RANDOM THRESHOLD DRIVEN TAIL DEPENDENCE MEASURES WITH APPLICATION TO PRECIPITATION DATA ANALYSIS

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Supplementary Materials

Supplementary materials include simulation examples (Section S1) and all technical derivations (Section S2).

S1. Simulation examples

We carry out simulation studies to evaluate the performance of TQCC in testing the null hypothesis of tail independence versus the alternative hypothesis of tail dependence. The performance of the proposed test is compared with two newly published tests in Bacro *et al.* (2010) and Hüsler and Li (2009).

The approach proposed by Hüsler and Li (2009) originated from an extreme value condition of maxima domain of an extreme value distribution which can be characterized by a dependence function $l(x, y)$ satisfying the inequalities

$$x \vee y \leq l(x, y) \leq x + y, \ \ x, y > 0.$$

Testing asymptotic independence is to test $H_0: \ l(x, y) = x+y$ for all $x, y > 0$. Nonparametric estimators of $l(x, y)$ were constructed and the limit distributions were derived. Two tests, the integral test and the supremum test, were derived with critical quantile values being numerically calculated. In our simulation examples, the integral test is used, sub-sample sizes are all set as $n/2$ of the total sample size n in each case, and the tuning parameter k is set as $0.075n$ respectively.

The test proposed by Bacro *et al.* (2010) is called the madogram test named after the well-known madogram used in spatial statistics. The madogram test considers the random variable

$$W = \frac{1}{2}|F(X) - F(Y)|,$$

and an empirical distribution based estimator of $v_W = E(W)$ as

$$\widehat{v_W} = \frac{1}{2n}\sum_{i=1}^{n}|F(X_i) - F(Y_i)|,$$

which results in the limit distribution

$$\sqrt{n}\frac{\widehat{v_W} - \frac{1}{6}}{\widehat{\sigma_W}} \xrightarrow{\mathcal{L}} N(0,1),$$

and a normal test.

Six typical examples of simulated bivariate sequences are chosen as follows:

(1) Componentwise maxima over 10,000 realizations of bivariate normal random variables with ρ = 0.2, 0.4, 0.6, 0.8. Details of this example can be seen in Bacro *et al.* (2010) Example (D4).

(2) $\{(X_i, Y_i)\}$ in Example 1, where (L_{ni}, Q_{ni}) follow (2.6) with $\rho = 0.2$, 0.4, 0.6, 0.8.

(3) Bivariate random samples drawn from $(1/U, 1/(1-U))$, where $U \sim \text{Uniform}(0, 1)$.

(4) Bivariate random samples drawn from $(Z_1 E_1, Z_2 E_1)$, where

$E_1 \sim \text{Exponential}(1)$, and Z_1 and Z_2 are independent unit Fréchet, and independent of E_1.

(5) Bivariate random samples with the joint distribution specified by a Gumbel copula,

$C_\theta(H(u_1), H(u_2)) = e^{-(\sum_{i=1}^{2}[-\log\{H(u_i)\}]^{1/\theta})^\theta}$, $u_1 > 0$, $u_2 > 0$, where $H(u)$ is a unit Fréchet distribution function, and $\theta = \sqrt{1-\rho}$ with $\rho = 0.2$, 0.4, 0.6, 0.8.

(6) Bivariate random samples drawn from two t_4 (Student's t distribution with 4 degrees of freedom) random variables with correlation coefficients $\rho = 0.2$, 0.4, 0.6, 0.8.

Cases of tail independence are given in Examples (1)-(4), while cases of tail dependence are in Examples (5) and (6). Example (5) is also used in Bacro *et al.* (2010) and Hüsler and Li (2009). In our simulation study, for the simulated sample $\{(X_i, Y_i)\}$ in each example, the threshold value is automatically chosen at the smaller one of two $100p$th percentiles of $\{X_i\}$ and $\{Y_i\}$, where $p = .80$, .825, .85, .875, .90, .925, .95, .975, respectively. The number of simulation replications is 1000.

Tables 5-10 report the proportions of rejecting the null hypothesis of tail independence with different sample sizes at the nominal level $\alpha = 0.05$. Column HLT stands for the approach of Hüsler and Li (2009), while column MaD stands for the madogram approach of Bacro *et al.* (2010). One can immediately see that HLT method well controls Type I error rates within its nominal level. Their method is relatively conservative and requires sample size as large as 2000 to get a good performance. In the mean time, MaD method is relatively aggressive, and Type I error rates are not controlled within its nominal level, which is also seen in Bacro *et al.* (2010) for Example (1). Overall, TQCC controls Type I error rates within its nominal level, and it has a better detection performance in tail dependent examples. These tables also show that the performances of TQCC are relatively less sensitive to choices of p in calculating TQCC. We recommend the use of the 95th percentile of transformed sample data. We note that in Example (5) with $\rho = 0.2$, corresponding to $\theta = 0.8944$, (note $\theta = 1$ corresponds to independence), the empirical testing powers are low although the results also show an increasing trend of the empirical testing powers as sample sizes increase. This example suggests that when the null hypothesis is not rejected, cautions should be taken, and a further analysis is recommended.

Table 5: *Empirical Type I error rates for Examples (1)-(4)*

Sample size n=300											
Example		HLT	MaD	TQCC							
				.80	.825	.85	.875	.90	.925	.95	.975
(1)	0.2	.03	.04	.04	.04	.03	.03	.03	.02	.02	.02
ρ	0.4	.04	.05	.03	.03	.03	.03	.03	.02	.02	.01
	0.6	.03	.16	.05	.04	.04	.03	.03	.03	.03	.02
	0.8	.03	.94	.11	.10	.09	.09	.07	.05	.04	.02
(2)	0.2	.02	.11	.03	.03	.03	.03	.03	.03	.03	.02
ρ	0.4	.03	.32	.04	.03	.03	.03	.03	.03	.03	.02
	0.6	.03	.65	.03	.03	.03	.03	.03	.02	.02	.02
	0.8	.04	.91	.03	.03	.03	.03	.03	.02	.02	.02
(3)		.03	0.00	.01	.01	.01	.01	.01	.01	.01	.01
(4)		.04	1.00	.07	.06	.06	.06	.05	.05	.04	.03

Table 6: *Empirical Type I error rates for Examples (1)-(4)*

Sample size n=500											
Example		HLT	MaD	TQCC							
				.80	.825	.85	.875	.90	.925	.95	.975
(1)	0.2	.03	.03	.02	.03	.03	.02	.02	.02	.02	.01
ρ	0.4	.03	.05	.04	.04	.04	.03	.03	.03	.03	.02
	0.6	.05	.19	.05	.05	.05	.05	.04	.04	.03	.03
	0.8	.05	1.00	.13	.12	.11	.09	.08	.07	.05	.03
(2)	0.2	.03	.14	.04	.04	.04	.04	.03	.03	.03	.02
ρ	0.4	.02	.45	.03	.03	.03	.03	.03	.03	.03	.02
	0.6	.03	.86	.04	.04	.04	.04	.04	.03	.03	.03
	0.8	.02	.99	.06	.06	.05	.05	.05	.05	.04	.03
(3)		.02	.00	.02	.02	.03	.03	.03	.03	.03	.03
(4)		.03	1.00	.06	.06	.06	.06	.05	.05	.04	.04

Table 7: *Empirical Type I error rates for Examples (1)-(4)*

Sample size n=1000												
Example		HLT	MaD	TQCC								
				.80	.825	.85	.875	.90	.925	.95	.975
(1)	0.2	.04	.02	.04	.04	.04	.04	.04	.04	.04	.03
ρ	0.4	.03	.05	.05	.05	.05	.05	.05	.04	.04	.03
	0.6	.04	.29	.06	.06	.06	.06	.05	.05	.05	.04
	0.8	.07	1.00	.18	.17	.14	.12	.11	.10	.08	.06
(2)	0.2	.03	.24	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.03
ρ	0.4	.03	.74	0.05	0.05	0.04	0.04	0.04	0.04	0.04	0.03
	0.6	.04	.99	0.04	0.04	0.03	0.04	0.03	0.03	0.03	0.03
	0.8	.03	1.00	0.06	0.06	0.05	0.05	0.05	0.05	0.05	0.04
(3)		.04	0.00	.03	.04	.04	.04	.04	.04	.04	.04
(4)		.05	1.00	.07	.07	.07	.06	.06	.05	.05	.04

Table 8: *Empirical Type I error rates for Examples (1)-(4)*

Sample size n=2000												
Example		HLT	MaD	TQCC								
				.80	.825	.85	.875	.90	.925	.95	.975
(1)	0.2	.06	.02	.04	.04	.04	.05	.05	.04	.04	.03
ρ	0.4	.06	.05	.06	.06	.06	.06	.06	.06	.06	.05
	0.6	.06	.43	.06	.06	.06	.06	.05	.04	.04	.03
	0.8	.14	1.00	.25	.24	.22	.19	.15	.14	.10	.06
(2)	0.2	.03	.40	.04	.04	.04	.04	.04	.04	.05	.04
ρ	0.4	.04	.96	.05	.05	.05	.04	.05	.05	.04	.04
	0.6	.05	1.00	.04	.04	.04	.05	.05	.05	.04	.04
	0.8	.04	1.00	.05	.05	.05	.06	.05	.05	.05	.05
(3)		.03	.00	.03	.04	.04	.04	.04	.04	.04	.04
(4)		.08	1.00	.09	.08	.07	.07	.07	.07	.06	.06

Table 9: *Empirical powers for Examples (5) and (6)*

Sample size n=300											
Example		HLT	MaD	TQCC							
				.80	.825	.85	.875	.90	.925	.95	.975
(5)	0.2	.03	.86	.14	.13	.12	.10	.09	.08	.07	.05
ρ	0.4	.06	1.00	.44	.42	.40	.37	.33	.27	.18	.10
	0.6	.07	1.00	.79	.77	.75	.70	.63	.55	.40	.20
	0.8	.12	1.00	1.00	.99	.99	.99	.97	.94	.85	.53
(6)	0.2	.03	1.00	.75	.75	.75	.74	.73	.72	.69	.65
ρ	0.4	.04	1.00	.85	.85	.84	.82	.81	.79	.76	.71
	0.6	.05	1.00	.95	.95	.94	.93	.93	.92	.89	.85
	0.8	.11	1.00	1.00	1.00	1.00	.99	.99	.99	.98	.95
Sample size n=500											
(5)	0.2	.04	.96	.17	.16	.15	.14	.13	.12	.10	.06
ρ	0.4	.07	1.00	.49	.46	.42	.39	.37	.31	.25	.14
	0.6	.15	1.00	.88	.87	.85	.81	.77	.70	.61	.38
	0.8	.25	1.00	1.00	1.00	1.00	.99	.99	.98	.95	.77
(6)	0.2	.03	1.00	0.80	0.79	0.78	0.77	0.77	0.76	0.74	0.72
ρ	0.4	.06	1.00	0.88	0.87	0.87	0.87	0.86	0.84	0.83	0.79
	0.6	.11	1.00	0.96	0.96	0.96	0.95	0.95	0.94	0.92	0.89
	0.8	.22	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.99	0.98

Table 10: *Empirical powers for Examples (5) and (6)*

Sample size n=1000											
Example		HLT	MaD	TQCC							
				.80	.825	.85	.875	.90	.925	.95	.975
(5)	0.2	.09	1.00	.20	.20	.20	.19	.18	.17	.15	.11
ρ	0.4	.15	1.00	.59	.58	.56	.53	.50	.46	.40	.28
	0.6	.34	1.00	.94	.93	.92	.90	.87	.84	.79	.61
	0.8	.63	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.97
(6)	0.2	.07	1.00	.84	.84	.83	.83	.83	.82	.81	.79
ρ	0.4	.14	1.00	.91	.91	.91	.91	.91	.90	.89	.86
	0.6	.29	1.00	.97	.97	.97	.96	.96	.96	.94	.92
	0.8	.52	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.99
Sample size n=2000											
(5)	0.2	.14	1.00	.25	.25	.24	.23	.22	.21	.19	.15
ρ	0.4	.35	1.00	.70	.68	.66	.64	.61	.57	.52	.42
	0.6	.70	1.00	.97	.96	.96	.95	.94	.92	.88	.77
	0.8	.93	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
(6)	0.2	.15	1.00	.86	.86	.86	.86	.86	.85	.84	.83
ρ	0.4	.30	1.00	.92	.92	.92	.92	.91	.91	.90	.88
	0.6	.59	1.00	.99	.99	.99	.98	.98	.98	.97	.97
	0.8	.86	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

S2. Appendix

Proof of Proposition 1. First, we have

$$\begin{aligned}
&\frac{P(\max(X,T_{nt})>u,\max(Y,T_{nt})>u)}{P(\max(X,T_{nt})>u)}\\
=&\frac{P(T_{nt}\le u)P(X>u,Y>u)/P(X>u)+P(T_{nt}>u)/P(X>u)}{P(T_{nt}\le u)+P(T_{nt}>u)/P(X>u)}\\
=&O\left(1/g(u)\right)+O\left(nu^{1-t}\right) \qquad \text{(S2.1)}\\
=&O\left(\max\left(1/g(u),nu^{1-t}\right)\right). \qquad \text{(S2.2)}
\end{aligned}$$

The first equality from the above identities directly proves (2.7). Notice that $g(u)\to\infty$ as $u\to\infty$. For any $h(u)$ satisfying $h(u)\to\infty$ when $u\to\infty$, we have

$$\begin{aligned}
g^*(u)=&\frac{P(\max(X',T_{nt})>h(u),\max(Y',T_{nt})>h(u))}{P(\max(X',T_{nt})>h(u))}\\
=&\frac{P(T_{nt}\le h(u))P(X'>h(u))+P(T_{nt}>h(u))/P(X'>h(u))}{P(T_{nt}\le h(u))+P(T_{nt}>h(u))/P(X'>h(u))}\\
=&O\left(\max\left(1/h(u),nh^{1-t}(u)\right)\right). \qquad \text{(S2.3)}
\end{aligned}$$

In particular, when $h(u)=\min\left(g(u),n^{-1}u^{t-1},(ng(u))^{\frac{1}{t-1}},u\right)$, we have

$$g^*(u)=O\left(\max\left(1/g(u),nu^{1-t}\right)\right). \qquad \text{(S2.4)}$$

Eq (S2.4) gives

$$\frac{P(X>u,Y>u)}{P(X>u)}=O(g^*(u)). \qquad \text{(S2.5)}$$

Combining Eq (S2.2) and (S2.4), we have

$$\frac{P(\max(X,T_{nt})>u,\max(Y,T_{nt})>u)}{P(\max(X,T_{nt})>u)}=O\left(g^*(u)\right). \qquad \text{(S2.6)}$$

■

Proof of Proposition 2. Denote $K(u) = 1/g(u)$. The results follow from proving

$$P_{nt} = P\Big(\cup_{i=1}^{n}\big\{\frac{X_i}{n} > \frac{T_{n,t}}{n},\ \frac{Y_i}{n} > \frac{T_{n,t}}{n}\big\}\Big) = 1 - P\Big(\cap_{i=1}^{n}\big\{\frac{X_i}{n} > \frac{T_{n,t}}{n},\ \frac{Y_i}{n} > \frac{T_{n,t}}{n}\big\}^{c}\Big)$$
$$= 1 - P_{nt}^{c} \to \begin{cases} 0, & \text{when } t\eta < 1; \\ \\ 1, & \text{when } \eta = 1,\ t > 1, \end{cases}$$

as $n \to \infty$. We have

$$\begin{aligned} P_{nt}^{c} &= P\Big(\cap_{i=1}^{n}\big\{\frac{X_i}{n} > \frac{T_{n,t}}{n},\ \frac{Y_i}{n} > \frac{T_{n,t}}{n}\big\}^{c}\Big) \\ &= \int_0^{\infty} \big[1 - P(\frac{X_i}{n} > x,\ \frac{Y_i}{n} > x)\big]^{n} de^{-\frac{n^{1-t}}{x^t}} \\ &= \int_0^{\infty} \big[1 - P(X_i > nx,\ Y_i > nx)\big]^{n} de^{-\frac{n^{1-t}}{x^t}} \\ &= \int_0^{\infty} \big[1 - K(nx)(1 - e^{-\frac{1}{nx}})\big]^{n} de^{-\frac{n^{1-t}}{x^t}},\ \text{set } nx = n^{1/t}y \\ &= \int_0^{\infty} \big[1 - K(n^{\frac{1}{t}}y)(1 - e^{-\frac{1}{n^{1/t}y}})\big]^{n} de^{-\frac{1}{y^t}} \\ &= \int_0^{\infty} f_{nt}(y) de^{-\frac{1}{y^t}}. \end{aligned}$$

Note that $f_{nt}(y) \le 1$, and $n^{1/t}y(1 - e^{-\frac{1}{n^{1/t}y}}) \to 1$ as $n \to \infty$, we have

$$\begin{aligned} f_{nt} &= \big[1 - \frac{K(n^{\frac{1}{t}}y)}{n^{1/t}y} n^{1/t}y(1 - e^{-\frac{1}{n^{1/t}y}})\big]^{n} \\ &= \big[1 - \frac{K(n^{\frac{1}{t}}y)}{n^{1/t}y} n^{1/t}y(1 - e^{-\frac{1}{n^{1/t}y}})\big]^{\frac{n^{1/t}y}{K(n^{\frac{1}{t}}y)n^{1/t}y(1-e^{-\frac{1}{n^{1/t}y}})} \frac{n^{1-\frac{1}{t}}K(n^{\frac{1}{t}}y)n^{1/t}y(1-e^{-\frac{1}{n^{1/t}y}})}{y}}. \end{aligned}$$

The proof is then completed by noticing the limit of $n^{1-\frac{1}{t}}K(n^{\frac{1}{t}}y)$ and the limit of P_{nt}.

■

Proof of Theorem 1. For $z > 0$, we have

$$P(T_{n,t}/n^{1/t} < z) = P(T_{n,t} < n^{1/t}z) = \exp\{-n/(nz^t)\} = \exp(-1/z^t).$$

Denote $X_i^* = X_i/n^{1/t}$, $Y_i^* = Y_i/n^{1/t}$, $T_t^* = T_{n,t}/n^{1/t}$ and $b_n = n^{1/t}$. We have

$$P\Big(\frac{T_{n,t}}{n}\Big\{\max_{1\le i\le n}\frac{\max(X_i,T_{n,t})}{\max(Y_i,T_{n,t})}+1\Big\}<z\Big) = P\Big(T_t^*\Big\{\max_{1\le i\le n}\frac{\max(X_i^*,T_t^*)}{\max(Y_i^*,T_t^*)}+1\Big\}<n^{1-1/t}z\Big)$$
$$= \int_0^\infty \Big[P\Big\{\frac{\max(X_i^*,x)}{\max(Y_i^*,x)}<\frac{n^{1-1/t}z}{x}-1\Big\}\Big]^n d\exp(-1/x^t). \qquad \text{(S2.7)}$$

We now calculate the integrand in (S2.7). For $w>1$ and $x>0$, we have

$$\begin{aligned} P\Big\{\frac{\max(X_i^*,x)}{\max(Y_i^*,x)}<w\Big\} &= P(X_i^*/Y_i^*<w,\ X_i^*>x,\ Y_i^*>x)+P(X_i^*/x<w,\ X_i^*>x,\ Y_i^*\le x)\\ &\quad +P(x/x<w,\ X_i^*\le x,\ Y_i^*\le x)+P(x/Y_i^*<w,\ X_i^*\le x,\ Y_i^*>x)\\ &\equiv I_1+I_2+I_3+I_4, \end{aligned}$$

where

$$I_1 = \frac{w}{1+w}[1-\exp\{-(1+w)/(wb_nx)\}]-\exp\{-1/(b_nx)\}+\exp\{-2/(b_nx)\}.$$

Similarly,

$I_2 = \exp\{-1/(b_nx)\}[\exp\{-1/(wb_nx)\}-\exp\{-1/(b_nx)\}] = \exp\{-(1+w)/(wb_nx)\}-\exp\{-2/(b_nx)\}$, $I_3 = \exp\{-2/(b_nx)\}$, $I_4 = \exp\{-1/(b_nx)\}[1-\exp\{-1/(b_nx)\}] = \exp\{-1/(b_nx)\}-\exp\{-2/(b_nx)\}$, thus

$$I_1+I_2+I_3+I_4 = 1-\frac{1}{1+w}[1-\exp\{-(1+w)/(wb_nx)\}].$$

Setting $w = nz/(b_nx)-1$ gives

$$(I_1+I_2+I_3+I_4)^n = \Big[1-\frac{1}{nz-b_nx}\{1+o(1)\}\Big]^n \to \exp(-1/z).$$

This concludes

$$\lim_{n\to\infty}\Big[P\Big\{\frac{\max(X_i^*,x)}{\max(Y_i^*,x)}<n^{1-1/t}z/x-1\Big\}\Big]^n = \exp(-1/z).$$

This, together with the dominated convergence theorem and $T_{n,t}/nA_{n,t} \stackrel{\mathrm{P}}{\to} 1$ proves the first part of (i). The proof of the second part of (i) uses similar arguments. For part (ii), letting $\Delta_i = A_{n,t}^{-1}\max(X_i, T_{n,t})/\max(Y_i, T_{n,t})$ and $\Theta_i = A_{n,t}^{-1}\max(Y_i, T_{n,t})/\max(X_i, T_{n,t})$, we have

$$A_{n,t}\mathrm{q}_{T_{n,t}} = \frac{\max(\Delta_i) + \max(\Theta_i) - 2/A_{n,t}}{\max(\Delta_i)\max(\Theta_i) - 1/A_{n,t}^2} = \frac{\{\max(\Delta_i) + \max(\Theta_i)\}\{1 + o_P(1)\}}{\max(\Delta_i)\max(\Theta_i)\{1 + o_P(1)\}}.$$

This together with part (i) and Slutsky's theorem proves part (ii). ■

Proof of Corollary 1. Following the proofs in Theorem 1, we get

$$P\Big\{\frac{\max(X_i^*, x)}{x} < w\Big\} = \exp\{-\frac{1}{xwn^{1/t}}\}.$$

Setting $w = n^{1-1/t}z/x$, then all proofs follow the same proofs as in Theorem 1.

Proof of Theorem 2. Before proving Theorem 2, we need Lemma 1.

Lemma 1. *Suppose that X, $X_1, X_2, \ldots$ are positive random variables. Then $X_n \stackrel{\mathrm{P}}{\to} X$ if and only if there are two sequences of positive random variables $\xi_n^{(1)}$ and $\xi_n^{(2)}$ such that $\xi_n^{(1)} \stackrel{\mathrm{P}}{\to} 1$, $\xi_n^{(2)} \stackrel{\mathrm{P}}{\to} 1$, and $\xi_n^{(1)}X \le X_n \le \xi_n^{(2)}X$, $n = 1, 2, \ldots$.*

Proof. The sufficient part is obvious. For the necessary part, define $\widetilde{X}_n^{(1)} = \max(X_n, X)$ and $\widetilde{X}_n^{(2)} = \min(X_n, X)$. Then for $j = 1, 2$, $\widetilde{X}_n^{(j)}$ are measurable and $\widetilde{X}_n^{(j)} \stackrel{\mathrm{P}}{\to} X$ as $n \to \infty$. Setting $\xi_n^{(j)} = \widetilde{X}_n^{(j)}/X$, $j = 1, 2$, completes the proof. ■

We now show Theorem 2. By Lemma 1, there exist $\xi_n^{(j)} > 0$, $\xi_n^{(j)} \stackrel{\mathrm{P}}{\to} 1$, $j = 1, 2$, as $n \to \infty$, and $\xi_n^{(1)}u^* \le u_n^* \le \xi_n^{(2)}u^*$, which imply

$$\min(1, \xi_n^{(1)})\max(X_i, u^*a_n) \quad \le \quad \max(X_i, u_n) \le \max(1, \xi_n^{(2)})\max(X_i, u^*a_n),$$

$$\min(1,\xi_n^{(1)})\max(Y_i,u^*a_n) \quad\le\quad \max(Y_i,u_n)\le \max(1,\xi_n^{(2)})\max(Y_i,u^*a_n).$$

Then we have

$$\begin{aligned}\frac{\min(1,\xi_n^{(1)})\max(X_i,u^*a_n)}{\max(1,\xi_n^{(2)})\max(Y_i,u^*a_n)} &\le \frac{\max(X_i,u_n)}{\max(Y_i,u_n)}\le \frac{\max(1,\xi_n^{(2)})\max(X_i,u^*a_n)}{\min(1,\xi_n^{(1)})\max(Y_i,u^*a_n)},\\ \frac{\min(1,\xi_n^{(1)})\max(Y_i,u^*a_n)}{\max(1,\xi_n^{(2)})\max(X_i,u^*a_n)} &\le \frac{\max(Y_i,u_n)}{\max(X_i,u_n)}\le \frac{\max(1,\xi_n^{(2)})\max(Y_i,u^*a_n)}{\min(1,\xi_n^{(1)})\max(X_i,u^*a_n)}.\end{aligned}$$

Noticing that for $b_n = n(1-\exp\{-1/(u^*a_n)\})$, we have

$$\begin{aligned} & P\Big\{\max_{1\le i\le n}\frac{\min(1,\xi_n^{(1)})\max(X_i,u^*a_n)}{\max(1,\xi_n^{(2)})\max(Y_i,u^*a_n)}\le b_nx-1,\ \max_{1\le i\le n}\frac{\min(1,\xi_n^{(1)})\max(Y_i,u^*a_n)}{\max(1,\xi_n^{(2)})\max(X_i,u^*a_n)}\le b_ny-1\Big\}\\ \ge\ & P\Big\{\max_{1\le i\le n}\frac{\max(X_i,u_n)}{\max(Y_i,u_n)}\le b_nx-1,\ \max_{1\le i\le n}\frac{\max(Y_i,u_n)}{\max(X_i,u_n)}\le b_ny-1\Big\}\\ \ge\ & P\Big\{\max_{1\le i\le n}\frac{\max(1,\xi_n^{(2)})\max(X_i,u^*a_n)}{\min(1,\xi_n^{(1)})\max(Y_i,u^*a_n)}\le b_nx-1,\ \max_{1\le i\le n}\frac{\max(1,\xi_n^{(2)})\max(Y_i,u^*a_n)}{\min(1,\xi_n^{(1)})\max(X_i,u^*a_n)}\le b_ny-1\Big\}.\end{aligned}$$

Since $\max(1,\xi_n^{(k)})/\min(1,\xi_n^{(j)}) \xrightarrow{\text{P}} 1$ for all $j,k=1,2$, and by Theorem 5.2 of Zhang (2008b), $\max_{1\le i\le n}\{\max(X_i,u^*a_n)/\max(Y_i,u^*a_n)\}$ and $\max_{1\le i\le n}\{\max(Y_i,u^*a_n)/\max(X_i,u^*a_n)\}$ are tail independent, we have by Slutsky's theorem that both the first and the last probability in the above inequalities converge to $\exp(-1/x-1/y)$ as $n\to\infty$, hence the middle one converges to the same limit. The rest of the proof is similar to the proof in Theorem 1. ■

Proof of Corollary 2 is obvious.

Proof of Theorem 3. It can be shown that $c_1\le X_i/Y_i\le c_2$. Using the fact $\min(c_1,1)\le \max(X_i,u_n)/\max(Y_i,u_n)\le\max(1,c_2)$, we have that with probability tending to 1,

$$1\le \max_{1\le i\le n}\frac{\max(X_i,u_n)}{\max(Y_i,u_n)}\le \max(1,c_2),\quad 1\le \max_{1\le i\le n}\frac{\max(Y_i,u_n)}{\max(X_i,u_n)}\le \max(1,1/c_1).$$

So $\mathrm{q}_{u_n} \geq f(\max(1, c_2), \max(1, 1/c_1)) > 0$ (for f in (2.5)) completes the proof. ■

Proof of Proposition 3. Under **Condition A1** and $\alpha_1 + \alpha_2 = 1$ and $\beta_1 + \beta_2 = 1$ in (2.15), let $\alpha = \alpha_1, \beta = 1 - \beta_1$, then the corresponding bivariate distribution is

$$F(x, y) = \exp\big[- \max\{\alpha/x,\ (1-\beta)/y\} - \max\{(1-\alpha)/x,\ \beta/y\}\big].$$

Notice that $\alpha, \beta \geq 0$, $\alpha + \beta < 1$ imply $\beta/(1-\alpha) < (1-\beta)/\alpha$. Thus,

$$F(x, y) = \begin{cases} \exp(-1/y), & y/x \leq \beta/(1-\alpha), \\ \exp\{-(1-\beta)/y - (1-\alpha)/x\}, & \beta/(1-\alpha) < y/x < (1-\beta)/\alpha, \\ \exp(-1/x), & (1-\beta)/\alpha \leq y/x. \end{cases} \tag{S2.8}$$

We have

$$\begin{aligned} \lim_{u\to\infty} \frac{P(X > u,\ Y > u)}{P(X > u)} &= \lim_{u\to\infty} \frac{1 - F(u) - F(u) + F(u, u)}{1 - F(u)} = 1 - \lim_{u\to\infty} \frac{F(u) - F(u, u)}{1 - F(u)}, \\ &= 1 - \lim_{u\to\infty} \frac{e^{-1/u} - e^{-(2-(\alpha+\beta))/u}}{1 - e^{-1/u}} \\ &= 1 - \lim_{u\to\infty} \frac{-\frac{1}{u^2} e^{-1/u} + \frac{2-(\alpha+\beta)}{u^2} e^{-(2-(\alpha+\beta))/u}}{\frac{1}{u^2} e^{-1/u}} \\ &= 1 + \lim_{u\to\infty} (1 - (2 - (\alpha + \beta)) e^{1-(\alpha+\beta)/u}) = \alpha + \beta. \end{aligned}$$

By (S2.8), we have

$$\max \frac{Y_i}{X_i} \xrightarrow{\mathrm{P}} \frac{1-\beta}{\alpha}, \qquad \max \frac{X_i}{Y_i} \xrightarrow{\mathrm{P}} \frac{1-\alpha}{\beta}.$$

Then we have

$$\mathrm{q}_{u_n=0} \xrightarrow{\mathrm{P}} \frac{\frac{1-\beta}{\alpha} + \frac{1-\alpha}{\beta} - 2}{\frac{(1-\alpha)(1-\beta)}{\alpha\beta} - 1} = \alpha + \beta.$$

Similar to the proof of Theorem 3, we have that with probability tending to 1,

$$1 \le \max_{1\le i\le n} \frac{\max(X_i, u_n)}{\max(Y_i, u_n)} \le \frac{1-\beta}{\alpha}, \quad 1 \le \max_{1\le i\le n} \frac{\max(Y_i, u_n)}{\max(X_i, u_n)} \le \frac{1-\alpha}{\beta},$$

which shows $\mathrm{q}_{u_n} \to \alpha+\beta$. ■

Proof of Proposition 4. Notice that $\{\epsilon_{i1}/\epsilon_{i2}\}$ is a Cauchy random variable having its density function and distribution function as:

$$p(z) = \frac{1}{\pi}\frac{\sqrt{1-\rho^2}}{(z-\rho)^2+(1-\rho^2)}, \ F(z) = \frac{1}{2}+\frac{1}{\pi}\tan^{-1}\frac{z-\rho}{\sqrt{1-\rho^2}}.$$

Considering $1-F(\gamma_n) = \frac{1}{n}$, we have $1-\frac{1}{n} = \frac{1}{2}+\frac{1}{\pi}\tan^{-1}\frac{\gamma_n-\rho}{\sqrt{1-\rho^2}}$, i.e., $\tan(\frac{\pi}{2}-\frac{\pi}{n}) = \frac{\gamma_n-\rho}{\sqrt{1-\rho^2}}$, which implies

$$\frac{\sqrt{1-\rho^2}}{\gamma_n-\rho} = \cot(\frac{\pi}{n}) \sim \frac{\pi}{n}.$$

Then we have

$$\gamma_n \approx \frac{\sqrt{1-\rho^2}n}{\pi}+\rho \approx \frac{\sqrt{1-\rho^2}n}{\pi},$$

which implies

$$\frac{\pi}{n\sqrt{1-\rho^2}}\max_{1\le i\le n}\{\epsilon_{i1}/\epsilon_{i2}\} \xrightarrow{\mathcal{L}} e^{-1/x}, \text{ for } x>0,$$

and the proof follows the same steps of the proof for Theorem 1. ■

Proof of Theorem 4. The following lemma facilitates the proof of Theorem 4.

Lemma 2. *Suppose that $\{X_i\}_{i=1}^n$ is a random sample from the distribution*

$F_{\gamma_0}(x) = \exp(-1/x^{\gamma_0})$, with $x>0$, and the true shape parameter γ_0. Suppose that the estimator of γ_0 is $\widehat{\gamma} = \widehat{\gamma}(X_1,\dots,X_n)$ satisfying $n^\alpha(\widehat{\gamma}-\gamma_0) \xrightarrow{\mathcal{L}} W$, for some $\alpha>0$

and some random variable W. Then as $n \to \infty$,

$$\max_{1\le i\le n} |\log\{F_{\gamma_0}(X_i)\}/\log\{F_{\widehat{\gamma}}(X_i)\} - 1| \xrightarrow{\text{P}} 0.$$

Proof. We only prove the case of $\gamma_0 > 0$; cases of $\gamma_0 = 0$ and $\gamma_0 < 0$ can be similarly treated. We have that for any finite integer $k > 0$,

$$\begin{aligned}
&\max_{1\le i\le n} |\log(F_{\gamma_0}(X_i))/\log(F_{\widehat{\gamma}}(X_i)) - 1| = \max_{1\le i\le n} |X_i^{\widehat{\gamma}}/X_i^{\gamma_0} - 1| \\
&\le n^{\alpha}|\widehat{\gamma} - \gamma_0| \cdot \frac{1}{n^{\alpha}} \max_{1\le i\le n} |\log(X_i)| + \cdots \\
&\quad + n^{(k-1)\alpha}\frac{|\widehat{\gamma} - \gamma_0|^{k-1}}{(k-1)!} \cdot \frac{1}{n^{(k-1)\alpha}} \max_{1\le i\le n} |\log(X_i)|^{k-1} \\
&\quad + n^{k\alpha}\frac{|\widehat{\gamma} - \gamma_0|^{k}}{k!} \cdot \frac{1}{n^{k\alpha}} \max_{1\le i\le n} |\log(X_i)|^{k} X_i^{\gamma-\gamma_0},
\end{aligned}$$

where γ is between γ_0 and $\widehat{\gamma}$. It suffices to show two parts:

$$\frac{1}{n^{\alpha}} \max_{1\le i\le n} |\log(X_i)| \xrightarrow{\text{P}} 0, \quad \frac{1}{n^{k\alpha}} \max_{1\le i\le n} |\log(X_i)|^k X_i^{\gamma-\gamma_0} \xrightarrow{\text{P}} 0.$$

For the first part, we can show a more general result. For any $\epsilon > 0$, $\beta > 0$ and $1 \le \ell \le k$, we have that as $n \to \infty$,

$$\begin{aligned}
&P\Big(\frac{1}{n^{\ell\beta}} \max_{1\le i\le n} |\log(X_i)|^{\ell} \ge \epsilon\Big) = 1 - [P\{\exp(-\epsilon^{1/\ell} n^{\beta}) \le X_i \le \exp(\epsilon^{1/\ell} n^{\beta})\}]^n \\
= \quad &1 - [\exp\{-\exp(-\epsilon^{\gamma_0/\ell}\, n^{\gamma_0\beta})\} - \exp\{-\exp(\epsilon^{\gamma_0/\ell}\, n^{\gamma_0\beta})\}]^n \to 0. \qquad \text{(S2.9)}
\end{aligned}$$

Now we show the second part. We assume that s is a large enough positive constant.

Then

$$\begin{aligned}
&P(\max_{1\le i\le n} X_i^{\gamma-\gamma_0} \ge n^{s\alpha}\epsilon) = P(\max_{1\le i\le n} X_i^{\gamma-\gamma_0} \ge n^{s\alpha}\epsilon,\ \gamma-\gamma_0>0)\\
&\quad +P(\max_{1\le i\le n} X_i^{\gamma-\gamma_0} \ge n^{s\alpha}\epsilon,\ \gamma-\gamma_0<0)\\
&= P(\gamma-\gamma_0>0) - P(\max_{1\le i\le n} X_i^{\gamma-\gamma_0} \le n^{s\alpha}\epsilon,\ \gamma-\gamma_0>0)\\
&\quad +P(\gamma-\gamma_0<0) - P(\max_{1\le i\le n} \frac{1}{X_i^{\gamma_0-\gamma}} \le n^{s\alpha}\epsilon,\ \gamma-\gamma_0<0)\\
&= P(\widehat{\gamma}-\gamma_0>0) - P(\max_{1\le i\le n} X_i^{\gamma-\gamma_0} \le n^{s\alpha}\epsilon,\ \widehat{\gamma}-\gamma_0>0)\\
&\quad +P(\widehat{\gamma}-\gamma_0<0) - P(\max_{1\le i\le n} \frac{1}{X_i^{\gamma_0-\gamma}} \le n^{s\alpha}\epsilon,\ \widehat{\gamma}-\gamma_0<0). \qquad \text{(S2.10)}
\end{aligned}$$

For sufficiently large n, we have $n\epsilon>1$. Denote $s=s^*\alpha+1$, where $s^*>0$. Then

$$\begin{aligned}
&P(\widehat{\gamma}-\gamma_0>0) \ge P(\max_{1\le i\le n} X_i^{\gamma_0} \le n^{\frac{s^*\alpha\gamma_0}{\gamma-\gamma_0}}(n\epsilon)^{\frac{\gamma_0}{\gamma-\gamma_0}},\ \widehat{\gamma}-\gamma_0>0)\\
\ge\ & P(\max_{1\le i\le n} X_i^{\gamma_0} \le n^{\frac{s^*\alpha\gamma_0}{|\widehat{\gamma}-\gamma_0|}}(n\epsilon)^{\frac{\gamma_0}{|\widehat{\gamma}-\gamma_0|}},\ \widehat{\gamma}-\gamma_0>0) \qquad \text{(S2.11)}\\
=\ & P(n^{-1}\max_{1\le i\le n} X_i^{\gamma_0} \le n^{\frac{n^\alpha s^*\alpha\gamma_0}{n^\alpha|\widehat{\gamma}-\gamma_0|}-1}(n\epsilon)^{\frac{n^\alpha\gamma_0}{n^\alpha|\widehat{\gamma}-\gamma_0|}},\ n^\alpha(\widehat{\gamma}-\gamma_0)>0);
\end{aligned}$$

similarly,

$$\begin{aligned}
&P(\widehat{\gamma}-\gamma_0<0) \ge P(\max_{1\le i\le n} X_i^{-\gamma_0} \le n^{\frac{s^*\alpha\gamma_0}{\gamma_0-\gamma}}(n\epsilon)^{\frac{\gamma_0}{\gamma_0-\gamma}},\ \widehat{\gamma}-\gamma_0<0)\\
\ge\ & P(\max_{1\le i\le n} X_i^{-\gamma_0} \le n^{\frac{s^*\alpha\gamma_0}{|\gamma_0-\widehat{\gamma}|}}(n\epsilon)^{\frac{\gamma_0}{|\gamma_0-\widehat{\gamma}|}},\ \widehat{\gamma}-\gamma_0<0) \qquad \text{(S2.12)}\\
=\ & P(\max_{1\le i\le n} X_i^{-\gamma_0}-\log(n) \le n^{\frac{n^\alpha s^*\alpha\gamma_0}{n^\alpha|\gamma_0-\widehat{\gamma}|}}(n\epsilon)^{\frac{n^\alpha\gamma_0}{n^\alpha|\gamma_0-\widehat{\gamma}|}}-\log(n),\ n^\alpha(\widehat{\gamma}-\gamma_0)<0).
\end{aligned}$$

Note that $X_i^{\gamma_0}$ and $X_i^{-\gamma_0}$ are standard Fréchet and exponential random variables respectively, and we have $n^{-1}\max_{1\le i\le n} X_i^{\gamma_0} \xrightarrow{\mathcal{L}} X_1^{\gamma_0}$, and $\max_{1\le i\le n} X_i^{-\gamma_0} - \log(n) \xrightarrow{\mathcal{L}} \log(X_1^{\gamma_0})$.

It is obvious that $n^{\frac{n^{\alpha}s^{*}\alpha\gamma_0}{n^{\alpha}|\widehat{\gamma}-\gamma_0|}-1}(n\epsilon)^{\frac{n^{\alpha}\gamma_0}{n^{\alpha}|\widehat{\gamma}-\gamma_0|}} \xrightarrow{\mathrm{P}} \infty$, and $n^{\frac{n^{\alpha}s^{*}\alpha\gamma_0}{n^{\alpha}|\gamma_0-\widehat{\gamma}|}}(n\epsilon)^{\frac{n^{\alpha}\gamma_0}{n^{\alpha}|\gamma_0-\widehat{\gamma}|}}-\log(n) \xrightarrow{\mathrm{P}} \infty$. Applying these relations to (S2.10)-(S2.12), we have $P(\max_{1\leq i\leq n} X_i^{\gamma-\gamma_0} \geq n^{s\alpha}\epsilon) \to 0$ as $n \to \infty$. Note that as $n \to \infty$, for $k > s$, we have

$$
\begin{aligned}
&P(\max_{1\leq i\leq n} |\log(X_i)|^k X_i^{\gamma-\gamma_0} \leq n^{k\alpha}\epsilon) \\
&\geq P(\max_{1\leq i\leq n} |\log(X_i)|^k \leq n^{k\alpha-s\alpha}\epsilon^{1/2},\ \max_{1\leq i\leq n} X_i^{\gamma-\gamma_0} \leq n^{s\alpha}\epsilon^{1/2}) \to 1
\end{aligned}
$$

which completes the proof. ■

We now prove Theorem 4. By Lemma 2, for any $\epsilon > 0$, as $n \to \infty$,

$$P(1-\epsilon < X_i^{\widehat{\xi}_X}/X_i^{\xi_{0,X}} < 1+\epsilon,\ i = 1,\ldots,n) \to 1$$

which implies that for any $u_n > 0$,

$$P(1-\epsilon < \max(X_i^{\widehat{\xi}_X}, u_n)/\max(X_i^{\xi_{0,X}}, u_n) < 1+\epsilon,\ i = 1,\ldots,n) \to 1$$

and thus

$$P\Big(1-\epsilon < \min_{1\leq i\leq n} \frac{\max(X_i^{\widehat{\xi}_X}, u_n)}{\max(X_i^{\xi_{0,X}}, u_n)} \leq \max_{1\leq i\leq n} \frac{\max(X_i^{\widehat{\xi}_X}, u_n)}{\max(X_i^{\xi_{0,X}}, u_n)} < 1+\epsilon\Big) \to 1.$$

Therefore, by a similar argument in Lemma 1, there exist two sequences of positive random variables $\xi_n^{(1)}$ and $\xi_n^{(2)}$ such that $\xi_n^{(1)} \xrightarrow{\mathrm{P}} 1$, $\xi_n^{(2)} \xrightarrow{\mathrm{P}} 1$, and $\xi_n^{(1)} \max(X_i^{\xi_{0,X}}, u_n) \leq \max(X_i^{\widehat{\xi}_X}, u_n) \leq \xi_n^{(2)} \max(X_i^{\xi_{0,X}}, u_n)$, $i = 1,\ldots,n;\ n = 1,2,\ldots$ A similar argument is true for marginally transformed $Y_i^{\widehat{\xi}_Y}$. With these established notations, the proof of the theorem can be completed by following the same procedure used in the proof of Theorem 2. ■

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