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ON CONSTRUCTION OF MARGINALLY COUPLED DESIGNS

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Abstract: Intended for computer experiments with both qualitative and quantitative factors, marginally coupled designs were introduced by Deng, Hung and Lin (2015) as a more economical strategy than the original, sliced space-filling designs. Among the designs constructed in Deng, Hung and Lin (2015), the corresponding designs for quantitative factors possess only the one-dimensional space-filling property with respect to each level of any factor in designs for qualitative factors. In addition, their designs for quantitative factors have clustered points. To avoid clustered points and enhance two- and higher-dimensional space-filling property in designs for quantitative factors, we propose three approaches to construct marginally coupled designs. Theoretical results of marginally coupled designs are also derived.

Key words and phrases: Cascading Latin hypercube, completely resolvable, Latin hypercube, orthogonal array, projection, Rao-Hamming construction, space-filling.

1. Introduction

Computer experiments are becoming ubiquitous in science, engineering, and service for studying complex phenomena that might otherwise be too time-consuming, expensive, or unethical to observe (Santner, Williams and Notz (2003); Fang, Li and Sudjianto (2006)). They use computer codes to represent and implement the underlying physical mechanisms. It is not uncommon that computer codes have both qualitative and quantitative factors (Qian, Wu and Wu (2008); Han et al. (2009); Zhou, Qian and Zhou (2011)). For example, Schmidt, Cruz and Iyengar (2005) describe a data center computer experiment that involves qualitative factors (such as diffuser location and hot-air return-vent location) and quantitative factors (such as rack power and diffuser flow rate).

A commonly used class of designs for computer experiments with quantitative variables is space-filling designs, which aim to locate the design points evenly over the entire design space. Space-filling properties are typically achieved by using such criteria as distance-based optimality (Johnson, Moore and Ylvisaker (1990)), the low-dimensional projection property (Tang (1993)), discrepancy (Fang and Lin (2003)), and orthogonality (Bingham, Sitter and Tang (2009)). A special class of space-filling designs consists of Latin hypercube designs

(McKay, Beckman and Conover (1979)). Their key feature is that, if the range of each input variable is divided into the same number of equally-spaced intervals as the design run size, there is precisely one point in each interval; this is known as one-dimensional uniformity, or maximum stratification. There are numerous Latin hypercube designs of a given run size. For a detailed account of Latin hypercubes and other space-filling designs, we refer to the book chapter Lin and Tang (2015).

For computer experiments with qualitative and quantitative factors, a naive approach is to use an orthogonal array (known also as fractional factorial design, see Hedayat, Sloane and Stufken (1999); Wu and Hamada (2011)) for qualitative factors and a space-filling design for quantitative factors, and randomly combine runs of an orthogonal array and runs of a space-filling design, where each run of an orthogonal array is a level combination of the qualitative factors. Such an approach is, in general, ineffective in accurately estimating interactions between qualitative and quantitative factors.

As the first systematic solution to designs for computer experiments with qualitative and quantitative factors, Qian and Wu (2009) introduced sliced space-filling designs. Such a design consists of slices of space-filling designs with each slice corresponding to a level combination of the qualitative factors. The run sizes of such designs can be very large even for a moderate number of qualitative factors. To address this issue, Deng, Hung and Lin (2015) proposed a new type of designs, called *marginally coupled designs*, with the design points for quantitative factors forming a Latin hypercube design, and for each level of any qualitative factor, the corresponding design points for quantitative factors forming a small Latin hypercube design. In their construction, this small Latin hypercube design does not guarantee two- or higher- dimensional projection property and the whole design for quantitative factors can have clustered points. The two- or higher-dimensional projection property is desirable because it can be viewed as a stepping stone to construct space-filling designs, and designs with such a property can achieve higher variance reduction in numerical integration (Tang (1993)). Clustered points should be avoided in general, and they can create an ill-conditioning issue (Ranjan, Haynes and Karsten (2011)). Motivated by these observations, we introduce three methods for constructing marginally coupled designs with improved space-filling property in designs for quantitative factors.

The first method offers a better two- or higher-dimensional projection property in the design for quantitative factors in a marginally coupled design. This construction couples a completely resolvable orthogonal array with a smaller orthogonal array. The second method provides marginally coupled designs in which designs for quantitative factors do not have clustered points. A key tool in this method is the Rao-Hamming construction (Hedayat, Sloane and Stufken (1999)). Our third method is a generalization of the second, and it allows marginally coupled designs constructed to accommodate more quantitative factors while maintaining the property that the whole design for the quantitative factors has no clustered points.

We also provide some new existence results for s-level completely resolvable orthogonal arrays of λs^2 runs, and summarize the existence of orthogonal arrays of λs runs (Some of existence results are due to Hedayat, Sloane and Stufken (1999)). These existence results are useful in obtaining the catalogue of marginally coupled designs with better projection properties in designs for quantitative factors.

This paper is organized as follows. Section 2 introduces definitions, notation, and background. Section 3.1 provides a new way to characterize marginally coupled designs. Sections 3.2 and 3.3 introduce three classes of marginally coupled designs that have improved space-filling property in designs for quantitative factors. Section 3.2 also discusses the existence of completely resolvable orthogonal arrays used in the construction. Concluding remarks are provided in Section 4. Some proofs and tables of designs constructed are relegated to appendices.

2. Notation, Definitions and Background

An *orthogonal array* A of strength t is an $n \times m$ matrix with the jth column taking s_j distinct levels and, for every $n \times t$ submatrix of A, each of all possible level combinations appears equally often (Hedayat, Sloane and Stufken (1999)). When not all s_j's are equal, the orthogonal array is mixed or asymmetric, and is denoted by $\text{MOA}(n, s_1^{m_1} \cdots s_k^{m_k}, t)$, where the first m_1 columns have s_1 levels, the next m_2 columns have s_2 levels, and so on. If all s_j's are equal to s, the orthogonal array is symmetric and denoted by $\text{OA}(n, m, s, t)$. Throughout, we denote the s levels by $0, \ldots, s-1$. An $\text{OA}(n, m, s, 2)$, say A, is said to be α-resolvable if it can be expressed as $A = (A_1^{\mathrm{T}}, \ldots, A_{n/(s\alpha)}^{\mathrm{T}})^{\mathrm{T}}$ such that each of $A_1, \ldots, A_{n/(s\alpha)}$ is an $\text{OA}(s\alpha, m, s, 1)$. If $\alpha = 1$, the orthogonal array is called *completely resolvable* and denoted by $\text{CROA}(n, m, s, 2)$. Table 1 displays an $\text{OA}(9, 4, 3, 2)$ whose last three columns form a $\text{CROA}(9, 3, 3, 2)$.

A relevant concept for orthogonal arrays is difference schemes. It can be used to construct orthogonal arrays and completely resolvable orthogonal arrays. An $r \times c$ array is called a difference scheme if the entries are taken from s elements, and for each vector of difference between any two distinct columns of the array, each of s levels appears equally often (Bose and Bush (1952)). Such a difference scheme is denoted by $\text{D}(r, c, s)$. For example, consider columns $1, 2, 4$ (or columns $1, 3, 4$)

Table 1. An orthogonal array $\text{OA}(9,4,3,2)$ and a Latin hypercube $\text{L}(9,4)$.

$\text{OA}(9,4,3,2)$				$\text{L}(9,4)$			
0	0	0	0	0	0	0	0
0	1	1	2	1	3	3	6
0	2	2	1	2	6	6	3
1	0	1	1	3	1	4	4
1	1	2	0	4	4	7	1
1	2	0	2	5	7	1	7
2	0	2	2	6	2	8	8
2	1	0	1	7	5	2	5
2	2	1	0	8	8	5	2

of the orthogonal array in Table 1, with rows 1−3, 4−6 or 7−9 corresponding to a $\text{D}(3,3,3)$.

A Latin hypercube $L=(l_{ij})$ with n runs and k factors, denoted by $\text{L}(n,k)$, is an $n \times k$ matrix in which each column is a random permutation of equally-spaced n levels (McKay, Beckman and Conover (1979)). Table 1 provides an example of a Latin hypercube design of 9 runs for 4 factors. Without loss of generality, we use $0,\dots,n-1$ to denote the n levels. A Latin hypercube has the feature that each dimension achieves maximum stratification. To guarantee higher-dimensional stratification, Tang (1993) introduced orthogonal array-based Latin hypercubes. The procedure works as follows. Let A be an $\text{OA}(n,m,s,t)$, and replace the $r=n/s$ positions having level i by a random permutation of $\{ir,\dots,(i+1)r-1\}$, for $i=0,\dots,s-1$. The resulting design, known as an $\text{OA}(n,m,s,t)$-based Latin hypercube, achieves t-dimensional stratification. For the orthogonal array $\text{OA}(9,4,3,2)$, say A, in Table 1, the Latin hypercube design in the table is an example of Latin hypercubes based on A. Although the approach in Tang (1993) was only applied to orthogonal arrays, it in principle can be applied to any array to obtain Latin hypercubes. For simplicity, we refer this approach as to *level replacement-based Latin hypercube* approach.

For a computer experiment with m qualitative factors and k quantitative factors, let $D_1=\text{MOA}(n,s_1\cdots s_m,t)$ and $D_2=\text{L}(n,k)$ be the designs for qualitative and quantitative factors, respectively. A design $D=(D_1,D_2)$ is called a marginally coupled design if for $j=1,\dots,m$, and for each level of the jth factor in D_1, the corresponding rows in D_2 have the property that when projected into each quantitative factor, the resulting points have exactly one level from each of the n/s_j equally-spaced intervals. We use $\text{MCD}(D_1,D_2)$ to denote such a design. Lemma 1 is from Deng, Hung and Lin (2015).

Lemma 1. *Given D_1 is an $\text{OA}(n,m,s,2)$, a marginally coupled design exists if and only if D_1 is a completely resolvable orthogonal array.*

3. Main Results

This section provides a new way to characterize marginally coupled designs and, inspired by this, we introduce three approaches to constructing marginally coupled designs that have improved space-filling property in designs for quantitative factors.

3.1. Characterization of marginally coupled designs

Lemma 1 establishes the condition of D_1 in an $\text{MCD}(D_1, D_2)$ when D_1 is an s-level orthogonal array. We further study the condition of D_2 and provide a unified result of the necessary and sufficient condition for a marginally coupled design. To derive the result, we define a matrix $\tilde{D}_2$ based on a $D_2 = \text{L}(n, k)$. For $i = 1, \ldots, n$ and $j = 1, \ldots, k$, let

$$\tilde{D}_{2,ij} = \left\lfloor \frac{D_{2,ij}}{s} \right\rfloor, \tag{3.1}$$

where $D_{2,ij}$ and $\tilde{D}_{2,ij}$ are the (i, j)th entry of D_2 and $\tilde{D}_2$, respectively, and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Conversely, D_2 can be obtained from $\tilde{D}_2$ via the *level replacement-based Latin hypercube* approach. Here $\tilde{D}_2$ is an $\text{OA}(n, k, n/s, 1)$.

Proposition 1. *Given D_1 is an $OA(n, m, s, 2)$, D_2 is an $\text{L}(n, k)$ and $\tilde{D}_2$ is defined via (3.1), then (D_1, D_2) is a marginally coupled design if and only if for $j = 1, \ldots, k$, $(D_1, \tilde{d}_j)$ is an $MOA(n, s^m(n/s), 2)$, where $\tilde{d}_j$ is the jth column of $\tilde{D}_2$.*

Proposition 1 follows by the definition of $\text{MCD}(D_1, D_2)$ and the relationship in (3.1) between $\tilde{D}_2$ and D_2. This proposition provides the necessary and sufficient condition on both D_1 and D_2 for (D_1, D_2) to be a marginally coupled design when D_1 is an s-level orthogonal array. The condition in Proposition 1 contains the one in Lemma 1, because $(D_1, \tilde{d}_j) = \text{MOA}(n, s^m(n/s), 2)$ for $j = 1, \ldots, k$ implies that D_1 is a completely resolvable orthogonal array.

Constructions 1 and 2 in Deng, Hung and Lin (2015) provide marginally coupled designs when D_1's are s-level orthogonal arrays of s^2 runs and λs^2 runs, $\lambda \geq 2$, respectively. Their Construction 1 yields $(D_1, \tilde{D}_2) = \text{OA}(s^2, m + k, s, 2)$, and thus has restrictive run sizes and numbers of factors. Their Construction 2 obtains columns of D_2 by applying the *level replacement-based Latin hypercube* approach independently to the last column of the same $\text{MOA}(n, s^m(n/s), 2)$. Thus the columns of the corresponding $\tilde{D}_2$ can be obtained from one to another by level permutations. This results in clustered points in D_2. Motivated by this observation, we propose a method in Section 3.3 for constructing D_2 whose columns cannot be obtained from one to another by level permutations. We introduce a method for constructing D_2 such that, for each level of each factor in D_1, the corresponding rows can achieve stratification in any two- or higher-dimensional projections in Section 3.2. Throughout, two columns are said to be equivalent if one can be obtained from the other by level permutations.

3.2. Construction of MCDs with low dimensional stratification

This section presents a construction for $\mathrm{MCD}(D_1, D_2)$'s of n runs, m qualitative factors, and k quantitative factors through an $\mathrm{MOA}(n, s^m(n/s), 2)$ and an $\mathrm{OA}(n/s, k, s_1, 2)$, where s_1 and s can be different. The key feature of such a design is that, with respect to each level of any column in D_1, design D_2 achieves stratification in any two- or higher-dimensional projection. Let A be an $\mathrm{MOA}(n, s^m(n/s), 2)$ and B be an $\mathrm{OA}(n/s, k, s_1, 2)$. *Construction 1* is as follows.

Step 1. Obtain an orthogonal array-based Latin hypercube $\mathrm{L}(n/s, k)$, say L, based on a given B via the *level replacement-based Latin hypercube* approach.

Step 2. Obtain an $n \times k$ matrix $\tilde{D}_2$ by replacing the levels $0, \ldots, n/s-1$ of the last column of a given A with the 1st, 2nd, ..., and the (n/s)th row of the L from Step 1, respectively.

Step 3. Obtain an $n \times k$ matrix D_2 based on $\tilde{D}_2$ from Step 2 by replacing the s entries with level i in each column of $\tilde{D}_2$ by a random permutation of $\{is, \ldots, (i+1)s-1\}$ for $i = 0, \ldots, n/s-1$.

Let D_1 be the first m columns of A and $D = (D_1, D_2)$.

Theorem 1. *For design $D = (D_1, D_2)$ obtained in Construction* 1*,*

(i) *design D is a marginally coupled design for m qualitative factors and k quantitative factors; and*

(ii) *the rows in D_2 corresponding to each level of any factor in D_1 can achieve stratification on the $s_1 \times s_1$ grids in any two-dimensional projection.*

Proof. Each column of $\tilde{D}_2$ can be obtained by level permutations from the last column of A, so column-combining D_1 and any column of $\tilde{D}_2$ yields an $\mathrm{MOA}(n, s^m(n/s), 2)$. Part (i) follows immediately by Proposition 1. We now show part (ii). The L from Step 1 is an orthogonal array-based Latin hypercube based on a $B = \mathrm{OA}(n/s, k, s_1, 2)$, so L achieves stratification on the $s_1 \times s_1$ grids in any two dimensions. Corresponding to each level of any factor of D_1, the last column of the $\mathrm{MOA}(n, s^m(n/s), 2)$ is a permutation of $\{0, 1, \ldots, n/s-1\}$. In addition, in Step 2, this permutation with n/s symbols is replaced by the n/s rows of an $\mathrm{L}(n/s, k)$, which is an $\mathrm{OA}(n/s, k, s_1, 2)$-based Latin hypercube design. This means corresponding to each level of any factor of D_1, the design points in $\tilde{D}_2$ is an $\mathrm{OA}(n/s, k, s_1, 2)$-based Latin hypercube design. Because $\tilde{D}_2 = \lfloor D_2/s \rfloor$, the corresponding design in D_2 has stratifications of $s_1 \times s_1$ in any two dimensions.

Table 2. Array $\mathrm{MOA}(8, 2^4 4^1, 2)$ and $\mathrm{L}(4, 3)$ in Example 1.

A: $\mathrm{MOA}(8, 2^4 4^1, 2)$					L: $\mathrm{L}(4, 3)$		
0	0	0	0	0	2	3	1
1	1	1	1	0	3	1	2
0	0	1	1	1	0	2	3
1	1	0	0	1	1	0	0
0	1	0	1	2			
1	0	1	0	2			
0	1	1	0	3			
1	0	0	1	3			

Remark 1. The space-filling property of the small $\mathrm{L}(n/s, k)$ plays a critical role on the space-filling property of D_2. More precisely, if the small $\mathrm{L}(n/s, k)$ is based on an orthogonal array of strength t, the rows in the corresponding D_2 with respect to each level of any factor in D_1 will have stratification in any t dimensions; if it is based on a strong orthogonal array $\mathrm{SOA}(n/s, k, s_1^3, 3)$ (He and Tang (2013)), the rows of D_2 corresponding to each level of any column of D_1 will achieve stratification on the $s_1 \times s_1 \times s_1$ grids in any three dimensions, in addition, it can achieve stratification on the $s_1^2 \times s_1$ and the $s_1 \times s_1^2$ grids in any two dimensions.

Example 1. Let A be an $\mathrm{MOA}(8, 2^4 4^1, 2)$ and B be an $\mathrm{OA}(4, 3, 2, 2)$. Consider constructing an eight-run marginally coupled design $\mathrm{MCD}(D_1, D_2)$ for four qualitative factors and three quantitative factors via *Construction 1* using A and B. In this case, we have $m = 4, k = 3$, and $s = s_1 = 2$. In Step 1, we obtain an $L(4, 3)$, say L, based on B. One such an L is given in the right part of Table 2.

In Step 2, an 8×3 $\tilde{D}_2$ is obtained by replacing levels $0, 1, 2, 3$ in the last column of A by the first, second, third and fourth rows of L, respectively. Thus

$$\tilde{D}_2 = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 3 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In Step 3, for each column of $\tilde{D}_2$, replace the two positions having level i by a random permutation of $\{2i, 2i + 1\}$ for $i = 0, 1, 2, 3$. The resulting matrix

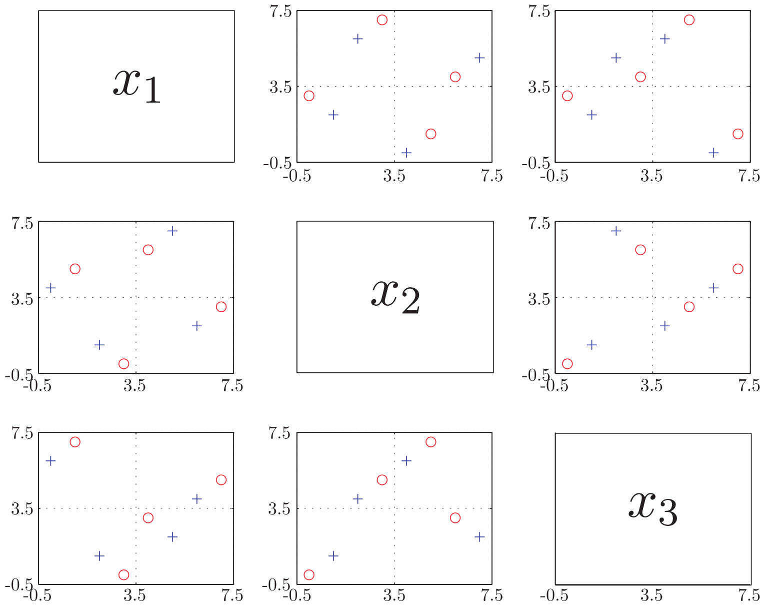

Figure 1. Bivariate projections among the three columns x_1, x_2, x_3 of D_2. Projected points of D_2 corresponding to levels 0 and 1 of the first column of D_1 are represented by "+" and "o", respectively.

is denoted by D_2. Now let D_1 consist of the first four columns of A. The MCD(D_1, D_2) constructed is as follows.

$$
\begin{array}{cccc}
 & D_1 & & \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{ccc}
 & D_2 & \\
5 & 7 & 2 \\
4 & 6 & 3 \\
6 & 2 & 4 \\
7 & 3 & 5 \\
0 & 4 & 6 \\
1 & 5 & 7 \\
2 & 1 & 1 \\
3 & 0 & 0
\end{array}
$$

The rows in D_2 corresponding to each level of any factor in D_1 achieve stratification on 2×2 grids in any two dimensions. (See Figure 1, for saving spaces we only plot the rows in D_2 corresponding to each level of the first factor in D_1).

Theorem 1 has it that mixed orthogonal arrays MOA$(n, s^m(n/s), 2)$'s and orthogonal arrays OA$(n/s, k, s_1, 2)$'s can be used to produce marginally coupled designs with Property (ii). Theorem 2 shows that the reverse of the argument holds.

Theorem 2. *Suppose that there exists a marginally coupled design $D = (D_1, D_2)$ with D_1 and D_2 satisfying*

(i) *D_1 is an OA$(n, m, s, 2)$; and*

(ii) *the rows in D_2 corresponding to each level of any factor in D_1 can achieve stratification on $s_1 \times s_1$ grids in any two dimensions.*

Then both MOA$(n, s^m(n/s), 2)$ and OA$(n/s, k, s_1, 2)$ exist.

Proof. First we show the existence of an MOA$(n, s^m(n/s), 2)$. According to Proposition 1, the existence of an MCD(D_1, D_2) with $D_1 =$ OA$(n, m, s, 2)$ implies the existence of an MOA$(n, s^m(n/s), 2)$. Thus (i) follows. Now we show the existence of an OA$(n/s, k, s_1, 2)$. Without loss of generality, let D^* be the submatrix of D_2 consisting of those rows of D_2 corresponding to level 0 of the first factor of D_1. Then D^* is an $(n/s) \times k$ matrix with entries from $\{0, 1, \ldots, n-1\}$. Let $\lambda^* = n/s_1$. Because, (ii) of Theorem 2, that is, D^* achieves stratification on $s_1 \times s_1$ grids in any two dimensions, $\lfloor D^*/\lambda^* \rfloor$ must be an OA$(n/s, k, s_1, 2)$. Thus an OA$(n/s, k, s_1, 2)$ exists.

We use *Construction* 1 to construct marginally coupled designs and tabulate the designs obtained. We first discuss the existence of orthogonal arrays, MOA$(n, s^m(n/s), 2)$'s and OA$(n/s, k, s_1, 2)$'s, used in the construction. The existence of an MOA$(n, s^m(n/s), 2)$ is equivalent to the existence of a CROA$(n, m, s, 2)$. Thus, we first establish the existence of CROA$(n, m, s, 2)$'s. According to the degrees of freedom, an MOA$(n, s^m(n/s), 2)$ satisfies $(s-1)m + (n/s - 1) \leq n - 1$, and thus we have

$$m \leq \frac{n}{s}, \tag{3.2}$$

which is an upper bound for the number of columns in D_1.

Theorem 3. *For a prime power $s = p^v$, there exist four types of completely resolvable orthogonal arrays:* (i) *CROA$(s^u, s^{u-1}, s, 2)$;* (ii) *CROA$(2s^u, 2s^{u-1}, s, 2)$;* (iii) *CROA$(4s^u, 4s^{u-1}, s, 2)$; and* (iv) *CROA$(p^w s^2, p^w s, s, 2)$, where p is a prime, and v, u, w are positive integers, with $u \geq 2$.*

The proof of Theorem 3 uses Theorems 6.6, 6.19, and 6.63, and Corollary 6.39 of Hedayat, Sloane and Stufken (1999). We relegate the proof, along with the necessary Lemmas, to Appendix 1.

The four types of completely resolvable orthogonal arrays in Theorem 3 reach the upper bound in (3.2). To employ *Construction* 1, we need small orthogonal arrays OA$(s^{u-1}, k, s_1, 2)$, OA$(2s^{u-1}, k, s_1, 2)$, OA$(4s^{u-1}, k, s_1, 2)$, and OA$(p^w s, k, s_1, 2)$ corresponding to types (i) - (iv) of completely resolvable orthogonal arrays, respectively. That is, s_1 and k in such orthogonal arrays can be chosen as listed in Table 3.

Table 3. Marginally coupled designs MCD(D_1, D_2)'s constructed by *Construction* 1.

Type	CROA$(n,m,s,2)$			OA$(n/s,k,s_1,2)$		MCD(D_1,D_2)	
	s	n	m	s_1	k	D_1	D_2
1	p^v	s^u	s^{u-1}	$s_1^{u_1}=s^{u-1}$	$(s_1^{u_1}-1)/(s_1-1)$	$OA(s^u,s^{u-1},s,2)$	$L(s^u,k)$
2	p^v	$2s^u$	$2s^{u-1}$	$s_1^{u_1}=s^{u-1}$	$2(s_1^{u_1}-1)/(s_1-1)-1^*$	$OA(2s^u,2s^{u-1},s,2)$	$L(2s^u,k)$
3	p^v	$4s^u$	$4s^{u-1}$	$s_1^{u_1}=s^{u-1}$	$4(s_1^{u_1}-1)/(s_1-1)-3^*$	$OA(4s^u,4s^{u-1},s,2)$	$L(4s^u,k)$
4	p^v	p^ws^2	p^ws	$s_1^{u_1}=p^ws$	$(s_1^{u_1}-1)/(s_1-1)$	$OA(p^ws^2,p^ws,s,2)$	$L(p^ws^2,k)$
5	2	8λ	4λ	2	$4\lambda-1$	$OA(8\lambda,4\lambda,2,2)$	$L(8\lambda,k)$

Note: p is a prime, u,u_1,v,w are positive integers and $u,u_1\geq 2$.
*: the two families of orthogonal arrays refer to Theorems 6.40 and 6.63 of Hedayat, Sloane and Stufken (1999).

In addition to the four types of completely resolvable orthogonal arrays in Theorem 3, there is the fifth type, CROA$(8\lambda,4\lambda,2,2)$, provided a Hadamard matrix of order 4λ exists (see Corollary 2 of Deng, Hung and Lin (2015)). This fifth type implies the existence of an MOA$(8\lambda,2^{4\lambda}(4\lambda),2)$. Coupled with an OA$(4\lambda,4\lambda-1,2,2)$, *Construction* 1 provides the fifth type of marginally coupled designs with 8λ runs in which D_1 is a two-level design. Table 3 summarizes the designs that can be constructed via *Construction 1* based on the five types of completely resolvable orthogonal arrays. For practical use, we consider the possible settings of parameters in Table 3, and provide the corresponding marginally coupled designs up to 100 runs in Table B.1 in Appendix 2. The actual designs can be obtained from authors upon request.

Here is a property of a Latin hypercube D_2 in an MCD(D_1,D_2) in Table 3: it is a cascading Latin hypercube of n points with levels $(n/s,\ s)$ (Handcock (1991)). In Step 2 of Construction 1, we substitute the entries in the last column of the MOA$(n,s^m(n/s),2)$ by rows of the L$(n/s,k)$ from Step 1. This implies that the columns in the substituted array can be obtained from each other by level permutations. That is, columns in $\tilde{D}_2$ are equivalent to each other, and thus D_2 is a cascading Latin hypercube.

A cascading Latin hypercube has clustered points which are undesirable for computer experiments. This leads to the questions of when D_2 must be cascading, and when D_2 can be non-cascading. Before answering the questions, we cite a useful lemma on saturated orthogonal arrays MOA$(n,s_1^{m_1}s_2^{m_2},2)$'s. Such a saturated orthogonal array has $(s_1-1)m_1+(s_2-1)m_2=n-1$.

Lemma 2 (Mukerjee and Wu (1995))**.** *Consider any two distinct rows of a saturated orthogonal array MOA$(n,s_1^{m_1}s_2^{m_2},2)$. For $i=1,2$, let Δ_i be the number of coincidences between these two rows arising from the s_i-symbol columns. Then Δ_1 and Δ_2 are nonnegative integers satisfying $\Delta_1\leq m_1,\Delta_2\leq m_2$, and*

$$s_1\Delta_1+s_2\Delta_2=m_1+m_2-1.$$

Proposition 2. *Let A be a* CROA$(n, n/s, s, 2)$. *Then there is a unique (up to equivalence) (n/s)-level column d such that (A, d) is an* MOA$(n, s^m(n/s), 2)$ *with* $m = n/s$.

Proof. If d_1 and d_2 are two columns of n/s levels such that both (A, d_1) and (A, d_2) are MOA$(n, s^{n/s}(n/s), 2)$'s, then both are saturated orthogonal arrays. Let $\delta_{ij}(A)$, $\delta_{ij}(d_1)$, and $\delta_{ij}(d_2)$ be the number of coincidences between the ith row and the jth row of A, d_1, and d_2, respectively, for $1 \leq i \neq j \leq n$. By Lemma 2, we have

$$s\delta_{ij}(A) + (\frac{n}{s})\delta_{ij}(d_1) = \frac{n}{s} + 1 - 1, \tag{3.3}$$

$$s\delta_{ij}(A) + (\frac{n}{s})\delta_{ij}(d_2) = \frac{n}{s} + 1 - 1. \tag{3.4}$$

Because $\delta_{ij}(A)$ in (3.3) and (3.4) is fixed for a given A, we have $\delta_{ij}(d_1) = \delta_{ij}(d_2)$ for $1 \leq i \neq j \leq n$. That is, d_1 and d_2 are equivalent up to level permutations.

Proposition 2 implies that a marginally coupled design MCD(D_1, D_2) with D_1 a CROA$(n, n/s, s, 2)$ satisfies that the columns of the corresponding $\tilde{D}_2$ are equivalent up to level permutations. Thus, D_2 must be a cascading Latin hypercube.

3.3. Construction of MCDs with non-cascading D_2's

This section introduces a method for constructing MCD(D_1, D_2)'s of s^u runs, where D_1 is an s-level orthogonal array for $(s+1-k)s^{u-2}$ qualitative factors and D_2 is a Latin hypercube for up to k $(1 \leq k \leq s)$ quantitative factors. The key feature of such marginally coupled designs is that D_2 is not a cascading Latin hypercube. The proposed method makes use of the Rao-Hamming construction. For detail of the construction, refer to Section 3.4 of Hedayat, Sloane and Stufken (1999).

For an integer $u \geq 2$, a prime power s and $j = 1, \ldots, u$, let e_j be an s-level column of length s^u with the entries taken from $GF(s)$, the Galois field of order s. Suppose that columns $\{e_1, \ldots, e_u\}$ are independent. The Rao-Hamming construction provides OA$(s^u, (s^u-1)/(s-1), s, 2)$'s using these columns. We apply the Rao-Hamming construction to obtain orthogonal arrays $B_0, \ldots, B_{s+1}$ in the following way. The array B_0 is generated by applying it to $u-2$ independent columns $\{e_1, \ldots, e_{u-2}\}$, $B_0 = (e_1, \ldots, e_{u-2})C_0$ with C_0 a $(u-2) \times [(s^{u-2}-1)/(s-1)]$ matrix, by collecting all the nonzero column vectors $(l_1, l_2, \ldots, l_{u-2})^{\mathrm{T}}$ where $l_j \in GF(s)$, $j = 1, \ldots, u-2$, and the first nonzero entry in $(l_1, l_2, \ldots, l_{u-2})$ is one. For $i = 1, \ldots, s+1$, the array B_i is generated by applying the Rao-Hamming construction to $u-1$ independent columns $\{e_1, \ldots, e_{u-2}, w_i\}$, $B_i = (e_1, \ldots, e_{u-2}, w_i)C$ with C a $(u-1) \times [(s^{u-1}-1)/(s-1)]$ matrix, by collecting all the nonzero column vectors $(l_1, l_2, \ldots, l_{u-1})^{\mathrm{T}}$, $l_j \in GF(s)$, $j = 1, \ldots, u-1$, and the first nonzero entry in $(l_1, l_2, \ldots, l_{u-1})$ is one, where $w_i = (e_{u-1}, e_u)c_i^{\mathrm{T}}$, $c_i = (1, \alpha_{i-1})$ for $i = 1, \ldots, s$ and $c_{s+1} = (0, 1)$, $\alpha_i \in GF(s) = \{\alpha_0, \ldots, \alpha_{s-1}\}$ with $\alpha_0 = 0$. Lemma 3 discusses the properties of the $s+2$ orthogonal arrays $B_0, \ldots, B_{s+1}$. For ease of expression, $B_i \setminus B_0$ is the array that consists of all columns in B_i but not in B_0.

Lemma 3. *For $B_0, \ldots, B_{s+1}$ as above,*

(i) *B_0 is an $OA(s^u, (s^{u-2}-1)/(s-1), s, 2)$, that consists of s^2 replicates of $OA(s^{u-2}, (s^{u-2}-1)/(s-1), s, 2)$;*

(ii) *for $i = 1, \ldots, s+1$, B_i is an $OA(s^u, (s^{u-1}-1)/(s-1), s, 2)$ that consists of s replicates of $OA(s^{u-1}, (s^{u-1}-1)/(s-1), s, 2)$;*

(iii) *for $1 \le i \le s+1$, $B_0 \subset B_i$, and for $1 \le i' \ne i \le s+1$, none of columns in $B_{i'} \setminus B_0$ can be generated by any linear combinations of columns in B_i;*

(iv) *if $f_1, \ldots, f_{u-1}$ are any $u-1$ independent columns from B_i and f is any column of $B_{i'} \setminus B_0$, then $(f_1, \ldots, f_{u-1}, f)$ is an $OA(s^u, u, s, u)$, where $i \ne i'$; and*

(v) *$\{B_0, (B_1 \setminus B_0), (B_2 \setminus B_0), \ldots, (B_{s+1} \setminus B_0)\}$ form an $OA(s^u, (s^u-1)/(s-1), s, 2)$, and they are disjoint.*

Given the orthogonal arrays $B_0, \ldots, B_{s+1}$, we propose *Construction* 2 to construct MCD(D_1, D_2)'s.

Construction 2: For given u and s, let $r = (s^{u-2}, s^{u-3}, \ldots, s, 1)^{\mathrm{T}}$. For a given k $(1 \le k \le s)$ and $i = 1, \ldots, k$, let $(f_{i,1}, f_{i,2}, \ldots, f_{i,u-1})$ be $u-1$ independent columns from $B_i \setminus B_0$ and $\tilde{d}_i = (f_{i,1}, f_{i,2}, \ldots, f_{i,u-1})r$. Obtain $\tilde{D}_2 = (\tilde{d}_1, \ldots, \tilde{d}_k)$ and $D_1 = \cup_{i=k+1}^{s+1}(B_i \setminus B_0)$.

Theorem 4. *For D_1 and $\tilde{D}_2 = (\tilde{d}_1, \ldots, \tilde{d}_k)$ generated by Construction 2 and $m = (s+1-k)s^{u-2}$,*

(i) D_1 is an OA$(s^u, m, s, 2)$ and $\tilde{D}_2$ is an OA$(s^u, k, s^{u-1}, 1)$ for $u \ge 2$;

(ii) $(D_1, \tilde{d}_i)$ is an MOA$(s^u, s^m(s^{u-1}), 2)$ for $i = 1, \ldots, k$;

(iii) if $k \ge 2$, no two distinct columns in $\tilde{D}_2$ are equivalent; and

(iv) if $k \ge 2$, the $(\tilde{d}_i, \tilde{d}_{i'})$ achieves stratification on $s^{u-1} \times s$ grids and $s \times s^{u-1}$ grids in two dimensions, where $i, i' = 1 \ldots, k$ and $i \ne i'$.

Proof. Here (i) follows from (v) of Lemma 3 by noting that each $B_i \setminus B_0$ has s^{u-2} columns for $i = k+1, \ldots, s+1$. To show (ii), we need to show that, for $i = 1, \ldots, k$ and any column b in D_1, $(b, \tilde{d}_i)$ is an MOA$(s^u, s(s^{u-1}), 2)$. This follows from (iv) of Lemma 3 because it indicates that each of the s^u

level combinations of $(b, \tilde{d}_i)$ occurs exactly once. For (iii), consider any two columns $\tilde{d}_i = (f_{i,1}, f_{i,2}, \ldots, f_{i,u-1})r$ and $\tilde{d}_{i'} = (f_{i',1}, f_{i',2}, \ldots, f_{i',u-1})r$, $i \neq i'$, where $(f_{i,1}, f_{i,2}, \ldots, f_{i,u-1})$ and $(f_{i',1}, f_{i',2}, \ldots, f_{i',u-1})$ are independent columns from $B_i \setminus B_0$ and $B_{i'} \setminus B_0$, respectively. Then (iv) of Lemma 3 indicates that $(f_{i,1}, f_{i,2}, \ldots, f_{i,u-1}, f_{i',1})$ is an $\mathrm{OA}(s^u, u, s, u)$, so $(\tilde{d}_i, \lfloor \tilde{d}_{i'}/s^{u-2} \rfloor) = (\tilde{d}_i, f_{i',1})$ is an $\mathrm{MOA}(s^u, (s^{u-1})s, 2)$. Since each level of $\{0, 1, \ldots, s^{u-1} - 1\}$ appears in $\tilde{d}_i$ at s positions, without loss of generality suppose that the first s positions of $\tilde{d}_i$ have the identical level. If $\tilde{d}_i$ and $\tilde{d}_{i'}$ are equivalent, then the first s entries of $\tilde{d}_{i'}$ must be the same, and hence the first s entries of $\lfloor \tilde{d}_{i'}/s^{u-2} \rfloor$ must be the same. This contradicts that $(\tilde{d}_i, \lfloor \tilde{d}_{i'}/s^{u-2} \rfloor) = (\tilde{d}_i, f_{i',1})$ is an $\mathrm{MOA}(s^u, (s^{u-1})s, 2)$, where every possible level combination can only appear once. So (iii) follows. Then (iv) follows since both $(\tilde{d}_i, f_{i',1})$ and $(\tilde{d}_{i'}, f_{i,1})$ are $\mathrm{MOA}(s^u, (s^{u-1})s, 2)$'s.

Remark 2. Construction 1 in Deng, Hung and Lin (2015) corresponds to *Construction* 2 with $u = 2$, with B_0 an empty set.

Example 2. Consider the case $s = 3, u = 3$, $k = 2$, with $GF(3) = \{\alpha_0, \alpha_1, \alpha_2\} = \{0, 1, 2\}$. *Construction* 2 provides an $\mathrm{MCD}(D_1, D_2)$ with D_1 an $\mathrm{OA}(27, 6, 3, 2)$ and D_2 an $\mathrm{L}(27, 2)$, such that D_2 can achieve stratification on 9×3 and 3×9 grids in any two dimensions. To apply *Construction* 2, let e_1, e_2, e_3 be independent three-level columns of length 27, and $\omega_1 = e_2, \omega_2 = e_2 + e_3, \omega_3 = e_2 + 2e_3$, and $\omega_4 = e_3$. Then the orthogonal arrays $B_0, B_1, \ldots, B_4$ are obtained as $B_0 = \{e_1\}$, $B_i = \{e_1, \omega_i, \omega_i + e_1, \omega_i + 2e_1\}$, for $i = 1, 2, 3, 4$. Now applying the Rao-Hamming construction to e_1, e_2, and e_3, we obtain an $\mathrm{OA}(27, 13, 3, 2)$, whose column partition is displayed in Table 4(a). Let $\tilde{d}_1 = (\omega_1, \omega_1 + e_1)(3, 1)^{\mathrm{T}}$, $\tilde{d}_2 = (\omega_2, \omega_2 + e_1)(3, 1)^{\mathrm{T}}$, and $\tilde{D}_2 = (\tilde{d}_1, \tilde{d}_2)$. Obtain D_2 from $\tilde{D}_2$ by substituting the three entries with level i in each column by a permutation of $(3i, 3i + 1, 3i + 2)$, for $i = 0, 1, \ldots, 8$. For ease of presentation, denote B_0 by P_0, and $B_i \setminus B_0$ by P_i for $i = 1, \ldots, 4$. If $D_1 = (P_3, P_4)$, then (D_1, D_2) is a marginally coupled design, see Table 4(b).

Construction 2 provides k columns for D_2 in $\mathrm{MCD}(D_1, D_2)$'s for $1 \leq k \leq s$. Some marginally coupled designs $\mathrm{MCD}(D_1, D_2)$, up to 100 runs, constructed by *Construction* 2 are listed in Table B.2 in Appendix B. *Construction* 3 provides $\mathrm{MCD}(D_1, D_2)$'s in which D_2's have considerably more columns than those provided in *Construction 2*. Specifically, when constructing an $\mathrm{MCD}(D_1, D_2)$ with s^u runs, a D_2 in *Construction 3* has $(u-1)k$ columns for a given k.

Construction 3: For a given k, $1 \leq k \leq s$, and $i = 1, \ldots, k$, let $(f_{i,1}, f_{i,2}, \ldots, f_{i,u-1})$ be $u-1$ independent columns from $B_i \setminus B_0$ and $\tilde{D}_2 = \big((f_{1,1}, f_{1,2}, \ldots, f_{1,u-1})R, \ldots,$

Table 4. A marginally coupled design of 27 runs in Example 2.

(a) The partition of $\mathrm{OA}(27, 13, 3, 2)$.

$P_0^{\#}$	P_1^{*}	P_2^{*}	P_3^{*}	P_4^{*}
0	0 0 0	0 0 0	0 0 0	0 0 0
0	0 0 0	1 1 2	2 2 1	1 1 2
0	0 0 0	2 2 1	1 1 2	2 2 1
0	1 1 2	1 1 2	1 1 2	0 0 0
0	1 1 2	2 2 1	0 0 0	1 1 2
0	1 1 2	0 0 0	2 2 1	2 2 1
0	2 2 1	2 2 1	2 2 1	0 0 0
0	2 2 1	0 0 0	1 1 2	1 1 2
0	2 2 1	1 1 2	0 0 0	2 2 1
1	0 1 1	0 1 1	0 1 1	0 1 1
1	0 1 1	1 2 0	2 0 2	1 2 0
1	0 1 1	2 0 2	1 2 0	2 0 2
1	1 2 0	1 2 0	1 2 0	0 1 1
1	1 2 0	2 0 2	0 1 1	1 2 0
1	1 2 0	0 1 1	2 0 2	2 0 2
1	2 0 2	2 0 2	2 0 2	0 1 1
1	2 0 2	0 1 1	1 2 0	1 2 0
1	2 0 2	1 2 0	0 1 1	2 0 2
2	0 2 2	0 2 2	0 2 2	0 2 2
2	0 2 2	1 0 1	2 1 0	1 0 1
2	0 2 2	2 1 0	1 1 1	2 1 0
2	1 0 1	1 0 1	1 0 1	0 2 2
2	1 0 1	2 1 0	0 2 2	1 0 1
2	1 0 1	0 2 2	2 1 0	2 1 0
2	2 1 0	2 1 0	2 1 0	0 2 2
2	2 1 0	0 2 2	1 0 1	1 0 1
2	2 1 0	1 0 1	0 2 2	2 1 0

$^{\#}$: $P_0 = B_0$; * : $P_i = B_i \setminus B_0$ for $i = 1, 2, 3, 4$.

(b) $\mathrm{MCD}(D_1, D_2)$.

$\tilde{D}_2$		D_2		D_1					
0	0	2	0	0	0	0	0	0	0
0	4	1	12	2	2	1	1	1	2
0	8	0	26	1	1	2	2	2	1
4	4	13	14	1	1	2	0	0	0
4	8	14	24	0	0	0	1	1	2
4	0	12	2	2	2	1	2	2	1
8	8	24	25	2	2	1	0	0	0
8	0	25	1	1	1	2	1	1	2
8	4	26	13	0	0	0	2	2	1
1	1	3	4	0	1	1	0	1	1
1	5	5	16	2	0	2	1	2	0
1	6	4	20	1	2	0	2	0	2
5	5	15	15	1	2	0	0	1	1
5	6	17	18	0	1	1	1	2	0
5	1	16	5	2	0	2	2	0	2
6	6	18	19	2	0	2	0	1	1
6	1	19	3	1	2	0	1	2	0
6	5	20	17	0	1	1	2	0	2
2	2	8	8	0	2	2	0	2	2
2	3	6	10	2	1	0	1	0	1
2	7	7	22	1	0	1	2	1	0
3	3	11	9	1	0	1	0	2	2
3	7	10	23	0	2	2	1	0	1
3	2	9	7	2	1	0	2	1	0
7	7	23	21	2	1	0	0	2	2
7	2	22	6	1	0	1	1	0	1
7	3	21	11	0	2	2	2	1	0

$(f_{k,1}, f_{k,2}, \ldots, f_{k,u-1})R\big)$, where

$$R = \begin{pmatrix} s^{u-2} & 1 & \cdots & s^{u-4} & s^{u-3} \\ s^{u-3} & s^{u-2} & \cdots & s^{u-5} & s^{u-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s & s^2 & \cdots & s^{u-2} & 1 \\ 1 & s & \cdots & s^{u-3} & s^{u-2} \end{pmatrix}.$$

Furthermore, let $D_1 = \cup_{i=k+1}^{s+1} (B_i \setminus B_0)$.

Theorem 5 summarizes the properties of D_1 and $\tilde{D}_2$ in *Construction* 3. It follows directly from Lemma 3 and Theorem 4 and thus we skip the proof. Our result has it that $\tilde{D}_2$ in *Construction* 3 can be partitioned into groups such that any two columns from the same group are equivalent and any two columns from different groups are not equivalent. In other words, D_2 in *Construction* 3 is not a cascading Latin hypercube, but its columns can be partitioned into groups of cascading Latin hypercubes. The parameter settings of marginally coupled designs constructed by *Construction* 3 are the same as those in Table B.2 in Appendix B except that the number of quantitative factors should be $(u-1)k$ for a given k in the table.

Theorem 5. *For D_1 and $\tilde{D}_2$ generated by Construction 3 and $m = (s+1-k)s^{u-2}$,*

(i) *D_1 is an OA$(s^u, m, s, 2)$ and $\tilde{D}_2$ is an OA$(s^u, k(u-1), s^{u-1}, 1)$ for $u \geq 2$;*

(ii) *$(D_1, \tilde{d}_i)$ is an MOA$(s^u, s^m(s^{u-1}), 2)$ for $i = 1, \ldots, k(u-1)$;*

(iii) *for any two distinct columns $\tilde{d}_i$ and $\tilde{d}_{i'}$ from $\tilde{D}_2$ with $\lfloor (i-1)/(u-1) \rfloor = \lfloor (i'-1)/(u-1) \rfloor$, $\tilde{d}_i$ and $\tilde{d}_{i'}$ are equivalent and they achieve stratification on $s \times s$ grids in two dimensions; and*

(iv) *if $\tilde{d}_i$ and $\tilde{d}_{i'}$ are from $\tilde{D}_2$ with $\lfloor (i-1)/(u-1) \rfloor \neq \lfloor (i'-1)/(u-1) \rfloor$, then $\tilde{d}_i$ and $\tilde{d}_{i'}$ are not equivalent and they achieve stratification on $s^{u-1} \times s$ grids and $s \times s^{u-1}$ grids in two dimensions.*

Example 3 (Example 2 continued)**.** Applying *Construction* 3, we have

$$\tilde{D}_2 = \Big((\omega_1, \omega_1 + e_1)R, (\omega_2, \omega_2 + e_1)R\Big), \text{ where } R = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Obtain D_2 from $\tilde{D}_2$ by substituting the three entries with level i in each column by a permutation of $(3i, 3i+1, 3i+2)$, for $i = 0, 1, \ldots, 8$. As in Example 2, if $D_1 = (P_3, P_4)$, then (D_1, D_2) is a marginally coupled design for six qualitative factors and four quantitative factors with these properties of $\tilde{D}_2 = (\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_{4)})$: $(\tilde{d}_i, \tilde{d}_j)$ achieves stratification on 3×3 grids in two dimensions for $\lfloor (i-1)/2 \rfloor = \lfloor (j-1)/2 \rfloor$; $(\tilde{d}_i, \tilde{d}_j)$ achieves stratification on 9×3 and 3×9 grids in two dimensions for $\lfloor (i-1)/2 \rfloor \neq \lfloor (j-1)/2 \rfloor$, $i \neq j$, $i, j = 1, \ldots, 4$. We do not list the generated design here.

4. Conclusion and Remarks

We provide three methods to construct MCD(D_1, D_2)'s, where D_1 is an s-level orthogonal array. The design feature of MCD(D_1, D_2)'s in the first method is that the rows in D_2 corresponding to each level of any factor in D_1 achieve stratification in any two or higher dimensions. The run size n of such designs is a multiple of s^2. In addition, D_1 can have as many as n/s factors. The disadvantage of such designs is that D_2 is a cascading Latin hypercube and thus has clustered points. To construct D_2 without clustered points, the second method allows marginally coupled designs of s^u runs to be constructed, where s is a prime power and $u \geq 2$. The third construction extends the second and it can accommodate more quantitative factors, where the whole design for quantitative factors keeps the non-cascading property.

For future work, one direction is to introduce methods for constructing designs with more flexible run sizes. Another direction is the construction of marginally coupled designs with improved space-filling properties in designs for quantitative factors when designs for qualitative factors are mixed orthogonal arrays. In addition, it is important to study the sampling property of the marginally coupled designs constructed (Qian (2012)).

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Appendix A

Before presenting the proof of Theorem 3, we cite four lemmas for later development; they are due to Theorem 6.6, Corollary 6.39, Theorem 6.63, and Theorem 6.19 of Hedayat, Sloane and Stufken (1999).

Lemma A.1. *A difference scheme $D(p^m, p^m, p^v)$ exists for any prime p and any integer $m \geq v \geq 1$.*

Lemma A.2. *A difference scheme $D(2s^m, 2s^m, s)$ exists for any prime power s and any integer $m \geq 1$.*

Lemma A.3. *A difference scheme $D(4s^m, 4s^m, s)$ exists for any prime power s and any integer $m \geq 1$.*

Lemma A.4. *The existence of a $D(r, c, s)$ implies the existence of a CROA$(rs, c, s,\ 2)$.*

Proof of Theorem 3. For a prime power $s = p^v$, by setting $m = uv$, or $m = v + w$, Lemma A.1 indicates that a difference scheme of $D(s^u, s^u, s)$, or $D(p^w s, p^w s, s)$ exists. Then combining them with Lemma A.1, one can obtain the existence of the first and fourth types of completely resolvable orthogonal arrays, respectively. Again, by combining Lemma A.4 with Lemmas A.2 and A.3, one can obtain the existence of completely resolvable orthogonal array of the second and third types. Therefore, the proof is complete.

Appendix B

Tables B.1 and B.2 present some marginally coupled designs MCD(D_1, D_2) up to 100 runs, using *Constructions* 1 *and* 2.

Table B.1. Some marginally coupled designs MCD(D_1, D_2) of n runs for m qualitative factors and k quantitative factors constructed by *Construction* 1, $n \leq 100$.

MOA($n, s^m(n/s), 2$)	OA($n/s, k, s_1, t$)	MCD(D_1, D_2)	
		D_1	D_2
MOA($8, 2^4 4^1$)	OA($4, 3, 2, 2$)	OA($8, 4, 2, 2$)	L($8, 3$)
MOA($16, 2^8 8^1$)	OA($8, 7, 2, 2$)	OA($16, 8, 2, 2$)	L($16, 7$)
	OA($8, 4, 2, 3$)	OA($16, 8, 2, 2$)	L($16, 4$)
MOA($24, 2^{12} 12^1$)	OA($12, 11, 2, 2$)	OA($24, 12, 2, 2$)	L($24, 11$)
MOA($27, 3^9 9^1$)	OA($9, 4, 3, 2$)	OA($27, 9, 3, 2$)	L($27, 4$)
MOA($32, 2^{16} 16^1$)	OA($16, 15, 2, 2$)	OA($32, 16, 2, 2$)	L($32, 15$)
	OA($16, 8, 2, 3$)	OA($32, 16, 2, 2$)	L($32, 8$)
	OA($16, 5, 2, 4$)	OA($32, 16, 2, 2$)	L($32, 5$)
	OA($16, 5, 4, 2$)	OA($32, 16, 2, 2$)	L($32, 5$)
MOA($32, 4^8 8^1$)	OA($8, 7, 2, 2$)	OA($32, 8, 4, 2$)	L($32, 7$)
	OA($8, 4, 2, 3$)	OA($32, 8, 4, 2$)	L($32, 4$)
MOA($40, 2^{20} 20^1$)	OA($20, 19, 2, 2$)	OA($40, 20, 2, 2$)	L($40, 19$)
MOA($48, 2^{24} 24^1$)	OA($24, 23, 2, 2$)	OA($48, 24, 2, 2$)	L($48, 23$)
	OA($24, 12, 2, 3$)	OA($48, 24, 2, 2$)	L($48, 12$)
MOA($54, 3^{18} 18^1$)	OA($18, 7, 3, 2$)	OA($54, 18, 3, 2$)	L($54, 7$)
MOA($56, 2^{28} 28^1$)	OA($28, 27, 2, 2$)	OA($56, 28, 2, 2$)	L($56, 27$)
MOA($64, 2^{32} 32^1$)	OA($32, 31, 2, 2$)	OA($64, 32, 2, 2$)	L($64, 31$)
	OA($32, 16, 2, 3$)	OA($64, 32, 2, 2$)	L($64, 16$)
	OA($32, 6, 2, 5$)	OA($64, 32, 2, 2$)	L($64, 6$)
MOA($64, 4^{16} 16^1$)	OA($16, 15, 2, 2$)	OA($64, 16, 4, 2$)	L($64, 15$)
	OA($16, 8, 2, 3$)	OA($64, 16, 4, 2$)	L($64, 8$)
	OA($16, 5, 2, 4$)	OA($64, 16, 4, 2$)	L($64, 5$)
	OA($16, 5, 4, 2$)	OA($64, 16, 4, 2$)	L($64, 5$)
MOA($72, 2^{36} 36^1$)	OA($36, 35, 2, 2$)	OA($72, 36, 2, 2$)	L($72, 35$)
MOA($80, 2^{40} 40^1$)	OA($40, 39, 2, 2$)	OA($80, 40, 2, 2$)	L($80, 39$)
	OA($40, 20, 2, 3$)	OA($80, 40, 2, 2$)	L($80, 20$)
MOA($81, 3^{27} 27^1$)	OA($27, 13, 3, 2$)	OA($81, 27, 3, 2$)	L($81, 13$)
	OA($27, 4, 3, 3$)	OA($81, 27, 3, 2$)	L($81, 4$)
MOA($88, 2^{44} 44^1$)	OA($44, 43, 2, 2$)	OA($88, 44, 2, 2$)	L($88, 43$)
MOA($96, 2^{48} 48^1$)	OA($48, 47, 2, 2$)	OA($96, 48, 2, 2$)	L($96, 47$)
	OA($48, 24, 2, 3$)	OA($96, 48, 2, 2$)	L($96, 24$)
	OA($48, 13, 4, 2$)	OA($96, 48, 2, 2$)	L($96, 13$)
MOA($100, 5^{20} 20^1$)	OA($20, 19, 2, 2$)	OA($100, 20, 5, 2$)	L($100, 19$)

In these MCD(D_1, D_2)'s, the rows in D_2 corresponding to each level of any factor in D_1 can achieve stratification on the $\underbrace{s_1 \times \cdots \times s_1}_{t}$ grids in any t-dimensional projection.

Table B.2. Some marginally coupled designs MCD(D_1, D_2) of $n = s^u$ runs for $m = (s+1-k)s^{u-2}$ qualitative factors and k quantitative factors, using *Construction* 2, $n < 100$.

$D_1 = \mathrm{OA}(s^u, m, s, 2)$	$\tilde{D}_2 = \mathrm{OA}(s^u, k, s^{u-1}, 1)$	Constraint
$\mathrm{OA}(8, 2(3-k), 2, 2)$	$\mathrm{OA}(8, k, 4, 1)$	$1 \le k \le 2$
$\mathrm{OA}(16, 4(3-k), 2, 2)$	$\mathrm{OA}(16, k, 8, 1)$	
$\mathrm{OA}(32, 8(3-k), 2, 2)$	$\mathrm{OA}(32, k, 16, 1)$	
$\mathrm{OA}(64, 16(3-k), 2, 2)$	$\mathrm{OA}(64, k, 32, 1)$	
$\mathrm{OA}(27, 3(4-k), 3, 2)$	$\mathrm{OA}(27, k, 9, 1)$	$1 \le k \le 3$
$\mathrm{OA}(81, 9(4-k), 3, 2)$	$\mathrm{OA}(81, k, 27, 1)$	
$\mathrm{OA}(64, 4(5-k), 4, 2)$	$\mathrm{OA}(64, k, 16, 1)$	$1 \le k \le 4$

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