow in Section 7.

2. Existing Methods

In this section, we review existing methods for quasi-independence testing:

conditional Kendall's tau (Tsai (1990)), conditional product-moment correlation coefficient (Chen, Tsai and Chao (1996)), and a weighted score test for the Cox proportional hazard model (Jones and Crowley (1992)).

Let (L, X, C) be a tuple of truncation, failure, and censoring times, with $T = \min(X, C)$ and $\delta = I(X \leq C)$, where $I(A)$ is the indicator function for an event A. The quasi-independence hypothesis is

$$H_0: \ \mathrm{pr}(L \leq l, X \leq x \,|\, L \leq X) = \frac{\mathrm{pr}(L \leq l)\mathrm{pr}(X \leq x)}{\alpha}, \quad l < x$$

for some constant α. We suppose that left-truncated data $\{(L_i, X_i)\}_{i=1}^n$ are independently and identically distributed given $L < X$. In the presence of right censoring, left-truncated right-censored data $\{(L_i, X_i, C_i)\}_{i=1}^n$ are assumed randomly sampled given $L < X$, where we observe $\{(L_i, T_i, \delta_i)\}_{i=1}^n$. We need some conditions,

Assumption 1. *X and C are independent conditionally on L,*

Assumption 2. *$L < C$ with probability 1.*

2.1. Conditional Kendall's tau

Kendall's tau is a popular nonparametric measure of association between two random variables. If (X_1, Y_1) and (X_2, Y_2) are independent copies of a bivariate random vector (X, Y), Kendall's tau is

$$\rho^{\mathrm{Tau}} = E\{\mathrm{sign}(X_1 - X_2)\mathrm{sign}(Y_1 - Y_2)\},$$

where sign(t) is $\mathrm{sign}(t) = I(t > 0) - I(t < 0)$. It is known that if X and Y are independent, then $\rho^{\mathrm{Tau}} = 0$. A consistent estimator of ρ^{Tau} can be obtained by the U-statistic

$$\widehat{\rho}^{\mathrm{Tau}} = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{sign}(X_j - X_i)\mathrm{sign}(Y_j - Y_i).$$

One can construct a test statistic based on $\widehat{\rho}^{\mathrm{Tau}}$ to test $\rho^{\mathrm{Tau}} = 0$, and its limiting distribution can be obtained using the central limit theorem for U-statistic (Randles and Wolfe (1991)).

Tsai (1990) defined conditional Kendall's tau to adjust the Kendall's tau for left-truncated data. Let (L_1, X_1) and (L_2, X_2) be independent copies from the distribution of (L, X) given $L < X$. The conditional Kendall's tau is

$$\begin{aligned}\rho^{\mathrm{CTau}} &= E\big\{\mathrm{sign}(L_1 - L_2)\mathrm{sign}(X_1 - X_2)|\Omega_{12}\big\} \\ &= \frac{E\big\{\mathrm{sign}(L_1 - L_2)\mathrm{sign}(X_1 - X_2)I(\Omega_{12})\big\}}{\mathrm{pr}(\Omega_{12})}, \end{aligned} \tag{2.1}$$

where Ω_{12} is an event defined by $\Omega_{12} = \{\max(L_1, L_2) < \min(X_1, X_2)\}$. We call Ω_{12} the comparable region. In this region, there is overlap between the two durations. The conditional Kendall's tau measures concordance only in Ω_{12}. It is easy to show that $\rho^{\text{CTau}} = 0$ between L and X under quasi-independence. A consistent estimator of ρ^{CTau} (2.1) is

$$\begin{aligned}\widehat{\rho}^{\text{CTau}} &= \frac{\left[\left\{1/\binom{n}{2}\right\}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\text{sign}(L_j - L_i)\text{sign}(X_j - X_i)I(\Omega_{ij})\right]}{\left\{W_n/\binom{n}{2}\right\}} \\ &= \frac{1}{W_n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\text{sign}(L_j - L_i)\text{sign}(X_j - X_i)I(\Omega_{ij}),\end{aligned}$$

where $W_n = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} I(\Omega_{ij})$. As in $\widehat{\rho}^{\text{Tau}}$, the conditional version $\widehat{\rho}^{\text{CTau}}$ is approximated by a normal distribution, using the same central limit theorem. The asymptotic variance of $\widehat{\rho}^{\text{CTau}}$ can be obtained as the variance of an U-statistic using $O(n^3)$ of FLoating-point OPerations per Second (flops). Martin and Betensky (2005) introduced an analytic technique in which the computation reduces to $O(n^2)$ flops. Tsai (1990) derived another formula for the variance that only involves the numbers of risk sets of given data and requires at most $O(n^2)$ flops.

Conditional Kendall's tau was adapted to right-censoring by Tsai (1990), and Martin and Betensky (2005) in the multivariate case. Having (L_1, T_1, δ_1) and (L_2, T_2, δ_2) as independent copies conditioning on $L < X$, the quantity of interest is

$$\begin{aligned}\kappa^{\text{CTau}} &= E\big\{\text{sign}(L_1 - L_2)\text{sign}(T_1 - T_2)|\Lambda_{12}\big\} \\ &= \frac{E\big\{\text{sign}(L_1 - L_2)\text{sign}(T_1 - T_2)I(\Lambda_{12})\big\}}{\text{pr}(\Lambda_{12})},\end{aligned}$$

where Λ_{12} is a comparable region with left truncation and right censoring given by $\Lambda_{12} = \{\max(L_1, L_2) < \min(T_1, T_2)\} \cap [(\delta_1\delta_2 = 1) \cup \{\delta_1 \,\text{sign}(T_2 - T_1) = 1\} \cup \{\delta_2 \,\text{sign}(T_1 - T_2) = 1\}]$. Due to right censoring, the comparable region Λ_{12} is modified from Ω_{12} so that the minimum of T_1 and T_2 should be a failed one. As in Ω_{12}, there is overlap between the two durations in Λ_{12}. If L and X are quasi-independent, Martin and Betensky (2005) proved that $\kappa^{\text{CTau}} = 0$ under Assumptions 1 and 2. A consistent estimator of κ^{CTau} is

$$\widehat{\kappa}^{\text{CTau}} = \frac{1}{V_n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\text{sign}(L_j - L_i)\text{sign}(T_j - T_i)I(\Lambda_{ij}), \qquad (2.2)$$

where $V_n = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} I(\Lambda_{ij})$.

2.2. Conditional product-moment correlation

Chen, Tsai and Chao (1996) proposed a test based on conditional product-moment correlation coefficient for left-truncated data. Similarly to Tsai (1990), they first defined the population version of the correlation coefficient conditioning on Ω_{12},

$$\rho^{\text{CProd}} = E\big\{(L_1 - L_2)(X_1 - X_2)|\Omega_{12}\big\},$$

and proved $\rho^{\text{CProd}} = 0$ under quasi-independence. They also suggested a consistent estimator

$$\widehat{\rho}^{\text{CProd}} = \frac{1}{W_n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(L_j - L_i)(X_j - X_i)I(\Omega_{ij}).$$

Chen, Tsai and Chao (1996) did not extend the test to the right-censored data. One might have considered an analogue of ρ^{CProd} for right-censoring, for example, say $\kappa^{\text{CProd}} = E\big\{(L_1-L_2)(T_1-T_2)|\Lambda_{12}\big\}$. While, in general, $\kappa^{\text{CProd}} \neq 0$ under the quasi-independence, and a similar extension with conditional Kendall's tau fails, in Section 3, we find that $\kappa^{\text{CProd}} = 0$ with additional assumptions.

2.3. Weighted score tests from the Cox proportional hazard model

Jones and Crowley (1992) adopted a class of weighted score tests from the Cox proportional hazard model proposed in Jones and Crowley (1989) to quasi-independence testing. Consider a conditional hazard function with a time-varying covariate given by

$$\lambda(t\,|\,Z_i) = \lambda_0(t)\,\exp\{Z_i(t)\beta\}, \quad i = 1, \ldots, n,$$

where $\lambda(t\,|\,Z)$ is the hazard function for X given a predictable covariate function $Z(t)$ and the truncation event $L < X$. Testing $\beta = 0$ yields a score-type test statistic and Jones and Crowley (1989) proposed its weighted version,

$$T^{\text{Cox}}(q, Z) = \sum_{i=1}^{n}\delta_i\, q(T_i)\left\{Z_i(T_i) - \frac{\sum_{j=1}^{n} Y_j(T_i)Z_j(T_i)}{Y(T_i)}\right\},$$

where $q(t)$ is a predictable weight function, $Y_j(t) = I(L_j < t \leq T_j)$ is the at-risk indicator for the j-th observation at time t, and $Y(t) = \sum_{j=1}^{n} Y_j(t)$ is the number of subjects at risk for time t. The asymptotic distribution of $T^{\text{Cox}}(q, Z)$ under $\beta = 0$ was derived using counting process theory. Jones and Crowley (1992) proposed to take a function of truncation time as a covariate. Then quasi-independence is equivalent to $\beta = 0$, which can be tested via the score test, T^{Cox}. The covariates considered in that paper were $Z_i(t) = L_i$ or $Z_i(t) = R_i^*(t)$, where $R_i^*(t) = R_i(t)/Y(t)$ and $R_i(t)$ is the rank of L_i in the risk set defined at t, explicitly, $R_i(t) = 1 + \sum_{j=1}^n Y_j(t)I(L_i < L_j)$. We denote the weighted score statistics with covariate L_i or $R_i^*(t)$ by $T^{\text{Cox}}(q, L)$ and $T^{\text{Cox}}(q, R^*)$, respectively.

3. A General Class for Quasi-Independence Tests

A common characteristic from the conditional Kendall's tau and the conditional product-moment correlation is that they have a form of U-statistics among the pairs in a comparable region defined in the presence of left truncation and right censoring. Here we observe that the tests from the weighted score tests from the Cox proportional hazard model can also be expressed as a U-statistic. Considering $q(t) = Y(t)$, the size of risk set at time t as the weight function, we have

$$T^{\text{Cox}}\{Y, a(L)\} = -\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{a(L_i) - a(L_j)\}\text{sign}(T_i - T_j)I(\Lambda_{ij}) \qquad (3.1)$$

for any real-valued function $a(\cdot)$ and

$$T^{\text{Cox}}(Y, R^*) = -\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\text{sign}(L_i - L_j)\text{sign}(T_i - T_j)I(\Lambda_{ij}) + \frac{1}{2}\sum_{i=1}^{n}\delta_i. \quad (3.2)$$

The derivation of (3.1) and (3.2) is given in the Appendix. Jones and Crowley (1992) discuss the relationship in (3.2), $T^{\text{Cox}}(Y, R^*)$ is equivalent to the conditional Kendall's tau. From (3.1) and (3.2), we see that $T^{\text{Cox}}(Y, L)$ and $T^{\text{Cox}}(Y, R^*)$ share the common characteristic of $\widehat{\rho}^{\text{CTau}}$, $\widehat{\kappa}^{\text{CTau}}$, and $\widehat{\rho}^{\text{CProd}}$ as U-statistics indexed among comparable pairs adjusting for left truncation and right censoring.

These statistics are all skew-symmetric functions of L_i and L_j, satisfying $g(L_i, L_j) = -g(L_j, L_i)$. To see why the skew-symmetry of g is crucial, consider the case of left-truncation without right-censoring. For all these U-statistics, we need their estimands to be zero under quasi-independence to have a valid asymptotic null distribution. In taking the expectation, the integral region Ω_{12} can be divided into the subevents $A_1 = (L_1 < L_2 < X_1 < X_2)$, $A_2 = (L_1 < L_2 < X_2 < X_1)$, $A_3 = (L_2 < L_1 < X_1 < X_2)$, and $A_4 = (L_2 < L_1 < X_2 < X_1)$, flipping the order of L_1 and L_2, and X_1 and X_2, under the restriction of a comparable region. Then the expectation can be expressed as a sum of four subintegrals associated with the four regions. The skew-symmetry of the mapping function for truncation times renders cancellation of subintegrals from A_1 and A_3, and similarly from A_2 and A_4.

Motivated by the common structure of the existing statistics, we propose a class of U-type test statistics indexed by skew-symmetric functions. We present the proposed method for the left-truncated data in Section 3.1 and for left-truncated and right-censored data in Section 3.2. The cancellation of subintegrals plays a key role when proving the asymptotic null distribution of the classes. In Section 3.3, we discuss several new tests that are special cases of the proposed class of statistics, including a version of Spearman's rank correlation coefficient.

3.1. Tests for left-truncated data without right-censoring

We assume $\{(L_i, X_i)\}_{i=1}^n$ are observed in region $\{(l, x) : l < x\}$. Let g and h satisfy $g(s,t) = -g(t,s)$ and $h(s,t) = -h(t,s)$ for all real s, t. This skew-symmetry of $g(\cdot,\cdot)$ is needed to cancel out subintegrals from subregions as described earlier. We impose skew-symmetry of $h(\cdot,\cdot)$ to satisfy symmetry of U-statistics. With $\Omega_{ij} = (\max(L_i, L_j) < \min(X_i, X_j))$, we propose

$$\widehat{\rho}(g,h) = \frac{1}{W_n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} g(L_i, L_j)h(X_i, X_j)I(\Omega_{ij}), \tag{3.3}$$

where $W_n = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} I(\Omega_{ij})$, to estimate

$$\begin{aligned}\rho(g,h) &= E\big\{g(L_1, L_2)h(X_1, X_2)|\Omega_{12}\big\} \\ &= \frac{E\big\{g(L_1, L_2)h(X_1, X_2)I(\Omega_{12})\big\}}{\mathrm{pr}(\Omega_{12})}.\end{aligned} \tag{3.4}$$

The special cases of $\rho(g,h)$ include ρ^{CTau}, by choosing $g(s,t) = h(s,t) = \mathrm{sign}(s-t)$ and ρ^{CProd}, $g(s,t) = h(s,t) = s - t$, respectively. Other choices of g and h generate new statistics, which we discuss in Section 3.3.

Theorem 1. *Let $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ be skew-symmetric bivariate functions and let $\widehat{\rho}(g,h)$ and $\rho(g,h)$ be defined as in* (3.3) *and* (3.4)*. As $n \to \infty$, $n\widehat{\rho}(g,h)^2/\{4\phi(g,h)/\mathrm{pr}(\Omega_{12})^2\}$ is asymptotically distributed as a chi-squared distribution with 1 degree of freedom under the quasi-independence between L and X, where $\phi(g,h) = E\{g(L_1, L_2)h(X_1, X_2)\ I(\Omega_{12})g(L_1, L_3)h(X_1, X_3)I(\Omega_{13})\}$, and (L_i, X_i), $i = 1, 2, 3$ are independent copies of the distribution of (L, X) conditioning on $L < X$.*

A proof is given in the Appendix. A key step in the proof is showing that $\rho(g,h) = 0$ under the quasi-independence between L and X using skew-symmetry of function g, following by the central limit theorem for one-sample U-statistic (Randles and Wolfe (1991)). We can replace $\mathrm{pr}(\Omega_{12})$ and $\phi(g,h)$ with $\widehat{\mathrm{pr}}(\Omega_{12}) = \sum_{i<j} I(\Omega_{ij})/\binom{n}{2}$ and $\widehat{\phi}(g,h) = \sum_{i<j<k} a_{ij}b_{ik}/\binom{n}{3}$, where $a_{ij} = b_{ij} = g(L_i, L_j)h(X_i, X_j)I(\Omega_{ij})$, and the result still holds after applying Slutsky's theo-

rem. Martin and Betensky (2005) reduced the burden of computation of $\widehat{\phi}(g,h)$ with $O(n^3)$ flops to $O(n^2)$ flops with

$$\sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i} a_{ij}b_{ik} = \sum_{i=1}^{n}(a_{i\cdot}b_{i\cdot} - c_{i\cdot}), \tag{3.5}$$

where $a_{i\cdot} = \sum_{j\neq i} a_{ij}$, $b_{i\cdot} = \sum_{j\neq i} b_{ij}$, and $c_{i\cdot} = \sum_{j\neq i} a_{ij}b_{ij}$.

3.2. Tests for left-truncated right-censored data

Given $\{(L_i, T_i, \delta_i)\}_{i=1}^n$, we propose the estimate

$$\widehat{\kappa}(g,h) = \frac{1}{V_n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} g(L_i, L_j)h(T_i, T_j)I(\Lambda_{ij}),$$

where $V_n = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} I(\Omega_{ij})$ and $\Lambda_{ij} = \{\max(L_i, L_j) < \min(T_i, T_j)\} \cap [(\delta_i\delta_j = 1) \cup \{\delta_i \operatorname{sign}(T_j - T_i) = 1\} \cup \{\delta_j \operatorname{sign}(T_i - T_j) = 1\}]$. Technically speaking, $\widehat{\rho}(g,h)$ is a special case of $\widehat{\kappa}(g,h)$ that assigns $\delta_i = 1$ for all i. As with $\widehat{\rho}(g,h)$, $\widehat{\kappa}(g,h)$ consistently estimates

$$\kappa(g,h) = \frac{E\big\{g(L_1, L_2)h(T_1, T_2)I(\Lambda_{12})\big\}}{\mathrm{pr}(\Lambda_{12})}.$$

Assumption 3. *Either* $(3A)$ $h(s,t) = sign(s-t)$ *or* $(3B)$ L *and* C *are quasi-independent.*

Theorem 2. *Let* $g(\cdot,\cdot)$ *and* $h(\cdot,\cdot)$ *be skew-symmetric bivariate functions and Assumption* 3 *holds. As* $n \to \infty$, $n\widehat{\kappa}(g,h)^2/\{4\varphi(g,h)/\mathrm{pr}(\Lambda_{12})^2\}$ *is asymptotically distributed as a chi-squared distribution with* 1 *degree of freedom under the quasi-independence between* L *and* X, *where* $\varphi(g,h) = E\{g(L_1, L_2)h(T_1, T_2)I(\Lambda_{12})$ $g(L_1, L_3)h(T_1, T_3)I(\Lambda_{13})\}$, *and* (L_i, T_i, δ_i), $i = 1, 2, 3$ *are independent copies of the distribution from* (L, T, δ) *conditioning on* $L < X$.

A sketch of a proof of is given in the Appendix.

3.3. New test statistics and a unifying framework

We mentioned that the test statistics $\widehat{\rho}(g,h)$ and $\widehat{\kappa}(g,h)$ embed the existing tests described in Section 2. Other choices of g and h also generate useful tests. We highlight some examples:

Example 1. Let $g(L_i, L_j) = L_i - L_j$ and $h(T_i, T_j) = T_i - T_j$. Then $\widehat{\kappa}(L_i - L_j, T_i - T_j)$ can be viewed as an extension of the conditional production-moment correlation test accommodating right censoring.

Example 2. Consider a rank difference function, $r(X_i, X_j) = \mathrm{rank}(X_i)/n -$

Table 1. Special cases of the proposed class. The "sign", "linear", and "rank" functions denote bivariate mappings $\mathrm{sign}(s-t)$, $s-t$, and $r(s,t)$ defined in Section 3.3, respectively. Tests requiring additional Assumption 3B are marked with asterisk(*).

Data (statistic)	$g(\cdot,\cdot)$	$h(\cdot,\cdot)$	Equivalent test statistic
Left-truncated	Sign	Sign	Tsai (1990), $\widehat{\rho}^{\mathrm{CTau}}$
($\widehat{\rho}(g,h)$)	Sign	Linear	A subclass of Jones and Crowley (1992), $T^{\mathrm{Cox}}(q,Z)$ with $q(t)=Y(t)$, $Z_i(t)=L_i$
	Linear	Linear	Chen, Tsai and Chao (1996), $\widehat{\rho}^{\mathrm{CProd}}$
	Rank	Sign	(Example 3)
	Rank	Rank	(Example 2)
Left-truncated and right-censored ($\widehat{\kappa}(g,h)$)	Sign	Sign	Tsai (1990), $\widehat{\kappa}^{\mathrm{CTau}}$
	Linear	Sign	A subclass of Jones and Crowley (1992), $T^{\mathrm{Cox}}(q,Z)$ with $q(t)=Y(t)$, $Z_i(t)=L_i$
	Linear	Linear*	(Example 1)
	Rank	Sign	(Example 3)
	Rank	Rank*	(Example 2)

$\mathrm{rank}(X_j)/n$, where $\mathrm{rank}(X_i)$ is the rank of X_i among a dataset $\{X_i\}_{i=1}^n$. The function $\mathrm{rank}(\cdot)/n$ approximates the cumulative distribution function of X, F_X. Then $\widehat{\rho}\{r(L_i,L_j),r(T_i,T_j)\}$ and $\widehat{\kappa}\{r(L_i,L_j),r(T_i,T_j)\}$ correspond to Spearman's rank correlation coefficient adapted to left-truncated data and to left-truncated and right-censored data, respectively, measuring the correlation among comparable pairs. The proposed test is likely to inherit advantages of Spearman's rank correlation compared for non-truncated data such as effectively detecting nonlinear dependence between two variables.

Example 3. Consider a hybrid of the Kendall's tau and the rank correlation, $\widehat{\rho}\{r(L_i,L_j),\mathrm{sign}(X_i-X_j)\}$ and $\widehat{\kappa}\{r(L_i,L_j),\mathrm{sign}(T_i-T_j)\}$. An advantage of the hybrid statistic $\widehat{\kappa}\{r(L_i,L_j),\mathrm{sign}(T_i-T_j)\}$ is that it does not require Assumption 3B, as does $\widehat{\kappa}(r(L_i,L_j),r(T_i,T_j)\}$ in Example 2.

Table 1 summarizes existing approaches and the new tests according to the choice of g and h, where "sign", "rank" and "linear" functions in the second and third column are defined by $\mathrm{sign}(s-t)$, $\mathrm{rank}(s)/n-\mathrm{rank}(t)/n$, and $s-t$, respectively, for real s and t. The three dependence measures on bivariate data, Kendall's tau, Spearman's rank correlation, and Pearson's product-moment correlation, are systemically adjusted in our proposed test statistics.

4. Asymptotic Relative Efficiencies

The fact that we can specify g and h leads to a natural question, which choice of g and h is suitable to improve power to reject the quasi-independent

Table 2. Asymptotic efficacies for the select cases of the proposed class. Notations are defined in the main body of the paper.

$\widehat{\kappa}(g,h)$		Equivalent form	$\mu(t)$ and $\sigma^2(t)$
$g(\cdot,\cdot)$	$h(\cdot,\cdot)$	in $T^{\mathrm{Cox}}(q,Z)$	
Sign	Sign	$T^{\mathrm{Cox}}(Y,R^*)$	$\mu(t)=\beta\int_0^t \bar{y}^2(u)\sigma_{a(L)R^*}(u)\lambda_1(u)du$ $\sigma(t)^2=\int_0^t \bar{y}(u)^3\sigma_{R^*R^*}(u)\lambda_1(u)\alpha_1(u)du$
Rank	Sign	$T^{\mathrm{Cox}}(Y,F_L(L))$	$\mu(t)=\beta\int_0^t \bar{y}(u)^2\sigma_{a(L)F(L)}(u)\lambda_1(u)du$ $\sigma(t)^2=\int_0^t \bar{y}(u)^3\sigma_{F(L)F(L)}(u)\lambda_1(u)\alpha_1(u)du$
Linear	Sign	$T^{\mathrm{Cox}}(Y,L)$	$\mu(t)=\beta\int_0^t \bar{y}(u)^2\sigma_{a(L)L}(u)\lambda_1(u)du$ $\sigma(t)^2=\int_0^t \bar{y}(u)^3\sigma_{LL}(u)\lambda_1(u)\alpha_1(u)du$

hypothesis. For complete bivariate data, Randles and Wolfe (1991) reported that Spearman's rank correlation and Kendall's tau had asymptotically comparable powers. The asymptotic relative efficiency of the rank correlation to the product-moment correlation was 0.91 when the assumptions for the latter were satisfied (Daniel (1990); Siegel and Castellan (1988)). For left-truncated survival data, Jones and Crowley (1992) reported that $T^{\mathrm{Cox}}(q,L)$ with suitable choice of weight function q is more efficient than the conditional Kendall's tau test. They used their (1990) formula for asymptotic power under local contiguous alternatives.

We use the approach of Jones and Crowley (1990) and Jones and Crowley (1992) to compare the theoretical powers of our proposed test statistics. If h is the sign function, the proposed $\widehat{\rho}(g,\mathrm{sign})$ and $\widehat{\kappa}(g,\mathrm{sign})$ has equivalent form to Jones and Crowley (1992)'s test statistic, T^{Cox}, as seen in (3.1) and (3.2) for several choices of g. We restrict our interest to the case of $g(L_i,L_j)$ being one of the followings: (1) $\mathrm{sign}(L_i-L_j)$ ("sign"); (2) $r(L_i,L_j)$ which is eventually approximated to $F_L(L_i)-F_L(L_j)$ ("rank"); and (3) L_i-L_j ("linear"). We consider a sequence of contiguous hazard alternatives,

$$H_1^n : \lambda(t\,|\,L_i)=\lambda_1(t)\left\{\alpha_1(t)+n^{-1/2}a(L_i)\beta+O(n^{-1})\right\},$$

where $a(\cdot)$ is a real-valued measurable function. The given alternative has been called a relative risk model if $\alpha_1(t)=1$ and an excess risk model if $\lambda_1(t)=1$. Following arguments in Jones and Crowley (1990), we can show that the three test statistics converge in distribution to the normal distribution with mean $\mu(\infty)/\sigma(\infty)$ and variance 1, where $\mu(t)$ and $\sigma(t)^2$ are calculated as in Table 2. We introduce further notation in the table: $\bar{y}(t)$ is the limit of proportion of subjects at risk at time t; $\sigma_{XY}(t)$ is the limit of covariance of $X_i(t)$'s and $Y_i(t)$'s at risk at time t for two processes $X(t)$ and $Y(t)$. Jones and Crowley (1992) have it that

$$\sigma_{XY}(t) = \int_0^t X(l)Y(l)f_t(l)dl - \left\{ \int_0^t X(l)f_t(l)dl \right\} \left\{ \int_0^t Y(l)f_t(l)dl \right\}, \text{ where}$$

$$f_t(l) = I(l < t)\frac{\exp\left(-\int_0^l \Gamma_0(x)dx - \int_l^t \Gamma_1(x)dx\right)\theta(l)\exp\left(-\int_0^l \theta(x)dx\right)}{\int_0^t \exp\left(-\int_0^l \Gamma_0(x)dx - \int_l^t \Gamma_1(x)dx\right)\theta(l)\exp\left(-\int_0^l \theta(x)dx\right)dl} \tag{4.8}$$

with $\Gamma_i(x) = \psi_i(x) + \lambda_i(x)$ for $i = 0, 1$. Here, $\theta(t)$ is the hazard function of L, $\lambda_0(t)$ is the hazard rate of failure time of subjects who did not enter the study, and $\psi_1(t)$ and $\psi_0(t)$ are the hazard rates of censoring times of subjects who did and did not enter the study. Another note is that $\bar{y}(t)$ is the determinator of (4.8). The efficacy of a given test is defined as $\mu(\infty)^2/\sigma(\infty)^2$. We compare the efficacies of the tests by Pitman asymptotic relative efficiency, the ratio of the efficacy of a test to another.

We considered submodels of the contiguous alternatives:

- Model 1 (M1): $\lambda(t \mid L_i) = 0.3(1 + n^{-1/2}L_i\beta)$,
- Model 2 (M2): $\lambda(t \mid L_i) = 0.3\left\{1 + n^{-1/2}(L_i^2 + \sin L_i)^{-1}\beta\right\}$,

Both models are special cases of a relative risk or an excess risk model; M1 was selected to consider a model linear in L, and M2 was for one nonlinear in L. We generated L from either the exponential distribution with rate 2 or the uniform distribution on $[0, 1]$. We chose $\lambda_0(t) = 0.3$ and $(\psi_0(t), \psi_1(t))$ as $(0, 0)$, $(0, 1)$, and $(1, 1)$ following Jones and Crowley (1992). If $\psi_1(t) = 0$, there is no censoring on the observed patients and the resulting efficacies corresponds to those from the proposed class with no censoring. Similarly, with $\psi_1(t) = 1$ we can compare the efficacies from censored data. In those settings, we can calculate the given $\mu(t)$'s and $\sigma(t)^2$'s under M1 by simple algebra and formulas given in Appendix 2 in Jones and Crowley (1992). For M2, we approximated them by numerical integration.

Table 3 displays asymptotic relative efficiencies to $\widehat{\kappa}\{\mathrm{sign}(L_i - L_j), \mathrm{sign}(T_i - T_j)\}$ equivalent to those of the conditional Kendall's tau. We first focus on the case of $L \sim \mathrm{Exp}(2)$. The choice of g as the rank function, $g(L_i, L_j) = F_L(L_i) - F_L(L_j)$, performed better than, and comparable to, the conditional Kendall's tau when under M1 and M2, respectively. For $g(L_i, L_j) = L_i - L_j$ (the linear function), it performed the best under M1 where the dependence is linear and worst under M2 that depends on L nonlinearly. With L uniform on $[0, 1]$, we have $F_L(L) = L$ and the efficacies from g as the rank and linear

Table 3. Asymptotic relative efficiencies of selected $\widehat{\kappa}(g,h)$'s relative to $\widehat{\kappa}\{\text{sign}(L_i - L_j), \text{sign}(T_i - T_j)\}$ (Kendall's tau).

Model	$g(\cdot,\cdot)$	$h(\cdot,\cdot)$	True distribution						
			L	Exp(2)			Unif(0, 1)		
			(ψ_0, ψ_1)	(0, 1)	(0, 1)	(1, 1)	(0, 0)	(0, 1)	(1, 1)
M1	Rank	Sign		1.162	1.210	1.325	0.998	1.047	1.116
	Linear	Sign		1.721	1.800	1.769	0.998	1.047	1.116
M2	Rank	Sign		1.028	1.039	1.010	1.001	1.008	1.018
	Linear	Sign		0.402	0.401	0.414	1.001	1.008	1.018

functions are theoretically the same. Their performance was slightly better than or similar to that of the conditional Kendall's tau. From this limited comparison, one has that if g is linear then one can have a more powerful test under a linear relationship between L and T, but lead to a poor performance under certain nonlinear alternatives. On the other hand, the choice of g as the rank function performed better in some nonlinear cases.

5. Simulation Study

We evaluated finite sample performances of the proposed test statistics $\widehat{\rho}(g,h)$ and $\widehat{\kappa}(g,h)$ with g and h chosen as in Table 1. Simulation scenarios mimic those of Jones and Crowley (1992) and Chen, Tsai and Chao (1996).

Pairs of (L, X) were generated from two null and three non-null scenarios. We considered three exponential models with L uniform on $[0, 5]$ and the hazard function x as $h(x|L) = 0.3$, $h(x|L) = 0.3\,(1-L/12)$, and $h(x|L) = 0.3\,\{(L-2.5)^2 + 2\}^{-1}$. The first of these represents the null case, the other two represent linear and nonlinear alternative cases. We also considered two normal models where $(L, X)^T$ was multivariate normal with mean $(-1, 0)^T$ and covariance matrices $[1, \rho; \rho, 1]$, for $\rho = 0, 0.15$, representing null and alternative cases, respectively.

For left-truncated and right-censored data, censoring time C was independently generated from an exponential distribution, keeping the censoring rate around 40%. Finally, observations with $L \geq \min(X, C)$ were discarded. Sample size after the truncation was set to 400. For the generated dataset, we calculated the five $\widehat{\rho}(g,h)$'s and $\widehat{\kappa}(g,h)$'s with the choices of g and h presented in Table 1 and conducted the quasi-independence hypothesis testing at the significance level 5% according to the asymptotic null distribution shown in the Theorems.

Table 4 reports empirical rejection rates over 5,000 replications for the tests under the five null and alternative scenarios. Under the null, all the tests showed the rejection rates close to the nominal level. Depending on the alternative, dif-

Table 4. Empirical rejection rates of the test statistics $\widehat{\rho}(g,h)$ and $\widehat{\kappa}(g,h)$ with 5,000 simulated datasets from two null and three alternative scenarios, with $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ chosen according to Table 1. Tests requiring Assumption 3B are marked with asterisk (*). Abbreviations: Exp, exponential model; Norm, normal model; L, linear; NL, nonlinear.

Data (statistic)	$g(\cdot,\cdot)$	$h(\cdot,\cdot)$	Empirical rejection frequency				
			Null		Alternative		
			Exp	Norm	Exp(L)	Exp(NL)	Norm
Left-truncated	Sign	Sign	0.047	0.055	0.573	0.168	0.375
($\widehat{\rho}(g,h)$)	Linear	Sign	0.043	0.052	0.641	0.104	0.412
	Linear	Linear	0.046	0.051	0.736	0.087	0.442
	Rank	Sign	0.043	0.053	0.628	0.177	0.383
	Rank	Rank	0.044	0.055	0.587	0.280	0.375
Left-truncated	Sign	Sign	0.049	0.049	0.250	0.291	0.279
and	Linear	Sign	0.050	0.050	0.343	0.162	0.300
right-censored	Linear	Linear*	0.047	0.051	0.384	0.115	0.314
($\widehat{\kappa}(g,h)$)	Rank	Sign	0.049	0.051	0.298	0.315	0.278
	Rank	Rank*	0.050	0.050	0.240	0.453	0.266
(freq. of censored data)			0.401	0.401	0.402	0.401	0.400

ferent tests had higher power than the others.The conditional product-moment correlation was the most powerful under the normal alternative and the exponential alternative when the relationship between L and X was linear. The conditional Spearman's rank correlation featured in Example 2 was the most powerful under the exponential alternative when the relationship between L and X was nonlinear. The results were consistent for both left-truncated and left-truncated and right-censored data.When we restricted comparisons among the tests not requiring Assumption 3B for left-truncated and right-censored data, the hybrid test with g as linear and h as sign, was the most powerful under the normal alternative and the exponential alternative with the linear link. The new test with g as rank and h as sign featured in Example 3 was the most powerful under the exponential alternative with nonlinear link and as powerful as the conditional Kendall's tau. These findings are consistent with those in Section 4. Summarizing the results, newly proposed tests can be more powerful than existing ones under certain alternatives.

6. Data Example: Channing House

We applied the proposed quasi-independence tests to the Channing House data (Hyde (1980)) that include ages at death of 97 males and 365 females who were residents of the Channing House retirement center from January 1964 to July 1975. The data are left-truncated since a subject who died before entering

Table 5. p-values of the test statistic $\widehat{\kappa}(g,h)$ with the Channing house datasets for testing truncation time and failure time.

$\widehat{\kappa}(g,h)$		Evaluated test statistic (p-value)	
$g(\cdot,\cdot)$	$h(\cdot,\cdot)$	Men	Women
Sign	Sign	3.972 (0.046)	0.600 (0.438)
Linear	Sign	3.248 (0.072)	0.663 (0.416)
Linear	Linear	7.142 (0.008)	11.682 (0.001)
Rank	Sign	3.749 (0.053)	0.521 (0.469)
Rank	Rank	7.315 (0.007)	8.287 (0.004)

Table 6. p-values of the test statistic $\widehat{\kappa}(g,h)$ with the Channing house datasets for testing truncation time and censoring time, by reversing the role of survival time and censoring time.

$\widehat{\kappa}(g,h)$		Evaluated test statistic (p-value)	
$g(\cdot,\cdot)$	$h(\cdot,\cdot)$	Men	Women
Sign	Sign	5.380 (0.020)	30.213 ($< 10^{-7}$)
Linear	Sign	7.490 (0.006)	37.393 ($< 10^{-7}$)
Rank	Sign	7.199 (0.007)	35.514 ($< 10^{-7}$)

the community could not be recruited. In addition, the ages at death were right-censored by the end of the study.

Table 5 shows the p-values of the five $\widehat{\kappa}(g,h)$'s. The tests with h as the sign function show that the quasi-independence is marginally significant among the male group but not significant among the female group. The other two tests where h is not the sign function show a strong association between ages at entrance and death. Since these tests are valid under quasi-independence between censoring and truncation times, we inspected validity of this assumption by reversing the role of survival time and censoring times and applying the tests $\widehat{\kappa}(g,h)$ with h as the sign function. Table 6 displays corresponding p-values and suggests that the quasi-independence between L and C can be rejected. Based on this finding, we adopted the results from the tests which do not require Assumption 3B, and concluded that among women, we fail to reject that the failure time and truncation time are quasi-independent.

7. Concluding Remarks

We proposed a general class of tests which can embed existing tests for quasi-independence of truncation time and survival time. The proposed class was built upon common characteristics of existing tests, namely, U-statistics of skew-symmetric transforms of all the pairs of comparable observations. For left-truncated and right-censored data, a subclass of tests in which $h(\cdot,\cdot)$ is not the sign function require an additional assumption of quasi-independence between truncation and censoring times. This subclass can be used after testing quasi-independence between truncation and censoring times via the proposed test with $h(\cdot,\cdot)$ as the sign function exchanging the role of failure and censoring times. We also compared the powers from several choices of g and h theoretically and empirically. Our results suggest that the choice of the linear function may give a good power when the data has linear relationship, and a new sets of tests utilizing the rank function can be powerful under certain nonlinear alternatives. The results resonate with those on Pearson's product-limit correlation, Spearman's rank correlation, and Kendall's tau in complete bivariate data. Further research is needed to better understand when which choices of g and h would be suitable in a variety of settings not considered in this paper.

Appendix

Proofs of the main results

Proof of (3.1)**.** Observe that

$$\begin{aligned}
T^{\text{Cox}}\{Y, a(L)\} &= \sum_{i=1}^{n} \delta_i\, Y(T_i) \left\{ a(L_i) - \frac{\sum_{j=1}^{n} Y_j(T_i) a(L_j)}{Y(T_i)} \right\} \\
&= \sum_{i=1}^{n} \delta_i \sum_{j=1}^{n} Y_j(T_i)\{a(L_i) - a(L_j)\} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i\, I(L_j < T_i) I(T_i \le T_j)\{a(L_i) - a(L_j)\}. \qquad \text{(A.1)}
\end{aligned}$$

For further simplification, we write $Q(i,j) = \delta_i\, I(L_j < T_i)I(T_i \le T_j)$. We see that $Q(i,i) = 0$, and $I(\Lambda_{ij}) = 1$ is equivalent to either $Q(i,j) = 1$ or $Q(j,i) = 1$ for $i \ne j$. Also, without ties, $I(T_i \le T_j) = I(T_i < T_j) = -I(T_i < T_j)\text{sign}(T_i - T_j)$. These identities leads (A.1) to

$$\begin{aligned}
T^{\text{Cox}}\{Y, a(L)\} &= -\sum_{i=1}^{n} \sum_{j=1}^{n} \{a(L_i) - a(L_j)\}\text{sign}(T_i - T_j) Q(i,j) \\
&= -\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \{a(L_i) - a(L_j)\}\text{sign}(T_i - T_j) I(\Lambda_{ij}).
\end{aligned}$$

Proof of (3.2)**.** With the definition of $R_i(t)$, $\sum_{i=1}^{n} Y_i(t) R_i(t) = 1 + \cdots + Y(t) = Y(t)\{Y(t)+1\}/2$. In addition, $2I(L_i < L_j) - 1 = -\text{sign}(L_i - L_j)$ from the no-tie assumption. Then we have

$$
\begin{aligned}
T^{\mathrm{Cox}}(Y, R^*) &= \sum_{i=1}^{n} \delta_i \, Y(T_i) \left\{ \frac{R_i(T_i)}{Y(T_i)} - \frac{\sum_{j=1}^{n} Y_j(T_i) R_j(T_i)}{Y(T_i)^2} \right\} \\
&= \frac{1}{2} \sum_{i=1}^{n} \delta_i \, \{2R_i(T_i) - Y(T_i) - 1\} \\
&= \frac{1}{2} \sum_{i=1}^{n} \delta_i \, \left[\sum_{j=1}^{n} Y_j(T_i) \big\{ 2I(L_i < L_j) - 1 \big\} + 1 \right] \\
&= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i \, I(L_j < T_i) I(T_i \le T_j) \mathrm{sign}(L_i - L_j) + \frac{1}{2} \sum_{i=1}^{n} \delta_i.
\end{aligned}
$$

The left term in the last equation has a similar form as (A.1) and it is easy to see that $\sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i \, I(L_j < T_i) I(T_i \le T_j) \mathrm{sign}(L_i - L_j) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{sign}(L_i - L_j) \mathrm{sign}(T_i - T_j) I(\Lambda_{ij})$.

Proof of Theorem 1. Suppose Assumptions 1 and 2 and quasi-independence hypothesis hold. For simplicity, we consider continuous random variables so that (L, X) has a joint density $f_{L,X}(l, x)$.

By the central limit theorem for one-sample U-statistics (Randles and Wolfe (1991)), as $n \to \infty$,

$$
\sqrt{n}(\widehat{\rho}(g, h) - \rho(g, h)) \xrightarrow{d} \mathcal{N}\left(0, \frac{4\zeta}{\mathrm{pr}(\Omega_{12})^2}\right),
$$

where $\zeta = \phi(g, h) - \{\rho(g, h)\mathrm{pr}(\Omega_{12})\}^2$. The theorem holds if $\rho(g, h) = 0$. We claim

$$
E\big\{g(L_1, L_2) h(X_1, X_2) I(\Omega_{12})\big\} = 0, \tag{A.2}
$$

which means that the numerator of $\rho(g, h)$ vanishes. Partition the event Ω_{12} into four disjoint events: $A_1 = (L_1 < L_2 < X_1 < X_2)$, $A_2 = (L_1 < L_2 < X_2 < X_1)$, $A_3 = (L_2 < L_1 < X_1 < X_2)$, and $A_4 = (L_2 < L_1 < X_2 < X_1)$. Note that $I(\Omega_{12}) = I(A_1) + I(A_2) + I(A_3) + I(A_4)$. In addition, the quasi-independence implies that $f_{L,X}(l, x) = f_L(l) f_X(x)/\alpha$ on $\{(l, x) : l < x\}$ for some α, where f_L and f_X are marginal densities of L and X, respectively. By the skew-symmetry of g, we have

$$
\begin{aligned}
&E\big\{g(L_1, L_2) h(X_1, X_2) I(A_1)\big\} \\
&= \int_{l_1 < l_2 < x_1 < x_2} g(l_1, l_2) h(x_1, x_2) f_{L,X}(l_1, x_1) f_{L,X}(l_2, x_2) d(l_1, l_2, x_1, x_2)
\end{aligned}
$$

$$= \frac{1}{\alpha^2}\int_{l_1=0}^{\infty}\int_{l_2=l_1}^{\infty}\int_{x_1=l_2}^{\infty}\int_{x_2=x_1}^{\infty} g(l_1,l_2)h(x_1,x_2)f_X(x_1)f_L(l_1)f_X(x_2)f_L(l_2)dx_2dx_1 \\ dl_2dl_1$$

$$= \frac{1}{\alpha^2}\int_{l_1=0}^{\infty}\int_{l_2=l_1}^{\infty}\int_{x_1=l_2}^{\infty}\int_{x_2=x_1}^{\infty} \{-g(l_2,l_1)\}h(x_1,x_2)f_X(x_1)f_L(l_1)f_X(x_2)f_L(l_2)dx_2 \\ dx_1dl_2dl_1$$

$$\overset{l_1\leftrightarrow l_2}{=} \frac{1}{\alpha^2}\int_{l_2=0}^{\infty}\int_{l_1=l_2}^{\infty}\int_{x_1=l_1}^{\infty}\int_{x_2=x_1}^{\infty} \{-g(l_1,l_2)\}h(x_1,x_2)f_X(x_1)f_L(l_2)f_X(x_2)f_L(l_1)dx_2 \\ dx_1dl_1dl_2$$

$$= -\int_{l_2<l_1<x_1<x_2} g(l_1,l_2)h(x_1,x_2)f_{L,X}(l_1,x_1)f_{L,X}(l_2,x_2)d(l_1,l_2,x_1,x_2)$$

$$= -E\big\{g(L_1,L_2)h(X_1,X_2)I(A_3)\big\},$$

which implies $E\big[g(L_1,L_2)h(X_1,X_2)\{I(A_1)+I(A_3)\}\big]=0$. A similar argument holds for A_2 and A_4. Thus (A.2) holds and the theorem follows.

Proof of Theorem 2. The proof is similar to that for Theorem 1. The asymptotic normality is straightforward from the central limit theorem. Our goal is to show

$$E\big\{g(L_1,L_2)\text{sign}(T_1,T_2)I(\Lambda_{12})\big\}=0 \tag{A.3}$$

which implies $\kappa(g,h)=0$. We suppose Assumption 1 and 2 along with the quasi-independence. For convenience, let (L,X,C) be continuous and have a joint density $f_{L,X,C}(l,x,c)$.

If Assumption 3A holds, (A.3) is verified by proving (a) $E\big\{g(L_1,L_2)\text{sign}(T_1-T_2)I(\Lambda_{12})\big\}=E\big\{g(L_1,L_2)\text{sign}(X_1-X_2)I(\Lambda_{12})\big\}$, and (b) $E\{g(L_1,L_2)\ h(X_1,X_2)\ I(\Lambda_{12})\}=0$ for any skew-symmetric g and h.

For (a), write $\Lambda_{12}=B_1\cup B_2\cup B_3\cup B_4$, where $B_1=(L_1<L_2<X_1<T_2)\cap(\delta_1=1)$, $B_2=(L_1<L_2<X_2<T_1)\cap(\delta_2=1)$, $B_3=(L_2<L_1<X_1<T_2)\cap(\delta_1=1)$, and $B_4=(L_2<L_1<X_2<T_1)\cap(\delta_2=1)$.

On B_1, we have $X_1=T_1<T_2<X_2$, which implies $\text{sign}(T_2-T_1)=\text{sign}(X_2-X_1)=1$. Similar arguments hold for B_2, B_3, and B_4. Thus $E\big\{g(L_1,L_2)\text{sign}(T_1-T_2)I(\Lambda_{12})\big\}=E\big\{g(L_1,L_2)\text{sign}(X_1-X_2)I(\Lambda_{12})\big\}$.

For (b), combine Assumption 1, 2, and quasi-independence to obtain $f_{L,X,C}(l,x,c)=f_X(x)f_{L,C}(l,c)/\alpha$ on $\{(l,x,c):l<x,l<c\}$ for some α, where f_X and $f_{L,C}$ are the marginal densities of X and (L,C), respectively. In addition, $B_1=(L_1<L_2<X_1<\min\{X_2,C_1,C_2\})$ and the other disjoint events can be written similarly. We now claim $E\big[g(L_1,L_2)h(X_1,X_2)\{I(B_1)+I(B_3)\}\big]=0$.

Observe that

$$
\begin{aligned}
&E\{g(L_1,L_2)h(X_1,X_2)I(B_1)\}\\
&=\int_{l_1<l_2<x_1<\min\{x_2,c_1,c_2\}} g(l_1,l_2)h(x_1,x_2)f_{L,X,C}(l_1,x_1,c_1)f_{L,X,C}(l_2,x_2,c_2)d(l_1,x_1,\\
&\quad c_1,l_2,x_2,c_2)\\
&=\frac{1}{\alpha^2}\int_{l_1=0}^{\infty}\int_{l_2=l_1}^{\infty}\int_{x_1=l_2}^{\infty}\int_{x_2=x_1}^{\infty}\int_{c_1=x_1}^{\infty}\int_{c_2=x_1}^{\infty} g(l_1,l_2)h(x_1,x_2)\times f_X(x_1)f_{L,C}(l_1,c_1)\\
&\quad f_X(x_2)f_{L,C}(l_2,c_2)dc_2dc_1dx_2dx_1dl_2dl_1\\
&=\frac{1}{\alpha^2}\int_{l_1=0}^{\infty}\int_{l_2=l_1}^{\infty}\int_{x_1=l_2}^{\infty} g(l_1,l_2)f_X(x_1)\mathrm{pr}(x_1)Q(l_1,x_1)Q(l_2,x_1)dx_1dl_2dl_1,
\end{aligned}
$$

where $Q(l,x)=\int_{c=x}^{\infty} f_{L,C}(l,c)dc$ and $\mathrm{pr}(x)=\int_{u=x}^{\infty} h(x,u)f_X(u)du$. The expectation on B_3 can be calculated as

$$
\begin{aligned}
&E\{g(L_1,L_2)h(X_1,X_2)I(B_3)\}\\
&=\frac{1}{\alpha^2}\int_{l_2=0}^{\infty}\int_{l_1=l_2}^{\infty}\int_{x_1=l_1}^{\infty} g(l_1,l_2)f_X(x_1)\mathrm{pr}(x_1)Q(l_1,x_1)Q(l_2,x_1)dx_1dl_1dl_2\\
&\overset{l_1\leftrightarrow l_2}{=}\frac{1}{\alpha^2}\int_{l_1=0}^{\infty}\int_{l_2=l_1}^{\infty}\int_{x_1=l_2}^{\infty} g(l_2,l_1)f_X(x_1)\mathrm{pr}(x_1)Q(l_2,x_1)Q(l_1,x_1)dx_1dl_2dl_1\\
&=\frac{1}{\alpha^2}\int_{l_1=0}^{\infty}\int_{l_2=l_1}^{\infty}\int_{x_1=l_2}^{\infty} \{-g(l_1,l_2)\}\, f_X(x_1)\mathrm{pr}(x_1)Q(l_2,x_1)Q(l_1,x_1)dx_1dl_2dl_1\\
&=-E\{g(L_1,L_2)h(X_1,X_2)I(B_1)\},
\end{aligned}
$$

where the skew-symmetry of g was used at the third inequality. Thus the claim holds, and similarly $E\big[g(L_1,L_2)h(X_1,X_2)\{I(B_2)+I(B_4)\}\big]=0$ which implies (A.3).

Suppose Assumption 3B holds. Integrating Assumption 1, 2, and 3B, $f_{L,X,C}(l,x,c)$ is equal to $f_L(l)f_X(x)f_C(c)/\alpha$ on $\{(l,x,c): l<x, l<c\}$ for some α, where f_L, f_X, f_C is the marginal densities of L, X and C, respectively. Then

$$
\begin{aligned}
&E\{g(L_1,L_2)h(T_1,T_2)I(B_1)\}\\
&=\frac{1}{\alpha^2}\int_{l_1=0}^{\infty}\int_{l_2=l_1}^{\infty}\int_{x_1=l_2}^{\infty}\int_{x_2=x_1}^{\infty}\int_{c_1=x_1}^{\infty}\int_{c_2=x_1}^{\infty} g(l_1,l_2)h(x_1,\min(x_2,c_2))\times\\
&\quad f_X(x_1)f_L(l_1)f_C(c_1)f_X(x_2)f_L(l_2)f_C(c_2)dc_2dc_1dx_2dx_1dl_2dl_1\\
&=\frac{1}{\alpha^2}\int_{l_1=0}^{\infty}\int_{l_2=l_1}^{\infty}\int_{x_1=l_2}^{\infty}\int_{x_2=x_1}^{\infty} g(l_1,l_2)f_X(x_1)f_L(l_1)f_X(x_2)f_L(l_2)\times\\
&\quad\left\{\int_{c_1=x_1}^{\infty} f_C(c_1)dc_1\right\}\left\{\int_{c_2=x_1}^{\infty} h(x_1,\min(x_2,c_2))f_C(c_2)dc_1\right\}dx_2dx_1dl_2dl_1
\end{aligned}
$$

On the other hand, on B_3, interchanging $l_1 \leftrightarrow l_2$ and the skew-symmetry of g yields

$$
\begin{aligned}
&E\big\{g(L_1, L_2)h(T_1, T_2)I(B_3)\big\} \\
&= \frac{1}{\alpha^2}\int_{l_2=0}^{\infty}\int_{l_1=l_2}^{\infty}\int_{x_1=l_1}^{\infty}\int_{x_2=x_1}^{\infty} g(l_1, l_2)f_X(x_1)f_L(l_1)f_X(x_2)f_L(l_2)\times \\
&\quad \left\{\int_{c_1=x_1}^{\infty} f_C(c_1)dc_1\right\}\left\{\int_{c_2=x_1}^{\infty} h(x_1, \min(x_2, c_2))f_C(c_2)dc_1\right\}dx_2dx_1dl_1dl_2 \\
&= \frac{1}{\alpha^2}\int_{l_1=0}^{\infty}\int_{l_2=l_1}^{\infty}\int_{x_1=l_2}^{\infty}\int_{x_2=x_1}^{\infty} g(l_2, l_1)f_X(x_1)f_L(l_2)f_X(x_2)f_L(l_1)\times \\
&\quad \left\{\int_{c_1=x_1}^{\infty} f_C(c_1)dc_1\right\}\left\{\int_{c_2=x_1}^{\infty} h(x_1, \min(x_2, c_2))f_C(c_2)dc_1\right\}dx_2dx_1dl_1dl_2 \\
&= -E\big\{g(L_1, L_2)h(T_1, T_2)I(B_1)\big\}.
\end{aligned}
$$

The rest of the proof goes in a same way as the previous proofs and we have (A.3).

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