A CENTRAL LIMIT THEOREM FOR NESTED OR SLICED LATIN HYPERCUBE DESIGNS

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Supplementary Material

In the supplementary material, we give the proofs of Propositions 1-4 and Lemma 7.

S1 The proof of Proposition 1

Proof. We first work on the cells $D = (D^1, \ldots, D^q)$, where $D^k \in \{[0, 1/n), [1/n, 2/n), \ldots, [1 - 1/n, 1)\}$, $k = 1, \ldots, q$.

From (2.1), for $k = 1, \ldots, q$ and $i = 1, \ldots, s-1$, $\pi_k(i) = \lfloor nX_i^k \rfloor + 1$. Because π_k is a uniform permutation on $\{1, \ldots, n\}$, $\pi_k(s)$ has probability $(n-s+1)^{-1}$ to be any of the $n-s+1$ elements of $\{1, \ldots, n\} \setminus \{\lfloor nX_i^k \rfloor + 1 : 1 \le i \le s-1\}$. Because the π_k are generated independently, the probability of $(d_1, \ldots, d_q) \in D$ is $(n-s+1)^{-q}$ if for any $1 \le i \le s-1$ and $1 \le k \le q$, $d_k \notin \delta_n(X_i^k)$.

Because the η_i^k are generated independently by the uniform distribution on $(0, 1]$, $(d_1, \ldots, d_q)$ is uniformly distributed in any D. Because D has volume n^{-q}, $g_{\mathrm{OLHD}}(d_1, \ldots, d_q) = \{n/(n-s+1)\}^q$ if for any $1 \le i \le s-1$ and $1 \le k \le q$, $d_k \notin \delta_n(X_i^k)$. □

S2 The proof of Proposition 2

Proof. First assume $Z_s = 1$. From (2.3), for $k = 1, \ldots, q$ and $i = 1, \ldots, s-1$ with $Z_i = 1$, $\gamma_k(\pi(i)) = \lfloor mX_i^k \rfloor + 1$. Because γ_k is a uniform permutation on $\{1, \ldots, m\}$ and γ_k is independent of ρ_k, $\gamma_k(\pi(s))$ has probability $(m - |\{i : 1 \le i \le s-1, Z_i = 1\}|)^{-1}$ to be any of the $m - |\{i : 1 \le i \le s-1, Z_i = 1\}|$ elements of $\{1, \ldots, m\} \setminus \{\lfloor mX_i^k \rfloor + 1 : 1 \le i \le s-1, Z_i = 1\}$.

Because τ_i^k is independent of ρ_k and the η_i^k are generated independently by the uniform distribution on $(0, 1]$, $(d_1, \ldots, d_q)$ is uniformly distributed in $D = (D^1, \ldots, D^q)$, where $D^k \in \{[(i-1)/m, i/m) \setminus (\cup_{j=1}^{s-1} \delta_n(X_j^k)) : 1 \le i \le m,$ for any $1 \le c \le s-1$ with $Z_c = 1, \lfloor mX_c^k \rfloor \ne i-1\}$. The volume of D is $\prod_k \{(l - |\{i : 1 \le i \le s-1, D^k \subseteq \delta_m(X_i^k)\}|)/n\}$. Thus, $g_{\mathrm{NLHD}}(d_1, \ldots, d_q) = \prod_{k=1}^q g_k(d_k)$ if $(d_1, \ldots, d_k)$ is in any of the D above and $g_{\mathrm{NLHD}}(d_1, \ldots, d_q) = 0$ otherwise.

Next, assume $Z_s = 2$. Let $\beta_k = \lceil \rho_k / l \rceil$. From (2.3), observe that β_k maps $l-1$ elements to any of the elements of $\{1, \ldots, m\}$ uniformly, and β_k is independent of $\gamma_k, \tau_1^k, \ldots, \tau_m^k$. Therefore, for $j = 1, \ldots, m$, the probability of $\beta_k(\pi(s) - m) = j$ is $(l - 1 - |\{i : 1 \le i \le s-1, Z_i = 2, \delta_m(X_i^k) = j, \}|)/(n - m - |\{i : 1 \le i \le s-1, Z_i = 2\}|)$.

Because ρ_k is independent of τ_i^k and the η_i^k are generated independently by the uniform distribution on $(0,1]$, $(d_1,\ldots,d_q)$ is uniformly distributed in $D=(D^1,\ldots,D^q)$, where $D^k \in \{[(i-1)/m, i/m) \setminus (\cup_{j=1}^{s-1}\delta_n(X_j^k)) : 1 \le i \le s-1\}$. The volume of D is $\prod_k \{(l - |\{i : 1 \le i \le s-1, D^k \subseteq \delta_m(X_i^k)\}|)/n\}$. Thus, $g_{\mathrm{NLHD}}(d_1,\ldots,d_q) = \prod h_k(d_k)$ if $(d_1,\ldots,d_k)$ is in any of the D above and $g_{\mathrm{NLHD}}(d_1,\ldots,d_q)=0$ otherwise. □

S3 The proof of Proposition 3

Proof. From (2.5), for $k=1,\ldots,q$ and $i=1,\ldots,s-1$ with $Z_i = Z_s$, $\gamma_{Z_i}^k(\pi(i) - m\lceil \pi(i)/m \rceil + m) = \lfloor mX_i^k \rfloor + 1$. Therefore, $\gamma_{Z_s}^k(\pi(s) - m\lceil\pi(s)/m\rceil + m)$ has probability $(m - |\{i : 1 \le i \le s-1, Z_i = Z_s\}|)^{-1}$ to be any of the $m - |\{i : 1 \le i \le s-1, Z_i = Z_s\}|$ elements of $\{1,\ldots,m\} \setminus \{\lfloor mX_i^k \rfloor + 1 : 1 \le i \le s-1, Z_i = Z_s\}$.

Because τ_b^k and γ_a^k are independent, and the η_i^k are generated independently by the uniform distribution on $(0,1]$, $(d_1,\ldots,d_q)$ is uniformly distributed in $D=(D^1,\ldots,D^q)$, where $D^k \in \{[(i-1)/m, i/m) \setminus (\cup_{j=1}^{s-1}\delta_n(X_j^k)) : 1 \le i \le m$, for any $1 \le c \le s-1$ with $Z_c = Z_s, \lfloor mX_c^k \rfloor \ne i-1\}$. The volume of D is $\prod_k\{(l - |\{i : 1 \le i \le s-1, D^k \subseteq \delta_m(X_i^k)\}|)/n\}$. Thus, $g_{\mathrm{SLHD}}(d_1,\ldots,d_q) = \prod g_k(d_k)$ if $(d_1,\ldots,d_k)$ is in any of the D above and $g_{\mathrm{SLHD}}(d_1,\ldots,d_q)=0$ otherwise. □

S4 The proof of Proposition 4

Proof. First, consider the case of OLHD. Let $b_s(i_1,\ldots,i_q) = \{n/(n-s+1)\}^q$ if $i_1 = \cdots = i_q = 0$ and $b_s(i_1,\ldots,i_q)=0$ otherwise. From Proposition 1, (2.7) is valid.

Second, consider the case of NLHD with $Z_s = 1$. Let $b_s(i_1,\ldots,i_q)=0$ if there is a k and $1 \le j \le s-1$ such that $i_k > 0$, $\lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor$ and $Z_j = 1$. Otherwise, let $w = \{k : i_k \ne 0\}$ and

$$b_s(i_1,\ldots,i_q) = \begin{cases} c, & i_1 = \cdots = i_q = 0, \\ p(w), & i_1,\ldots,i_q \ge 0, |w| > 0, \text{ for any } k \text{ such that } i_k > 0, i_k \text{ is the} \\ & \text{smallest among } \{j : 1 \le j \le s-1, \lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor\}, \\ -p(w), & i_1,\ldots,i_q \le 0, |w| > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{S4.1}$$

where $c = \{m/(m - |\{i : 1 \le i \le s-1, Z_i = 1\}|)\}^q$, and

$$p(w) = c \prod_{k \in w} \left(1 - |\{j : 1 \le j \le s-1, \lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor\}|/l\right)^{-1}.$$

Because $|\{j : 1 \le j \le s-1, \lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor\}| \le l-1$ for any $k \in w$, $p(w)$ is bounded as n goes to infinity. Therefore, $b_s(i_1,\ldots,i_q)$ is bounded as n goes to infinity and $b_s(0,\ldots,0) = c = 1 + O(n^{-1})$. Using an inclusion-exclusion argument, (2.7) is verified for NLHD with $Z_s = 1$.

Third, consider the case of NLHD with $Z_s = 2$. Let $w = \{k : i_k \neq 0\}$ and b_s be defined as in (S4.1), where $c = \{(n-m)/(n-m-|\{i : 1 \leq i \leq s-1, Z_i = 2\}|)\}^q$, $p(w) = 0$ if there is a $k \in w$ such that $|\{j : 1 \leq j \leq s-1, \lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor, Z_i = 2\}| = l-1$ and

$$p(w) = c \prod_{k \in w} \frac{n(l-1-|\{j : 1 \leq j \leq s-1, \lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor, Z_j = 2\}|)}{(n-m)(l-|\{j : 1 \leq j \leq s-1, \lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor\}|)}$$

otherwise. When $|\{j : 1 \leq j \leq s-1, \lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor, Z_j = 2\}| < l-1$, $|\{j : 1 \leq j \leq s-1, \lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor\}| \leq l-1$. Therefore, $b_s(i_1, \ldots, i_q)$ is bounded as n goes to infinity and $b_s(0, \ldots, 0) = c = 1 + O(n^{-1})$. Using an inclusion-exclusion argument, (2.7) is verified for NLHD with $Z_s = 2$.

Finally, consider the case for SLHD. Let $b_s(i_1, \ldots, i_q) = 0$ if there is a k and $1 \leq j \leq s-1$ such that $i_k > 0$, $\lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor$ and $Z_j = Z_s$. Otherwise, let $w = \{k : i_k \neq 0\}$ and b_s be defined as in (S4.1), where $c = \{m/(m-|\{i : 1 \leq i \leq s-1, Z_i = Z_s\}|)\}^q$, and

$$p(w) = c \prod_{k \in w} \left(1 - |\{j : 1 \leq j \leq s-1, \lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor\}|/l\right)^{-1}.$$

Because $|\{j : 1 \leq j \leq s-1, \lfloor mX_j^k \rfloor = \lfloor mX_{i_k}^k \rfloor\}| \leq l-1$ for any $k \in w$, $p(w)$ is bounded as n goes to infinity. Therefore, $b_s(i_1, \ldots, i_q)$ is bounded as n goes to infinity and $b_s(0, \ldots, 0) = c = 1 + O(n^{-1})$. Using an inclusion-exclusion argument, (2.7) is verified for SLHD. □

S5 The proof of Lemma 7

We first give the key steps of the proof. Let $r_i = r(X_i)$ by (3.2). Then

$$E\{(n^{1/2}\bar{R})^p\} = n^{-p/2} \sum_{a_1+\ldots+a_n=p, a_1,\ldots,a_n \geq 0} E\left(\prod_{i=1}^n r_i^{a_i}\right). \quad \text{(S5.2)}$$

Let t be the number of a_i's being one and s be the number of non-zero a_i's. There are at most $O(n^s)$ terms in (S5.2). Thus, it suffices to show that for any $s \leq p$,

$$E\left(\prod_{i=1}^s r_i^{a_i}\right) - E_{\text{IID}}\left(\prod_{i=1}^s r_i^{a_i}\right) = o(n^{p/2-s}).$$

If $t = 0$, then $s \leq p/2$. From Lemma 6,

$$E\left(\prod_{i=1}^s r_i^{a_i}\right) - E_{\text{IID}}\left(\prod_{i=1}^s r_i^{a_i}\right) = O(n^{-1}) = o(n^{p/2-s}).$$

If $t > 0$, $E_{\text{IID}}\left(\prod_{i=1}^s r_i^{a_i}\right) = 0$. Thus, it suffices to show that for any $1 \leq t \leq s \leq p$, $t + a_{t+1} + \ldots + a_s = p, a_{t+1}, \ldots, a_s > 1$,

$$E\left(\prod_{i=1}^{s-t} r_i^{a_i} \prod_{i=s-t+1}^{s} r_i\right) = o(n^{p/2-s}).$$

Because $t + 2(s-t) \le p$, $-t/2 \le p/2 - s$. Since $r_i = \sum_{|u|>1} f_u(X_i)$, it suffices to show for any $1 \le t \le s$, $|u_{s-t+1}|, \ldots, |u_s| > 1$, continuous functions f and bounded function $h(x)$,

$$E\left\{h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s} f_{u_i}(X_i)\right\} = o(n^{-t/2}). \tag{S5.3}$$

To show (S5.3), express

$$\begin{aligned} & E\left\{h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s} f_{u_i}(X_i)\right\} \\ = \; & E\left[h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s-1} f_{u_i}(X_i) E\left\{f_{u_s}(X_s) \mid X_1,\ldots,X_{s-1}\right\}\right]. \end{aligned}$$

From Proposition 4,

$$E\left\{f_{u_s}(X_s) \mid X_1,\ldots,X_{s-1}\right\} = \sum_{i_1,\ldots,i_q} b_s(i_1,\ldots,i_q)\left(\int_{D_s} f_{u_s}(y)dy\right),$$

where $i_1,\ldots,i_q = -(s-1),\ldots,s-1$, $b_s(i_1,\ldots,i_q)$ is a deterministic function on $n, m, i_1,\ldots,i_q, Z_1,\ldots,Z_s, M_{s-1}$ and bounded as n goes to infinity, $D_s = D^1_{i_1} \times \cdots \times D^q_{i_q}$ and

$$D^k_i = \begin{cases} [0,1) \setminus \cup_{j=1}^{s-1}\delta_m(X^k_j), & i = 0, \\ \delta_m(X^k_i), & i > 0, \\ \delta_n(X^k_{-i}), & i < 0. \end{cases}$$

By an inclusion-exclusion argument, rewrite

$$E\left\{f_{u_s}(X_s) \mid X_1,\ldots,X_{s-1}\right\} = \sum_{i_1,\ldots,i_q} \tilde{b}_s(i_1,\ldots,i_q)\left(\int_{\tilde{D}_s} f_{u_s}(y)dy\right),$$

where $i_1,\ldots,i_q = -(s-1),\ldots,s-1$, $\tilde{b}_s(i_1,\ldots,i_q)$ is a deterministic function on $n, m, i_1,\ldots,i_q, Z_1,\ldots,Z_s, M_{s-1}$ and bounded as n goes to infinity, $\tilde{D}_s = \tilde{D}^1_{i_1} \times \cdots \times \tilde{D}^q_{i_q}$ and

$$\tilde{D}^k_i = \begin{cases} [0,1), & i = 0, \\ \delta_m(X^k_i), & i > 0, \\ \delta_n(X^k_{-i}), & i < 0. \end{cases}$$

From (3.1),

$$\int_{\tilde{D}^1\times\cdots\times\tilde{D}^q} f_u(y)dy = 0$$

if there is at least one k such that $\tilde{D}^k = [0,1)$ and $k \in u$. Therefore, let $w(d_1,\ldots,d_q) = \{k : d_k \neq 0\}$, then $\int_{\tilde{D}_s} f_{u_s}(y)dy$ has order $O(n^{-|w(d_1,\ldots,d_q)\cup u_s|}) = O(n^{-2})$ and

$$\begin{aligned} & E\left\{h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s} f_{u_i}(X_i)\right\} \\ = \; & \sum_{i_1,\ldots,i_q=-(s-1)}^{s-1} E\left\{h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s-1} f_{u_i}(X_i)\tilde{b}_s(i_1,\ldots,i_q)\int_{\tilde{D}_s} f_{u_s}(y)dy\right\} \\ = \; & O(n^{-2}). \end{aligned} \tag{S5.4}$$

We can further reduce the order of (S5.4) if $t > 1$. For any term in the sum of (S5.4),

$$E\left\{h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s-1} f_{u_i}(X_i)\tilde{b}_s(i_1,\ldots,i_q)\int_{\tilde{D}_s} f_{u_s}(y)dy\right\}$$
$$= \sum_{j_1,\ldots,j_q=-(s-2)}^{s-2} E\left[h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s-2} f_{u_i}(X_i)b_{s-1}(j_1,\ldots,j_q)\right.$$
$$\left.\left\{\int_{D_{s-1}}\tilde{b}_s(i_1,\ldots,i_q)\left(\int_{\tilde{D}_s} f_{u_s}(y_s)dy_s\right)f_{u_{s-1}}(X_{s-1})dX_{s-1}\right\}\right],$$

where $b_{s-1}(i_1,\ldots,i_q)$ is a deterministic function on $n, m, i_1,\ldots,i_q, Z_1,\ldots,Z_{s-1}, M_{s-2}$ and bounded as n goes to infinity, $D_{s-1} = D^1_{j_1}\times\cdots\times D^q_{j_q}$ and

$$D^k_j = \begin{cases} [0,1)\setminus\cup_{i=1}^{s-2}\delta_m(X^k_i), & j=0,\\ \delta_m(X^k_j), & j>0,\\ \delta_n(X^k_{-j}), & j<0.\end{cases}$$

In any area of D_{s-1}, $\tilde{b}_s(i_1,\ldots,i_q)$ becomes a deterministic function on $n, m, i_1,\ldots,i_q, Z_1,\ldots,Z_{s-1}, M_{s-2}$ and bounded as n goes to infinity. Therefore,

$$E\left\{h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s-1} f_{u_i}(X_i)\tilde{b}_s(i_1,\ldots,i_q)\int_{\tilde{D}_s} f_{u_s}(y)dy\right\}$$
$$= \sum_{j_1,\ldots,j_q=-(s-2)}^{s-2} E\left[h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s-2} f_{u_i}(X_i)\hat{b}_{s-1}(j_1,\ldots,j_q)\right.$$
$$\left.\left\{\int_{D_{s-1}}\left(\int_{\tilde{D}_s} f_{u_s}(y_s)dy_s\right)f_{u_{s-1}}(X_{s-1})dX_{s-1}\right\}\right],$$

where $\hat{b}_{s-1}(i_1,\ldots,i_q)$ is a deterministic function on $n, m, i_1,\ldots,i_q, Z_1,\ldots,Z_{s-1}, M_{s-2}$ and bounded as n goes to infinity and D_{s-1} defined as before.

By an inclusion-exclusion argument, rewrite

$$E\left\{h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s-1} f_{u_i}(X_i)\tilde{b}_s(i_1,\ldots,i_q)\int_{\tilde{D}_s} f_{u_s}(y)dy\right\}$$
$$= \sum_{j_1,\ldots,j_q=-(s-2)}^{s-2} E\left[h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s-2} f_{u_i}(X_i)\tilde{b}_{s-1}(j_1,\ldots,j_q)\right.$$
$$\left.\left\{\int_{\tilde{D}_{s-1}}\left(\int_{\tilde{D}_s} f_{u_s}(y_s)dy_s\right)f_{u_{s-1}}(X_{s-1})dX_{s-1}\right\}\right], \quad \text{(S5.5)}$$

where $\tilde{b}_{s-1}(i_1,\ldots,i_q)$ is a deterministic function on $n, m, i_1,\ldots,i_q, Z_1,\ldots,Z_{s-1}, M_{s-2}$ and bounded as n goes to infinity, $\tilde{D}_{s-1} = \tilde{D}^1_{j_1}\times\cdots\times\tilde{D}^q_{j_q}$ and

$$\tilde{D}^k_j = \begin{cases} [0,1), & j=0,\\ \delta_m(X^k_j), & j>0,\\ \delta_n(X^k_{-j}), & j<0.\end{cases}$$

The first two steps shown above reduce the order of magnitudes for $E\{h(X_1,\ldots,X_{s-t})\prod_{i=s-t+1}^{s} f_{u_i}(X_i)\}$. In (S5.4), we took $f_{u_s}(X_s)$ out of the product and reached the $O(n^{-2})$ order. Continuing taking out the $f_{u_i}(X_i)$ terms as in (S5.5), we obtain on a more general formula given by

$$\left(\prod_{j=1}^{J}|D_j|\right)^{-1} E\left\{h(M_s)\prod_{i=s-t+1}^{s} f_{u_i}(X_i)\int_{\prod_{j=1}^{J}D_j}\prod_{j=1}^{J} f_{v_j}(y_j)dy_1\cdots dy_J\right\}. \tag{S5.6}$$

Suppose G is an arbitrary term by (S5.6) with the following parameters: $0 \le t \le s \le p$, $|u_{s-t+1}|,\ldots,|u_s| > 1$, J is a nonnegative integer, $h(M_s)$ is a deterministic function on $n, m, Z_1,\ldots,Z_n, X_1,\ldots,X_{s-t}, M_s$ and bounded as n goes to infinity, $v_j \subseteq \{1,\ldots,q\}$, $D_j = D_j^1\times\cdots\times D_j^q$, and D_j^k is either $[0,1)$, or $\delta_m(X_i^k)$ with $1\le i\le s$, or $\delta_n(X_i^k)$ with $1 \le i \le s$, or $\delta_m(y_i^k)$ with $j < i \le J$, or $\delta_n(y_i^k)$ with $j < i \le J$. Suppose that C is a $t\times q$ zero-one matrix with the (i,k)th element being one if and only if $k \in u_{i-s+t}$ and $D_j^k \not\subseteq \delta_m(X_{i-s+t}^k)$ for any $1 \le j \le J$. Let θ be the total number of ones. The following lemma gives the orders of G by θ.

Lemma S1. *The quantity G has order $O(n^{-\theta/2})$.*

Proof. We show this by induction on t. If $t = 0$, then $\theta = 0$ and the result holds. Next, assume the result holds for $t = 0,\ldots,z-1$ with $z \ge 1$. It suffices to show the result holds for $t = z$. Express

$$\begin{aligned} G &= \left(\prod_{j=1}^{J}|D_j|\right)^{-1} E\left\{h(M_s)\prod_{i=s-t+1}^{s} f_{u_i}(X_i)\int_{\prod_{j=1}^{J}D_j}\prod_{j=1}^{J} f_{v_j}(y_j)dy_1\cdots dy_J\right\} \\ &= \left(\prod_{j=1}^{J}|D_j|\right)^{-1} E\left[\prod_{i=s-t+1}^{s} f_{u_i}(X_i)\; E\{h(M_s)f_{u_s}(X_s)\right. \\ &\qquad \left.\left.\int_{\prod_{j=1}^{J}D_j}\prod_{j=1}^{J} f_{v_j}(y_j)dy_1\cdots dy_J \mid X_1,\ldots,X_{s-1}\right\}\right]. \end{aligned}$$

From Proposition 4 and similar to (S5.4) and (S5.5),

$$\begin{aligned} & E\left\{h(M_s)f_{u_s}(X_s)\left(\int_{\prod_{j=1}^{J}D_j}\prod_{j=1}^{J} f_{v_j}(y_j)dy_1\cdots dy_J\right) \mid X_1,\ldots,X_{s-1}\right\} \\ =& \sum_{i_1,\ldots,i_q=-(s-1)}^{s-1} b_s(i_1,\ldots,i_q)\int_{D_{J+1}} h(M_s)f_{u_s}(X_s)\left(\int_{\prod_{j=1}^{J}D_j}\prod_{j=1}^{J} f_{v_j}(y_j)dy_1\cdots dy_J\right)dX_s \\ =& \sum_{i_1,\ldots,i_q=-(s-1)}^{s-1} b_s(i_1,\ldots,i_q)\tilde{h}(M_{s-1})\int_{D_{J+1}} f_{u_s}(X_s)\left(\int_{\prod_{j=1}^{J}D_j}\prod_{j=1}^{J} f_{v_j}(y_j)dy_1\cdots dy_J\right)dX_s \\ =& \sum_{i_1,\ldots,i_q=-(s-1)}^{s-1} \tilde{b}_s(i_1,\ldots,i_q)\tilde{h}(M_{s-1})\int_{\tilde{D}_{J+1}} f_{u_s}(X_s)\left(\int_{\prod_{j=1}^{J}D_j}\prod_{j=1}^{J} f_{v_j}(y_j)dy_1\cdots dy_J\right)dX_s \end{aligned}$$

where $b_s(i_1,\ldots,i_q)$ is a deterministic function on $n, m, i_1, \ldots, i_q, Z_1, \ldots, Z_s, X_1, \ldots, X_{s-t}, M_{s-1}$ and bounded as n goes to infinity, $D_{J+1} = D^1_{J+1} \times \cdots \times D^q_{J+1}$,

$$D^k_{J+1} = \begin{cases} [0,1) \setminus \cup_{j=1}^{s-1} \delta_m(X^k_j), & i_k = 0, \\ \delta_m(X^k_{i_k}), & i_k > 0, \\ \delta_n(X^k_{-i_k}), & i_k < 0, \end{cases}$$

$\tilde{h}(M_{s-1})$ is a deterministic function on $n, m, Z_1, \ldots, Z_{s-1}, X_1, \ldots, X_{s-t}, M_{s-1}$ and bounded as n goes to infinity, $\tilde{b}_s(i_1,\ldots,i_q)$ is a deterministic function on $n, m, i_1, \ldots, i_q, Z_1, \ldots, Z_s, M_{s-1}$ and bounded as n goes to infinity, $\tilde{D}_{J+1} = \tilde{D}^1_{J+1} \times \cdots \times \tilde{D}^q_{J+1}$ and

$$\tilde{D}^k_{J+1} = \begin{cases} [0,1), & i_k = 0, \\ \delta_m(X^k_{i_k}), & i_k > 0, \\ \delta_n(X^k_{-i_k}), & i_k < 0. \end{cases}$$

Thus,

$$\begin{aligned} G \quad = \quad & \sum_{i_1,\ldots,i_q=-(s-1)}^{s-1} \left\{ \left(\prod_{j=1}^{J} |D_j| \right)^{-1} E \left(\prod_{i=s-t+1}^{s-1} f_{u_i}(X_i) \tilde{b}_s(i_1,\ldots,i_q) \tilde{h}(M_{s-1}) \right.\right. \\ & \left.\left. \int_{\tilde{D}_{J+1} \times \prod_{j=1}^{J} D'_j} \prod_{j=1}^{J+1} f_{v_j}(y_j) dy_1 \cdots dy_J \right) \right\}, \end{aligned} \tag{S5.7}$$

where $v_{J+1} = u_s$, $\tilde{D}_{J+1} = \tilde{D}^1_{J+1} \times \cdots \times \tilde{D}^q_{J+1}$,

$$\tilde{D}^k_{J+1} = \begin{cases} [0,1), & i_k = 0, \\ \delta_m(X^k_{i_k}), & i_k > 0, \\ \delta_n(X^k_{-i_k}), & i_k < 0, \end{cases}$$

$D'_j = D^{1\prime}_j \times \cdots \times D^{q\prime}_j$ and

$$D^{k\prime}_j = \begin{cases} \delta_m(y^k_{J+1}), & D^k_j = \delta_m(X^k_s), \\ \delta_n(y^k_{J+1}), & D^k_j = \delta_n(X^k_s), \\ D^k_j, & \text{otherwise.} \end{cases}$$

Therefore, G can be expressed as

$$G = \sum_{i_1,\ldots,i_q} \left(|D_{J+1}| G'_{i_1,\ldots,i_q} \right),$$

where $i_1, \ldots, i_q = -(s-1), \ldots, s-1$, $u_s \subseteq w(i_1,\ldots,i_q) = \{k : i_k \neq 0\}$, $|D_{J+1}| \leq n^{-|w(i_1,\ldots,i_q)|}$ and $G'_{i_1,\ldots,i_q}$ is a term by (S5.6) with the associated matrix $C'_{i_1,\ldots,i_q}$ and the total number of ones $\theta'_{i_1,\ldots,i_q}$. Furthermore, $C'_{i_1,\ldots,i_q}$ is a $(t-1) \times q$ matrix with equal or fewer elements of ones than the first $t-1$ rows of C. If $|i_k| > s-t$, the $(|i_k|-s+t, k)$th element of $C'_{i_1,\ldots,i_q}$ is zero. Other elements of $C'_{i_1,\ldots,i_q}$ are the same with that of the first $t-1$ rows of C. The last row of C has at most $|u_s|$ ones. Therefore, for any $(i_1,\ldots,i_q)$,

$$\theta'_{i_1,\ldots,i_q} \geq \theta - |w(i_1,\ldots,i_q)| - |u_s| \geq \theta - 2|w(i_1,\ldots,i_q)|.$$

By induction,

$$G = O(n^{-\theta'_{i_1,\ldots,i_q}/2} n^{-|w(i_1,\ldots,i_q)|}) = O(n^{-\theta/2}). \tag{S5.8}$$

Consequently, G has order $O(n^{-\theta/2})$. □

We now give the proof of Lemma 7.

Proof. We have argued in (S5.3) that it suffices to show for any $1 \leq t \leq s \leq p$, $|u_{s-t+1}|,\ldots,|u_s| > 1$, a continuous function f and a bounded function h,

$$E\left\{h(X_1,\ldots,X_{s-t}) \prod_{i=s-t+1}^{s} f_{u_i}(X_i)\right\} = o(n^{-t/2}).$$

Therefore, $E\left\{h(X_1,\ldots,X_{s-t}) \prod_{i=s-t+1}^{s} f_{u_i}(X_i)\right\}$ is a term by (S5.6) with $\theta = \sum_{i=s-t+1}^{s} |u_i| \geq 2t$. From Lemma S1, $E\left\{h(X_1,\ldots,X_{s-t}) \prod_{i=s-t+1}^{s} f_{u_i}(X_i)\right\} = O(n^{-t}) = o(n^{-t/2})$. □