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OUTLIER DETECTION VIA A MINIMUM RIDGE COVARIANCE DETERMINANT ESTIMATOR

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Abstract: In this paper, we propose an outlier detection procedure, based on a high-breakdown minimum ridge covariance determinant estimator that is especially useful for the large p/n scenario. The estimator is obtained from the subset of observations, after excluding potential outliers, by applying the so-called concentration steps. We explore the asymptotic distribution of the modified Mahalanobis distance related to the proposed estimator under certain moment conditions, and obtain a theoretical cutoff value for outlier identification. We also improve the outlier detection power by adding a one-step reweighting procedure. Lastly, we investigate the performance of the proposed methods using simulations and a real-data analysis.

Key words and phrases: High dimension, minimum covariance determinant estimator, random matrices.

1. Introduction

Data frequently contain one or more atypical observations, known as outliers, that is, observations that are well separated from the majority of the data, or in some way deviate from the general pattern of the data (Maronna et al. (2019)). Outliers cannot be avoided, and may make up between 1% and 10% of the data in a regular data set, possibly more in specific applications (Hampel et al. (2005)). The decreasing cost of collecting data means that modern data sets can be both large and complex, sometimes with a very high number of variables. The chance of contamination or imperfections in the data increases, both with the number of observations and their dimension. Thus, detecting potential outliers is important, either as a preprocessing step to avoid model misspecification and biased parameter estimation, or for some specific interest in finding anomalous observations.

Let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ be a random sample of the p-dimensional random vector $\boldsymbol{X}$ with mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)^\top$ and covariance matrix $\Sigma_p = (\sigma_{ij})_{p\times p}$. A common measure of outlyingness for an individual observation $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})^\top$ is the Mahalanobis distance

$$d_i^2(\boldsymbol{\mu}, \Sigma_p) = (\boldsymbol{x}_i - \boldsymbol{\mu})^\top \Sigma_p^{-1} (\boldsymbol{x}_i - \boldsymbol{\mu}). \tag{1.1}$$

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The well-known minimum covariance determinant algorithm (Rousseeuw and van Driessen (1999)) searches for a subsequence of $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ of size h, with $n/2 < h < n$, that has a sample covariance matrix with the smallest determinant. Thus, it obtains reliable estimates of $\boldsymbol{\mu}$ and Σ_p in (1.1). To determine the cutoff value for outlying points, Hardin and Rocke (2005) present a distributional result of (1.1) under a Gaussian assumption that is superior to the commonly used chi-square cutoff. The consistency and asymptotic normality of the minimum covariance determinant (MCD) estimator (Rousseeuw (1985)) are shown by Cator and Lopuhaä (2012). Based on the small-sample correction factors constructed by Pison, Van Aelst and Willems (2002), Cerioli (2010) proposes an iterated reweighted-MCD procedure that performs well for detecting multiple outliers.

However, when p/n increases, conventional outlier detection methods based on the MCD estimator become infeasible and suffer power loss (Adrover and Yohai (2002); Alqallaf et al. (2009)). In fact, the MCD approach is often recommended when $n > 5p$ (Boudt et al. (2019)). For outlier detection, Filzmoser, Maronna and Werner (2008) developed a computationally fast procedure by using a principal component analysis to identify outliers in a transformed space when $p/n \geq 1$. Ro et al. (2015) introduce the following alternative for (1.1):

$$d_i^2(\boldsymbol{\mu}, D_p) = (\boldsymbol{x}_i - \boldsymbol{\mu})^\top D_p^{-1} (\boldsymbol{x}_i - \boldsymbol{\mu}), \tag{1.2}$$

where $D_p = \operatorname{diag}(\sigma_{11}, \ldots, \sigma_{pp})$. By replacing (1.1) with (1.2), they propose a computationally efficient refined minimum diagonal product algorithm, and conduct simulation studies for autoregressive correlation and moving average models. Li and Jin (2022) consider a different alternative for (1.1):

$$d_i^2(\boldsymbol{\mu}, D_\Sigma) = (\boldsymbol{x}_i - \boldsymbol{\mu})^\top D_\Sigma^{-1} (\boldsymbol{x}_i - \boldsymbol{\mu}), \tag{1.3}$$

where D_Σ is the 2×2 block-diagonal partition of Σ_p. As such, they develop a high-breakdown block-diagonal product estimator. Other outlier detection techniques for high-dimensional data based on (1.2) include those of Yang, Wang and Zi (2018) and Wang et al. (2021).

Let $\bar{\boldsymbol{x}}_n = n^{-1} \sum_{i=1}^n \boldsymbol{x}_i$ and $S_n = n^{-1} \sum_{i=1}^n (\boldsymbol{x}_i - \bar{\boldsymbol{x}}_n)(\boldsymbol{x}_i - \bar{\boldsymbol{x}}_n)^\top$. Denote I_p as the $p \times p$ identity matrix. Motivated by the regularized Hotelling's T^2 test statistic used in the high-dimensional mean test (Chen et al. (2011); Ha et al. (2021)), we modify (1.1) as

$$d_i^2(\boldsymbol{\mu}, S_n(\lambda)) = (\boldsymbol{x}_i - \boldsymbol{\mu})^\top [S_n(\lambda)]^{-1} (\boldsymbol{x}_i - \boldsymbol{\mu}), \tag{1.4}$$

where $S_n(\lambda) = S_n + \lambda I_p$, and $\lambda > 0$ is a scalar tuning parameter. Here, the product λI_p is the perturbation that we add to the covariance estimator S_n, such that the matrix $S_n(\lambda)$ is positive definite, and hence invertible. Boudt et al.

(2019) suggest adding a preprocessing step to standardize each $\boldsymbol{x}_i$ as

$$\boldsymbol{u}_i = D_X^{-1}\left(\boldsymbol{x}_i - \boldsymbol{v}_X\right),$$

where D_X is a diagonal matrix in which the jth diagonal element is the Q_n estimator (Rousseeuw and Croux (1993)), and $\boldsymbol{v}_X$ is a location vector with elements that consist of the medians of all the variables. Then, they define the regularized sample covariance matrix by

$$K = \rho T + (1-\rho)c_\alpha S_U,$$

where S_U is the sample covariance matrix of $U = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$, T is a predetermined positive-definite target matrix, ρ is a regularization parameter selected to bound the condition number of K, and c_α is the consistency factor defined by Croux and Haesbroeck (1999). However, there is no distributional result or reweighting step in method of Boudt et al. (2019), and it is not easy to obtain appropriate standardized observations in high-dimensional settings.

Here, we consider outlier detection for the large p/n scenario, where $c_1 \leq p/n \leq c_2$, with c_1 and c_2 being some positive constants. By relaxing the Gaussian assumption, we derive the exact distribution of (1.4). We then propose a high-breakdown minimum ridge covariance determinant estimator. We explore the asymptotic distribution of the modified Mahalanobis distance related to the proposed estimator under certain moment conditions, and obtain a theoretical cutoff value for outlier identification, which is the basis for the proposed outlier detection procedure. We improve the outlier detection power by adding a one-step reweighting procedure. Lastly, we use simulation studies and an analysis of real data to show that the proposed procedure achieves higher detection power against sparse signals than that of its main competitors.

The remainder of the paper is organized as follows. In Section 2, we give our model assumptions, introduce the minimum ridge covariance determinant estimator, and present the main results. In Section 3, we examine the performance of the proposed methods using simulations and a real-data analysis. We conclude the paper in Section 4. All theoretical proofs are provided in the Appendix.

2. Methods and Properties

2.1. Model assumptions

Let $\boldsymbol{X}_n$ be a p_n-dimensional random vector admitting the independent components model

$$\boldsymbol{X}_n = T_{p_n}\boldsymbol{Z}_n + \boldsymbol{\mu}_n, \tag{2.1}$$

where $\boldsymbol{\mu}_n = (\mu_{1,n}, \ldots, \mu_{p_n,n})^\top$ is the location vector, T_{p_n} is a $p_n \times p_n$ full-rank transformation matrix, and $\boldsymbol{Z}_n$ is a p_n-dimensional random vector with

independent and identically distributed (i.i.d.) components. Denote the jth component of $\boldsymbol{Z}_n$ by $z_{j,n}$. For simplicity, we suppress the subscript n in the above notation if there is no confusion in the context.

Let F^{Σ_p} denote the empirical spectral distribution (ESD) of a matrix Σ_p (Bai and Silverstein (2010)), that is,

$$F^{\Sigma_p}(u) = \frac{1}{p}\sum_{j=1}^{p} \mathrm{I}_{[\lambda_j,\infty)}(u),$$

where λ_j, for $j = 1, \ldots, p$, are the eigenvalues of Σ_p, and $\mathrm{I}_A(\cdot)$ denotes the indicator function of the set A.

Our main assumptions are as follows:

Condition A1. $p, n \to \infty$ such that $c_n \triangleq p/n \to c \in (0, \infty)$.

Condition A2. $\Sigma_p \triangleq T_p T_p^\top$ is a $p \times p$ positive-definite matrix.

Condition A3. F^{Σ_p} converges to a proper probability measure F as $p \to \infty$.

Condition A4. $\limsup_{p\to\infty} \|\Sigma_p\| < \infty$ and $\limsup_{p\to\infty} \left\|\Sigma_p^{-1}\right\| < \infty$, where $\|\cdot\|$ denotes the spectral norm.

Condition A5. The first four moments of z_1 match those of the standard normal distribution $N(0, 1)$.

Conditions A1–A4 are common in research on the ESD of a high-dimensional sample covariance matrix; see, for example, Chen et al. (2011) and Ha et al. (2021). The four-moment matching condition in Condition A5 is required to obtain the limiting distribution of (1.4). Condition A5 is most closely related to the four-moment theorem for random covariance matrices of Tao and Vu (2012). The first and second moment conditions of z_1 are easy to meet in practice. The third moment condition is necessary for some of our lemmas, especially Lemma A.2. The fourth moment is essential for the proof of Lemma A.4, given in the Appendix. Extending the theoretical results using a relaxed version of Condition A5 is left to future research.

2.2. The minimum ridge covariance determinant estimate

The classical minimum covariance determinant procedure finds a subset of observations that have a sample covariance matrix with the smallest determinant by iteratively computing and sorting the Mahalanobis distances of each observation. To generalize this procedure to high-dimensional data sets, our method searches for a subset of h observations that minimizes the determinant of the ridge sample covariance matrix.

Let $\mathcal{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ be a collection of n observations of $\boldsymbol{X}_n$ in (2.1). Define $\mathcal{H} = \{H \subset \{1, \ldots, n\} : |H| = h,\ h > n/2\}$, the collection of all subsets of size

h, where $|H|$ denotes the cardinality of H. We set $h > n/2$, because potential outliers account for no more than half of the total observations. For any $H \in \mathcal{H}$, denote $\bar{\boldsymbol{x}}_H = |H|^{-1} \sum_{i \in H} \boldsymbol{x}_i$,

$$S_H = |H|^{-1} \sum_{i \in H} (\boldsymbol{x}_i - \bar{\boldsymbol{x}}_H) (\boldsymbol{x}_i - \bar{\boldsymbol{x}}_H)^\top,$$

and $S_H(\lambda) = S_H + \lambda I_p$, with $\lambda \in (0, \infty)$, a ridge sample covariance matrix in terms of $\{\boldsymbol{x}_i,\ i \in H\}$. It is easy to see that $S_H(\lambda)$ for a given λ is positive definite.

Definition 1. The minimum ridge covariance determinant (abbreviated as RICD) estimate of $\boldsymbol{\mu}$, the multivariate location parameter, for a given $\lambda > 0$, is defined as

$$\hat{\boldsymbol{\mu}}_{\text{RICD}} = \bar{\boldsymbol{x}}_{H_{\text{RICD}}} \text{ with } H_{\text{RICD}} = \underset{H \in \mathcal{H}}{\operatorname{argmin}} \det[S_H(\lambda)]. \tag{2.2}$$

Note that for $p > h$, the MCD estimate (Rousseeuw (1985)) becomes ill-defined, because $\det[S_H] = 0$ for such H. Denote the scatter estimate of Σ_p by $\hat{\Sigma}_{\text{RICD}} = S_{H_{\text{RICD}}}(\lambda)$. Note that $\hat{\boldsymbol{\mu}}_{\text{RICD}}$ and $\hat{\Sigma}_{\text{RICD}}$ can be shown to be location invariant and orthogonal equivariant, but not affine equivariant. See (Lopuhaä and Rousseeuw (1991)) for the definitions of location invariance, orthogonal equivariance, and affine equivariance of a covariance estimate.

When $\mathcal{X}$ is contaminated, there exist one or more $\boldsymbol{x}_i$ that are not observations of $\boldsymbol{X}_n$ in (2.1). These $\boldsymbol{x}_i$ may be arbitrary values, or go to ∞ as $n \to \infty$. Thus, $\bar{\boldsymbol{x}}_n$ is no longer an appropriate estimate of $\boldsymbol{\mu}$, and $\|\bar{\boldsymbol{x}}_n\|_F$ may be arbitrarily large, such that it "breaks down," where $\|\cdot\|_F$ denotes the Frobenius norm. The finite-sample breakdown point (Maronna et al. (2019)) ε_n of an estimate $\hat{\boldsymbol{\theta}}_n$ of the parameter $\boldsymbol{\theta}$ is the smallest proportion of observations from $\mathcal{X}$ that need to be replaced by arbitrary values to carry $\hat{\boldsymbol{\theta}}_n$ beyond all bounds:

$$\varepsilon_n(\hat{\boldsymbol{\theta}}_n, \mathcal{X}) = \min_{1 \le t \le n} \left\{ \frac{t}{n} : \sup_{\tilde{\mathcal{X}}} \left\| \hat{\boldsymbol{\theta}}_n(\mathcal{X}) - \hat{\boldsymbol{\theta}}_n\left(\tilde{\mathcal{X}}\right) \right\|_F = \infty \right\},$$

where $\tilde{\mathcal{X}} = \{\tilde{\boldsymbol{x}}_1, \ldots, \tilde{\boldsymbol{x}}_n\}$ is a data set with at least $(n - t)$ elements in common with $\mathcal{X}$, that is, $|\mathcal{X} \cap \tilde{\mathcal{X}}| \ge n - t$. It is easy to see that $\varepsilon_n(\bar{\boldsymbol{x}}_n, \mathcal{X}) = 1/n$. For the finite-sample breakdown point of the proposed estimates, we have the following theorem.

Theorem 1. *Suppose that $n/2 < h < n$ and $\lambda > 0$. Then, we have*

$$\varepsilon_n\left(\hat{\boldsymbol{\mu}}_{\text{RICD}}, \mathcal{X}\right) = \varepsilon_n(\hat{\Sigma}_{\text{RICD}}, \mathcal{X}) = \min\left\{ \frac{n - h + 1}{n}, 0.5 \right\}. \tag{2.3}$$

Theorem 1 shows that the proposed estimates can achieve the highest breakdown value, that is, 50%, when $h = [n/2] + 1$, where $[a]$ denotes the integer part of a. To achieve the best performance in practice, while ensuring $\varepsilon_n(\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}, \mathcal{X})$ and $\varepsilon_n(\hat{\Sigma}_{\mathrm{RICD}}, \mathcal{X})$ are as high as possible, we recommend a default choice of $h_{\mathrm{default}} = [n/2] + 1$.

To find H_{RICD} defined in (2.2), we modify the fast minimum covariance determinant algorithm (Rousseeuw and van Driessen (1999)) by replacing the Mahalanobis distance with its high-dimensional counterpart (1.4). However, when $n < p + 1$, the original algorithm requires a random initial subset H_{ini} containing $p + 1$ data points sampled from $\mathcal{X}$. To solve this problem, we set the size of the random initial subset to $h_{\mathrm{ini}} = h_{\mathrm{default}}$, given that $\varepsilon_n(\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}, \mathcal{X})$ does not depend on p.

Similarly to Rousseeuw and van Driessen (1999), we refer to the construction in the following theorem as a concentration step, consisting of two parts. This theorem illustrates the function of the second part of the concentration step, that is, sorting the distances of all $\boldsymbol{x}_i$ to the center of the subset obtained in the first part. By performing this part in the concentration step, we obtain a more concentrated h-sized subset, with a lower possibility of being contaminated by atypical points. This guarantees that an iteration process of repeating concentration steps leads to an optimal H, which, for convenience, is still denoted as H_{RICD}.

Theorem 2. *Let H be a subset of $\{1, \ldots, n\}$, with $|H| = h > n/2$. If $\tilde{H} \subset \{1, \ldots, n\}$ with $|\tilde{H}| = h$ is such that $\{d_i^2(\bar{\boldsymbol{x}}_H, S_H(\lambda)) : i \in \tilde{H}\} = \{d_{(1)}^2(\bar{\boldsymbol{x}}_H, S_H(\lambda)), \ldots, d_{(h)}^2(\bar{\boldsymbol{x}}_H, S_H(\lambda))\}$, where $d_{(1)}^2(\bar{\boldsymbol{x}}_H, S_H(\lambda)) \le \cdots \le d_{(n)}^2(\bar{\boldsymbol{x}}_H, S_H(\lambda))$ denote the order statistics of $\{d_i^2(\bar{\boldsymbol{x}}_H, S_H(\lambda)), \text{for } i = 1, \ldots, n\}$, then*

$$\det[S_{\tilde{H}}(\lambda)] \le \det[S_H(\lambda)]$$

with equality if and only if $\bar{\boldsymbol{x}}_H = \bar{\boldsymbol{x}}_{\tilde{H}}$ and $S_H(\lambda) = S_{\tilde{H}}(\lambda)$.

2.3. Asymptotic properties

The following theorem serves as a theoretical background for constructing a rule for identifying outliers. We first define additional notation:

$$\begin{aligned}\Theta^{(1)}(\lambda, c, A) &= \frac{1 - \lambda m_1(-\lambda)}{1 - c[1 - \lambda m_1(-\lambda)]},\\ \Theta^{(2)}(\lambda, c, A) &= \frac{1 - \lambda m_1(-\lambda)}{[1 - c + c\lambda m_1(-\lambda)]^3} - \lambda \frac{m_1(-\lambda) - \lambda m_2(-\lambda)}{[1 - c + c\lambda m_1(-\lambda)]^4},\end{aligned} \tag{2.4}$$

where c is a constant, A is a $p \times p$ nonnegative-definite matrix, $m_1(z)$ is defined as the Stieltjes transform of the ESD of A, $m_1(z) = \mathrm{tr}(A - zI_p)^{-1}/p$, and $m_2(z) = \mathrm{tr}(A - zI_p)^{-2}/p$.

Algorithm 1. The minimum ridge covariance determinant (RICD) procedure.

Step 1. Randomly sample c_s initial subsets $H_{j,\mathrm{ini}}$ from $\{1,\ldots,n\}$, with $|H_{j,\mathrm{ini}}| = [n/2]+1$, for $j = 1,\ldots,c_s$. Apply the concentration step to each initial subset three times, and obtain c_s concentrated subsets. Select l subsets from the above c_s concentrated subsets that have the lowest ridge covariance determinants.

Step 2. For each subset in the above l subsets, continue applying the concentration step until convergence, and obtain l final subsets. Select the best subset, with the minimum ridge covariance determinant as H_{RICD}.

Step 3. Compute $\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}$ and $\hat{\Sigma}_{\mathrm{RICD}}$, and $\Theta^{(1)}(\lambda, c_h, S_{H_{\mathrm{RICD}}})$ and $\Theta^{(2)}(\lambda, c_h, S_{H_{\mathrm{RICD}}})$, with $c_h = p/h$. For a given significance level of α, the kth observation is declared an outlier if

$$d_k^2\left(\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}, \hat{\Sigma}_{\mathrm{RICD}}\right) > p\Theta^{(1)}(\lambda, c_h, S_{H_{\mathrm{RICD}}}) + z_\alpha\left\{2p\Theta^{(2)}(\lambda, c_h, S_{H_{\mathrm{RICD}}})\right\}^{1/2}. \quad (2.5)$$

Theorem 3. *Assume that Conditions* A1–A5 *hold. Let* $\boldsymbol{X}_{1,n},\ldots,\boldsymbol{X}_{n,n}$ *be i.i.d. random vectors that have the same distribution as* $\boldsymbol{X}_n$ *in* (2.1)*. Then, for any* k *and* $\lambda > 0$*, we have*

$$\frac{\sqrt{p}\left((1/p)d_k^2(\bar{\boldsymbol{X}}_n, S_n(\lambda)) - \Theta^{(1)}(\lambda, c_n, S_n)\right)}{\sqrt{2\Theta^{(2)}(\lambda, c_n, S_n)}} \xrightarrow{D} \mathrm{N}(0,1), \quad \text{as } p \to \infty, \quad (2.6)$$

where $\bar{\boldsymbol{X}}_n = n^{-1}\sum_{i=1}^n \boldsymbol{X}_{i,n}$, $S_n(\lambda) = S_n + \lambda I_p$, *with* $S_n = n^{-1}\sum_{i=1}^n(\boldsymbol{X}_{i,n} - \bar{\boldsymbol{X}}_n)(\boldsymbol{X}_{i,n} - \bar{\boldsymbol{X}}_n)^\top$, *and "* $\xrightarrow{D}$ *" denotes convergence in distribution.*

Note that we can suppress the second subscript n in $\boldsymbol{X}_{1,n},\ldots,\boldsymbol{X}_{n,n}$ if there is no confusion in the context.

Because the computations of $\Theta^{(1)}(\lambda, c_n, S_n)$ and $\Theta^{(2)}(\lambda, c_n, S_n)$ do not require any knowledge of the true covariance matrix Σ_p beyond its positive definiteness, Theorem 3 provides a practical and efficient way for determining the cutoff value for identifying outliers.

2.4. The minimum ridge covariance determinant procedure

We adapt the procedure of the fast minimum covariance determinant approach (Rousseeuw and van Driessen (1999)) to solve the optimization problem (2.2) in a high-dimensional setting. We present a procedure to find H_{RICD} and the raw cutoff. We first explain what we mean by applying the concentration step described in Theorem 2 to a subset of $\{1,\ldots,n\}$, ℓ times: apply the concentration step to this subset, say $H_{(0)}$, and obtain a new subset of $\{1,\ldots,n\}$, say $H_{(1)}$; apply the concentration step to $H_{(1)}$, and obtain another new subset of $\{1,\ldots,n\}$, say $H_{(2)}$; repeat $\ell-2$ times, and obtain the final subset of $\{1,\ldots,n\}$, say $H_{(\ell)}$.

Denote z_α as the upper α-quantile of the standard normal distribution. Our procedure is given above.

2.5. Refined minimum RICD procedure

A one-step reweighting scheme is often an effective way of increasing the efficiency of an algorithm (Cerioli (2010); Ro et al. (2015)). Therefore, we improve the power of the proposed outlier test, described in Section 2.4, by adding a further reweighting step. Following Ro et al. (2015), we first assume that the parameters $\boldsymbol{\mu}$ and Σ_p are known, and define the weights

$$W_k = \begin{cases} 0 & \text{if } d_k^2\left(\boldsymbol{\mu}, S_n(\lambda)\right) > a_\delta, \\ 1 & \text{otherwise,} \end{cases} \tag{2.7}$$

where a_δ is the upper δ-quantile of the distribution of $d_k^2\left(\boldsymbol{\mu}, S_n(\lambda)\right)$. By (A.1) in Lemma A.4, given in the Appendix, it follows that

$$a_\delta = \operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right) + z_\delta\sqrt{2\operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right)^2}. \tag{2.8}$$

We have the following proposition.

Proposition 1. *Assume that Conditions* A1–A4 *hold. Let* $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ *be i.i.d. p-dimensional random vectors from* $N_p(\boldsymbol{\mu}, \Sigma_p)$*. Then,* $\mathrm{E}\left(X_{kj} \mid W_k = 1\right) = \mu_j$*, the* j*th element of* $\boldsymbol{\mu}$*, and*

$$\operatorname{Var}\left(X_{kj} \mid W_k = 1\right) = \sigma_{jj}\left[1 - \frac{2\phi\left(z_\delta\right)\left(\Sigma_p S_n(\lambda)^{-1}\Sigma_p\right)_{jj}}{\sigma_{jj}(1-\delta)\sqrt{2\operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right)^2}} + o(1)\right] \equiv \sigma_{jj}\tau_j, \tag{2.9}$$

where $\left(\Sigma_p S_n(\lambda)^{-1}\Sigma_p\right)_{jj}$ *is the* j*th diagonal element of* $\Sigma_p S_n(\lambda)^{-1}\Sigma_p$*, for* $j = 1, \ldots, p$*, and* ϕ *is the standard normal density function.*

This proposition reveals that $\operatorname{Var}\left(X_{kj} \mid W_k = 1\right)$ is smaller than the true scatter parameter σ_{jj}. Therefore, if too many observations are identified as outliers, we have a biased type-I error. Cerioli (2010) shows by simulation that multiplying the raw MCD scatter estimate by a proportionality constant $k_{\mathrm{MCD}}(h, n, v)$ improves the finite-sample performance of its algorithm. Denote $W_{\mathrm{RICD}} = \{k_1, \ldots, k_{n_w}\}$ as the set of indices of the observations $\boldsymbol{x}_k$ for which $w_k = 1$, where $w_k = 0$ if (2.5) holds, $w_k = 1$ otherwise, and $n_w = \sum_{k=1}^n w_k$. Following Cerioli (2010), we refine our estimates as follows:

$$\tilde{\boldsymbol{\mu}} = \bar{\boldsymbol{x}}_{W_{\mathrm{RICD}}}, \quad \tilde{S} = k_{\mathrm{RICD}}(h, p) S_{W_{\mathrm{RICD}}}, \tag{2.10}$$

where $k_{\mathrm{RICD}}(h, p)$ is an adjustment coefficient that depends on both h and p.

It is difficult to obtain a consistent estimate of τ_j in (2.9) in a high-dimensional setting for $j = 1, \ldots, p$. Nevertheless, it can be shown that

Algorithm 2. Refined minimum RICD procedure.

Step 1. Select the significance level α. Set $h = [n/2] + 1$. Choose c_s, for example, $c_s = 100$, and l, for example, $l = 10$. Apply Algorithm 1. Calculate the distance $d_k^2\left(\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}, \hat{\Sigma}_{\mathrm{RICD}}\right)$, and assign a weight to each observation according to (2.5), based on an appropriately chosen δ, for example, $\delta = \alpha/2$.

Step 2. Obtain n_w and W_{RICD}, and compute the refined location and scatter estimates $\tilde{\boldsymbol{\mu}}$ and $\tilde{S}$, respectively, using (2.10) and (2.12), respectively.

Step 3. Calculate the refined distance $d_k^2(\tilde{\boldsymbol{\mu}}, \tilde{S}(\lambda))$, update $\Theta^{(1)}(\lambda, c_{n_w}, \tilde{S})$ and $\Theta^{(2)}(\lambda, c_{n_w}, \tilde{S})$ according to (2.4) with $c_{n_w} = p/n_w$ and $\tilde{S}(\lambda) = \tilde{S} + \lambda I_p$. For a given significance level of α, the kth observation is declared an outlier if

$$d_k^2\left(\tilde{\boldsymbol{\mu}}, \tilde{S}(\lambda)\right) > p\Theta^{(1)}\left(\lambda, c_{n_w}, \tilde{S}\right) + z_\alpha\left\{2p\Theta^{(2)}\left(\lambda, c_{n_w}, \tilde{S}\right)\right\}^{1/2}. \tag{2.11}$$

$$\underset{1\le j\le p}{\mathrm{median}}\,\frac{1}{\tau_j} \approx \left[1 + \frac{2\phi\left(z_\delta\right)\mathrm{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right)}{p(1-\delta)\sqrt{2\,\mathrm{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right)^2}}\right]\{1 + o(1)\}, \quad p \to \infty,$$

where $\mathrm{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right)$ and $\mathrm{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right)^2$ can be estimated more easily. By Lemma A.2 in the Appendix, we can set the scaling factor $k_{\mathrm{RICD}}(h,p)$ in (2.10) as

$$k_{\mathrm{RICD}}(h,p) = 1 + \frac{2\phi\left(z_{\delta_w}\right)\Theta^{(1)}\left(\lambda, c_h, S_{H_{\mathrm{RICD}}}\right)}{(1-\delta_w)\sqrt{2p\Theta^{(2)}\left(\lambda, c_h, S_{H_{\mathrm{RICD}}}\right)}}, \tag{2.12}$$

where $\delta_w = 1 - n_w/n$ is the actual proportion of observations that are effectively excluded in the reweighting step. Our refined RICD procedure for outlier detection is summarized in Algorithm 2.

2.6. Choice of λ

Chen et al. (2011) suggest using the asymptotic approximation to choose the degree of regularization in their RHT test statistic (2). Based on the asymptotic properties of the modified Mahalanobis distance $d^2\left(\bar{\boldsymbol{x}}_n, S_n(\lambda)\right)$, we propose a data-driven approach for choosing the degree of regularization λ. Specifically, for each λ, we first calculate $\Theta^{(1)}\left(\lambda, c_n, S_n\right)$ and $\Theta^{(2)}\left(\lambda, c_n, S_n\right)$ based on the observed data $\mathcal{X}$. Then, for a target significance level α, the difference between $d^2\left(\bar{\boldsymbol{x}}_n, S_n(\lambda)\right)$ and its asymptotic approximation is measured by

$$D_\alpha(\lambda) = \underset{1\le k\le n}{\mathrm{median}}\, d_k^2\left(\bar{\boldsymbol{x}}_n, S_n(\lambda)\right) - p\Theta^{(1)}(\lambda, c_n, S_n) - z_\alpha\left\{2p\Theta^{(2)}\left(\lambda, c_n, S_n\right)\right\}^{1/2}.$$

We select λ as

$$\hat{\lambda} = \min\left\{\lambda : \lambda \in \Xi, |D_\alpha(\lambda)| \le \varrho\right\},$$

where Ξ is a prespecified selecting range for λ, and ϱ is a small positive value. We set $\alpha = 0.05$, $\Xi = [0.05, 200]$, and $\varrho = 1$ in our simulation studies. Note that the

optimal $\hat{\lambda}$ remains unchanged in the application of Algorithm 2 after it is chosen.

3. Numerical Studies

3.1. Simulations

In this section, we carry out simulation studies to evaluate the performance of the proposed procedure (refined RICD). We generate the data set $\mathcal{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ in two scenarios.

Scenario (I).

Here, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are independently distributed observations, where $\boldsymbol{x}_i$ is an observation from an ϵ-contaminated multivariate normal distribution $(1-\epsilon)N_p(\mathbf{0}, \Sigma_p) + (1/2)\epsilon N_p(\kappa\boldsymbol{\eta}_i, \Sigma_p) + (1/2)\epsilon N_p(-\kappa\boldsymbol{\eta}_i, \Sigma_p)$, unless stated otherwise. Two cases of $\boldsymbol{\eta}_i$ are considered: (i) (dense mean vector case): $\boldsymbol{\eta}_i$ is the normalized p-dimensional vector $\boldsymbol{\zeta}_i$ consisting of p i.i.d. random variables from the uniform distribution $\mathrm{U}(0,1)$, that is, $\boldsymbol{\eta}_i = \boldsymbol{\zeta}_i / \|\boldsymbol{\zeta}_i\|_F$; and (ii) (sparse mean vector case) $\boldsymbol{\eta}_i$ is the normalized p-dimensional vector $\boldsymbol{\zeta}_i$ in which $[p^{0.1}]$ randomly selected elements are i.i.d. from $\mathrm{U}(0,1)$, and the others are all zeros, that is, $\boldsymbol{\eta}_i = \boldsymbol{\zeta}_i / \|\boldsymbol{\zeta}_i\|_F$.

We fix the sample size $n = 100$, set the dimension p as 100, 200, and 400, and let the contamination ratio ϵ be 0.1 or 0.2. The two settings of the covariance structure and the magnitude of abnormality κ are given below:

Case (a). Autoregressive correlation structure setting. $\Sigma_p = (0.3^{|i-j|})_{p\times p}$; $\kappa = 8$, 9, 10, respectively, for $p = 100$, 200, 400;

Case (b). Random structure setting. $\Sigma_p = Q^\top D_0 Q$, with D_0 a diagonal matrix with diagonal elements $d_{jj} \overset{\text{i.i.d.}}{\sim} \mathrm{U}(1,5)$, for $j = 1, \ldots, p$, and Q an orthonormal matrix constructed from the spectral decomposition of $W^\top W$ ($W^\top W = Q^\top \Lambda Q$), with $W = (w_{ij})_{p\times p}$ being such that $w_{ij} \overset{\text{i.i.d.}}{\sim} \mathrm{U}(0,1)$; $\kappa = 12$, 14, 16, respectively, for $p = 100$, 200, 400.

Scenario (II). Non-Gaussian scenario

Case (c). Let the p-dimensional random vector $\boldsymbol{\xi} = 0.7827\boldsymbol{\gamma} + 0.6224\boldsymbol{\nu}$, where $\boldsymbol{\gamma}$ has i.i.d. elements with the common distribution $\mathrm{U}(-\sqrt{3}, \sqrt{3})$, and $\boldsymbol{\nu}$, independent of $\boldsymbol{\gamma}$, has i.i.d. elements with the common density function

$$f(\nu) = \begin{cases} \sqrt{2}e^{-\sqrt{2}\nu}/2, & \text{if } \nu \geq 0, \\ \sqrt{2}e^{\sqrt{2}\nu}/2, & \text{if } \nu < 0. \end{cases}$$

It can be shown that the distribution of ξ_1, the first element of $\boldsymbol{\xi}$, satisfies Condition A5. Denote the distribution of $\boldsymbol{\xi}$ by F_ξ. Replace the ϵ-contaminated multivariate normal distribution in Scenatio (I) with $(1-\epsilon)F_\xi + (1/2)\epsilon N_p(\kappa\boldsymbol{\eta}_i, I_p) + (1/2)\epsilon N_p(-\kappa\boldsymbol{\eta}_i, I_p)$; $\kappa = 8, 9$, or 10, respectively, for $p = 100, 200$, or 400.

Table 1. Average type-I error (%) by the proposed procedure for various p, ϵ, and α.

			$\epsilon = 0.1$			$\epsilon = 0.2$		
$\boldsymbol{\eta}_i$	Case	p	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
(i)	(a)	100	1.74	6.97	13.20	1.21	5.56	11.13
		200	1.41	6.59	13.15	1.00	5.28	11.16
		400	1.38	6.50	12.16	0.97	5.16	9.77
	(b)	100	1.62	6.86	12.89	1.22	5.44	10.75
		200	1.38	6.48	12.74	0.97	5.15	10.52
		400	1.26	6.05	11.02	0.83	4.50	8.03
	(c)	100	1.63	6.06	11.49	1.09	4.84	9.55
		200	1.33	6.00	11.73	0.92	4.74	9.81
		400	1.31	5.99	10.87	0.90	4.66	8.60
(ii)	(a)	100	1.75	7.00	13.17	1.24	5.60	11.16
		200	1.42	6.63	13.14	1.02	5.30	11.09
		400	1.41	6.52	12.19	1.02	5.15	9.54
	(b)	100	1.61	6.85	12.87	1.20	5.41	10.75
		200	1.39	6.49	12.71	0.98	5.17	10.50
		400	1.27	6.05	10.75	0.87	4.56	7.97
	(c)	100	1.61	6.08	11.52	1.08	4.84	9.56
		200	1.35	6.00	11.73	0.92	4.71	9.76
		400	1.29	5.99	10.76	0.92	4.65	8.40

We compare the performance of the proposed procedure (RICD) with that of several existing methods, namely, the refined minimum diagonal product procedure (RMDP) of Ro et al. (2015), the block diagonal product procedure (BDP) of Li and Jin (2022), and the principal component outlier detection procedure (PCout) of Filzmoser, Maronna and Werner (2008), for each setting. We evaluate the outlier identification performance using the type-I error rate, that is, the proportion of good observations that are incorrectly classified as outliers, and the detection power, that is, the proportion of contaminated observations that are correctly flagged. The average type-I error rate $\bar{\alpha}$ and the detection power $\bar{\beta}$ presented in this section are calculated from 500 replications.

The average type-I error rates (%) of the the proposed RICD procedure for various p and ϵ are displayed in Table 1, where the nominal significance level α is set to be 0.01, 0.05, or 0.1. The results show that the empirical type-I error rates are close to the nominal levels in most settings.

The simulation results for the four methods with $\alpha = 0.05$, $\epsilon = 0.1$, and 0.2 are summarized in Tables 2–3, showing that (i) the proposed method outperforms both the RMDP and the BDP procedures in terms of detection power in most cases, and (ii) the PCout method exhibits similar performance to that of our method in Case (i) for $\epsilon = 0.1$. However, the former has a conservative type-I error rate when the contamination ratio increases to 0.2, and suffers from some power loss in Case (ii).

Table 2. Average type-I error (%) and detection power (%), where $\alpha = 0.05$ and $\epsilon = 0.1$.

			RICD		RMDP		BDP		PCout	
$\boldsymbol{\eta}_i$	Case	p	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\alpha}$	$\bar{\beta}$
(i)	(a)	100	6.97	94.60	6.59	94.22	6.95	91.72	5.33	97.24
		200	6.59	92.53	6.19	92.69	7.34	90.92	5.01	97.52
		400	6.50	90.09	5.86	87.05	7.94	85.82	5.15	97.19
	(b)	100	6.86	88.47	6.10	86.72	6.61	83.31	5.65	92.91
		200	6.48	88.21	6.15	87.11	7.32	84.17	5.09	95.33
		400	6.05	83.56	5.85	83.92	8.42	83.67	4.71	95.73
	(c)	100	6.06	97.80	6.15	97.24	6.41	95.38	4.61	99.33
		200	6.00	96.61	6.12	96.29	7.57	94.98	4.15	100.00
		400	5.99	93.76	5.96	93.20	8.34	91.70	4.38	98.57
(ii)	(a)	100	7.00	97.35	6.33	92.80	6.79	95.07	6.86	30.79
		200	6.63	95.12	6.22	90.48	7.34	92.70	7.06	21.11
		400	6.52	92.42	6.17	83.68	8.47	84.60	7.62	17.64
	(b)	100	6.85	88.28	6.31	81.71	6.80	83.28	7.66	24.76
		200	6.49	88.13	6.40	81.72	7.60	83.19	7.20	16.59
		400	6.05	84.92	5.99	77.01	8.70	78.98	7.66	14.69
	(c)	100	6.08	97.66	6.16	95.18	6.59	95.66	6.98	39.96
		200	6.00	96.27	6.11	93.11	7.58	93.40	6.63	27.44
		400	5.99	94.27	5.78	86.49	8.57	85.84	7.03	19.46

In Scenario (I), we consider the following radial contamination scheme (Cerioli (2010)):

Case (d). Scatter outliers. $\boldsymbol{x}_i^{(\epsilon)}$ is an observation from $(1-\epsilon)N_p(\mathbf{0}, \Sigma_p) + \epsilon N_p(\mathbf{0}, \Sigma_{(i)})$, where Σ_p is set as in Case (a), $[p^{0.5}]$ random diagonal components of $\Sigma_{(i)}$ are 7.5, and the other entries are the same as those of Σ_p.

We fix the significance level $\alpha = 0.05$ in this case. A comparison of the results with different contamination ratios are reported in Table 4, which shows that the proposed method simultaneously maintains the desired type-I error rate and achieves high detection power. Similarly to the location outlier settings, the PCout procedure appears to be insensitive to sparse signals. The BDP procedure does not control the type-I error rate as well as the proposed method does for $p \geq 200$ and $\epsilon = 0.1$.

3.2. Real-data analysis

We illustrate the proposed method on an octane data set consisting of near-infrared absorbance spectra, with $p = 226$ wavelengths collected on $n = 39$ gasoline samples. The data set is described in Esbensen, Midtgaard and Schönkopf (1996), and is available in the R package `rrcov`. Because this data set has a large p/n ratio, we cannot compute the original minimum covariance determinant

Table 3. Average type-I error ($\bar{\alpha}$ %) and detection power ($\bar{\beta}$ %), where $\alpha = 0.05$ and $\epsilon = 0.2$.

			RICD		RMDP		BDP		PCout	
$\boldsymbol{\eta}_i$	Case	p	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\alpha}$	$\bar{\beta}$
(i)	(a)	100	5.56	93.38	4.89	91.80	5.25	87.92	2.22	98.12
		200	5.28	91.42	4.87	89.87	5.78	86.28	1.97	99.62
		400	5.19	88.53	4.31	84.26	6.17	81.13	1.69	99.85
	(b)	100	5.44	84.88	4.49	83.85	4.81	77.58	2.04	99.42
		200	5.15	85.12	4.55	83.43	5.73	78.54	1.71	99.96
		400	4.50	79.50	4.24	78.63	6.15	76.34	1.60	99.95
	(c)	100	4.84	96.84	4.67	96.76	5.31	93.89	1.55	99.99
		200	4.74	95.67	4.57	94.84	5.87	91.59	1.39	100.00
		400	4.66	92.22	4.46	90.84	6.62	87.62	1.26	100.00
(ii)	(a)	100	5.60	96.28	4.76	90.95	5.20	93.44	5.63	30.88
		200	5.30	94.10	4.73	86.91	5.78	89.30	6.57	19.99
		400	5.15	90.64	4.88	79.92	7.13	81.51	6.91	15.49
	(b)	100	5.41	85.48	4.90	78.20	5.31	79.20	6.24	23.06
		200	5.17	85.53	5.06	77.80	6.04	79.57	6.78	16.90
		400	4.56	81.06	4.91	72.98	7.28	74.80	7.14	14.66
	(c)	100	4.84	96.79	4.78	93.63	5.25	93.96	5.59	35.75
		200	4.71	94.92	4.82	90.21	6.23	90.76	5.53	24.93
		400	4.65	93.09	4.55	83.70	6.96	82.62	6.37	19.55

Table 4. Average type-I error ($\bar{\alpha}$ %) and detection power ($\bar{\beta}$ %) in Case (d), where $\alpha = 0.05$.

			RICD		RMDP		BDP		PCout	
Case	ϵ	p	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\alpha}$	$\bar{\beta}$
(d)	0.1	100	6.89	89.00	6.27	86.74	6.70	89.25	7.23	33.86
		200	6.55	91.25	6.36	89.23	7.48	91.96	7.08	26.64
		400	6.46	94.14	5.90	92.15	8.14	94.60	6.90	25.51
	0.2	100	5.62	87.30	4.76	83.35	5.21	86.20	5.61	33.02
		200	5.28	89.45	4.87	87.13	5.97	90.26	5.82	25.67
		400	5.15	93.00	4.30	90.17	6.17	93.12	5.79	23.64

estimate. Furthermore, because the 25th, 26th, 36th, 37th, 38th, and 39th samples contain added ethanol, they are outliers. We apply the proposed method to this data set at a significance level of 0.01, and record the distance measures $[d^2(\tilde{\boldsymbol{\mu}}, \tilde{S}(\lambda)) - p\Theta^{(1)}(\lambda, c_{n_w}, \tilde{S})]/[2p\Theta^{(2)}(\lambda, c_{n_w}, \tilde{S})]^{1/2}$ (2.11). The Q-Q plot of the distance measures is given in Figure 1, in which the dashed horizontal line indicates the cutoff value, "good" points are around the black solid line, and the true outliers are labeled as solid points. This figure clearly demonstrates that the proposed procedure correctly identifies all six outliers.

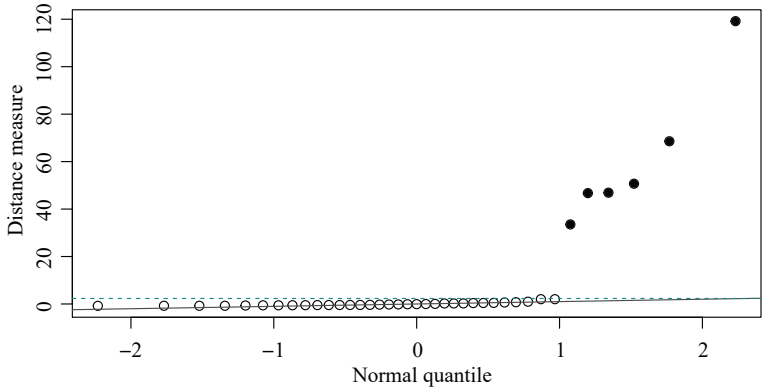

Figure 1. Q-Q plot of the distance measures based on the RICD.

Additional numerical studies are given in the Supplementary Material.

4. Conclusion

We have proposed a new outlier detection procedure based on the ridge sample covariance matrix. The resulting high-breakdown ridge covariance determinant estimate is well defined for high-dimensional data and contains more information on the correlations between the variables than the MDP estimate does (Ro et al. (2015)). We obtain the asymptotic distribution of the modified Mahalanobis distance by relaxing the commonly used Gaussian assumption. This novel outlier detection procedure first finds a clean subset by applying a concentration step, and then identifies outliers with modified distances that are above the cutoff value. The regularization parameter is selected adaptively based on the data, thus enhancing the robustness of the proposed method. Using simulations and a real-data example, we have shown that the proposed method is robust to the masking and swamping effects of the contaminated data, and outperforms the existing RMDP, BDP, and PCout methods in certain situations.

Supplementary Material

Supplementary Material available online includes additional simulation results and a real-data example.

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Appendix

First, we give some lemmas. Then we give the proofs of Theorems 1 and 2, Lemmas A.4 to A.6, Theorem 3, and Proposition 1.

Lemma A.1. Lemma 4 of Chen et al. (2011). *Given random variables* $\{x_n, y_n\}_{n=1}^{\infty}$. $f_n(x_n, y_n)$ *is a real function of* x_n *and* y_n. *If* $f_n \mid \mathcal{F}_n \xrightarrow{D} G$ *and distribution* G *is independent of* $\mathcal{F}_n$, *here* $\mid \mathcal{F}_n$ *denotes conditional on* $\mathcal{F}_n$ *and* $\mathcal{F}_n$ *is the* σ*-field generated by* $\{y_1, \ldots, y_n\}$, *then we have* $f_n \xrightarrow{D} G$.

Lemma A.2. Theorem 2.3 of Ha et al. (2021). *Assume that Conditions A1–A5 hold. Let* $\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}$ *be i.i.d. random vectors that have the same distribution as* $\boldsymbol{X}_n$ *in* (2.1). *For any* $\lambda > 0$, *we have*

$$\sqrt{p}\left|\frac{1}{p}\operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right) - \Theta^{(1)}\left(\lambda, c_n, S_n\right)\right| \xrightarrow{\mathrm{p}} 0$$

and

$$\frac{1}{p}\operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right)^2 - \Theta^{(2)}\left(\lambda, c_n, S_n\right) \xrightarrow{\mathrm{p}} 0, \quad \text{as } p \to \infty,$$

where " $\xrightarrow{\mathrm{p}}$ " *denotes convergence in probability,* $\Theta^{(i)}, i = 1, 2$, *are defined in* (2.4).

Lemma A.3. Lemmas 4.2–4.4 of Ha et al. (2021). *Assume that Condition* A1 *holds. Let* A *be a* $p \times p$ *nonrandom symmetric matrix with bounded spectral norm, and* $Z = (z_{ij})$ *a* $p \times n$ *random matrix whose entries are i.i.d., satisfying*

$$\mathrm{E}\, z_{11} = 0, \quad \mathrm{E}\, z_{11}^2 = 1, \quad \mathrm{E}\, z_{11}^4 < \infty, \quad \text{and} \quad |z_{11}| \le \eta_n\sqrt{n},$$

where $\{\eta_n\}$ *is a deterministic sequence with* $\eta_n \downarrow 0$ *whose convergence rate can be made arbitrarily slow. Then*

$$\mathrm{E}\left|\bar{\boldsymbol{z}}_k^{\top} A \bar{\boldsymbol{z}}_k\right|^v \le k_v, \quad \mathrm{E}\left|\frac{1}{n}\boldsymbol{z}_k^{\top} A \boldsymbol{z}_k\right|^v \le k_v, \quad \mathrm{E}\left|\bar{\boldsymbol{z}}_k^{\top} A \boldsymbol{z}_k\right|^v \le k_v, \quad v = 1, 2, \ldots$$

where $\bar{\boldsymbol{z}}_k = (1/n)\sum_{j\neq k}^{n} \boldsymbol{z}_j$, $k = 1, \ldots, n$, $\boldsymbol{z}_j$ *is the* j*th column of* Z, *and* k_v *is a constant depending on* v.

When $\boldsymbol{X}_{k,n} \in \{\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}\}$, it is difficult to obtain the universality of the CLT for the proposed estimator directly since $\boldsymbol{X}_{k,n}$ is not independent of the sample covariance S_n, and hence the ridge covariance $S_n(\lambda)$. Thus we divide the proof into two steps. Let $\Omega_n = \{\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}, \ldots\}$ denote the complete set of random vectors generated by model (2.1). At first, Lemma A.4 is derived to characterize the asymptotic distribution of the modified distance (1.4) when the objective $\tilde{\boldsymbol{X}} \notin \{\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}\}$. Define $\tilde{d}^2(\boldsymbol{\eta}, S_n(\lambda)) = (\tilde{\boldsymbol{X}} - \boldsymbol{\eta})^{\top} S_n(\lambda)^{-1}(\tilde{\boldsymbol{X}} - \boldsymbol{\eta})$.

Lemma A.4. *Assume that Conditions* A1–A5 *hold. Let* $\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}, \ldots$ *be i.i.d. random vectors that have the same distribution as* $\boldsymbol{X}_n$ *in* (2.1). *If the random vector* $\tilde{\boldsymbol{X}}$ *is independent of* $\{\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}\}$ *and* $\lambda > 0$, *we have*

$$\frac{\tilde{d}^2(\boldsymbol{\mu}, S_n(\lambda)) - \operatorname{tr}(S_n(\lambda)^{-1}\Sigma_p)}{\sqrt{2\operatorname{tr}(S_n(\lambda)^{-1}\Sigma_p)^2}} \xrightarrow{D} \mathrm{N}(0,1), \quad p \to \infty, \tag{A.1}$$

where $\tilde{d}^2(\boldsymbol{\mu}, S_n(\lambda)) = (\tilde{\boldsymbol{X}} - \boldsymbol{\mu})^\top S_n(\lambda)^{-1}(\tilde{\boldsymbol{X}} - \boldsymbol{\mu})$.

Lemma A.5. *Assume that Conditions* A1–A4 *hold. Let* $\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}, \ldots$ *be i.i.d. random vectors that have the same distribution as* $\boldsymbol{X}_n$ *in* (2.1) *satisfying that* $\mathrm{E}\, z_{11} = 0$ *and* $\mathrm{E}\, z_{11}^2 = 1$. *If the random vector* $\tilde{\boldsymbol{X}}$ *is independent of* $\{\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}\}$ *and* $\lambda > 0$, *we have*

$$\frac{\left|\tilde{d}^2(\bar{\boldsymbol{X}}_n, S_n(\lambda)) - \tilde{d}^2(\boldsymbol{\mu}, S_n(\lambda))\right|}{\sqrt{2\operatorname{tr}(S_n(\lambda)^{-1}\Sigma_p)^2}} = o_{\mathrm{p}}(1), \quad p \to \infty. \tag{A.2}$$

The asymptotic bias between $\tilde{d}^2(\bar{\boldsymbol{X}}_n, S_n(\lambda))$ and $\tilde{d}^2(\boldsymbol{\mu}, S_n(\lambda))$ is formally given in Lemma A.5, which ensures that we can use the raw location and scatter estimators to select a cutoff value for outlier identification. Next, instead of letting $\tilde{\boldsymbol{X}}$ be independent of $\{\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}\}$, we consider the modified distance (1.4) if $\tilde{\boldsymbol{X}} \in \{\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}\}$.

Let $\boldsymbol{X}_{k0,n} = (X_{1,n}, \ldots, X_{k-1,n}, 0, X_{k+1,n}, \ldots, X_{n,n})^\top$, $\bar{\boldsymbol{X}}_{k0} = (1/n)\boldsymbol{X}_{k0,n}^\top \mathbf{1}_n$, $S_{n,k0} = (1/n)\boldsymbol{X}_{k0,n}\boldsymbol{X}_{k0,n}^\top - \bar{\boldsymbol{X}}_{k0}\bar{\boldsymbol{X}}_{k0}^\top$ and $S_0(\lambda) = S_{n,k0} + \lambda I_p$. Here $\mathbf{1}_n$ denotes an n-dimensional vector consisting of 1s. The asymptotic bias between $d_k^2(\bar{\boldsymbol{X}}_n, S_n(\lambda))$ and $d_k^2(\bar{\boldsymbol{X}}_{k0}, S_0(\lambda))$ is given in the following lemma.

Lemma A.6. *Assume that Conditions* A1–A4 *hold. Let* $\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}$ *be i.i.d. random vectors that have the same distribution as* $\boldsymbol{X}_n$ *in* (2.1) *satisfying that* $\mathrm{E}\, z_{11} = 0$, $\mathrm{E}\, z_{11}^2 = 1$ *and* $\mathrm{E}\, z_{11}^4 < \infty$. *For any* $\boldsymbol{X}_{k,n} \in \{\boldsymbol{X}_{1,n}, \ldots, \boldsymbol{X}_{n,n}\}$ *and* $\lambda > 0$, *the following three arguments hold:*

$$\frac{\left|d_k^2\left(\bar{\boldsymbol{X}}_n, S_n(\lambda)\right) - d_k^2\left(\bar{\boldsymbol{X}}_{k0}, S_0(\lambda)\right)\right|}{\sqrt{2\operatorname{tr}(S_0(\lambda)^{-1}\Sigma_p)^2}} = o_p(1), \tag{A.3}$$

$$\operatorname{tr}\left(S_0(\lambda)^{-1}\Sigma_p\right)^2 - \operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right)^2 = O_p(1), \tag{A.4}$$

$$\operatorname{tr}\left(S_0(\lambda)^{-1}\Sigma_p\right) - \operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma_p\right) = O_p(1), \quad p \to \infty. \tag{A.5}$$

Although (A.1) presupposes that the estimate of $\boldsymbol{\mu}$ and $S_n(\lambda)$ are a sample without outliers, it is also expected to be roughly valid for the distance $d_k^2(\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}, \hat{\Sigma}_{\mathrm{RICD}})$, where $\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}$ and $\hat{\Sigma}_{\mathrm{RICD}}$ are reliable approximations to those obtained from a clean sample. This lemma, in conjunction with (A.1) and (A.2), suggests that we could use normal distributions to construct a threshold rule.

Note that both $\mathrm{tr}(S_n(\lambda)^{-1}\Sigma_p)$ and $\mathrm{tr}(S_n(\lambda)^{-1}\Sigma_p)^2$ in (A.1) involve the unknown covariance matrix Σ_p. Thus, Σ_p needs to be estimated in order to obtain the cutoff value for outlier identification. By the Stieltjes transform of the empirical spectral measure of a random matrix, we can simply adapt the estimates $\Theta^{(1)}(\lambda, c)$ and $\Theta^{(2)}(\lambda, c)$ from Ha et al. (2021).

The proofs of Theorems 1–2, Lemmas A.4–A.6, Theorem 3, and Proposition 1 are given below. For simplicity, we suppress the subscripts of Σ_p, S_n, $S_n(\lambda)$ and $\bar{\boldsymbol{X}}_n$, and suppress the second subscript n in the subscript $\{\ell, n\}$ if there is no confusion in the context.

Proof of Theorem 1. First we prove that $\varepsilon_n(\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}, \mathcal{X}) \leq (n-h+1)/n$. If we replace $(n-h+1)$ observations of the original data set $\mathcal{X}$, then the optimal subset $\tilde{H}_{\mathrm{RICD}}$ of $\tilde{\mathcal{X}}$ would contain at least one outlier, but the least square method breaks down even with one single outlier. Denote $\tilde{\boldsymbol{\mu}}_{\mathrm{RICD}} = \bar{\boldsymbol{x}}_{\tilde{H}_{\mathrm{RICD}}}$, it then follows that $\|\tilde{\boldsymbol{\mu}}_{\mathrm{RICD}}\|_F$ is not bounded.

On the other hand, to show $\varepsilon_n(\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}, \mathcal{X}) \geq (n-h+1)/n$, we prove that there exists a value M, which only depends on $\mathcal{X}$ and λ, such that for every $\tilde{\mathcal{X}}$ obtained by replacing at most $(n-h)$ observations in $\mathcal{X}$, the Frobenius norm of the RICD location estimate $\tilde{\boldsymbol{\mu}}_{\mathrm{RICD}}$ based on $\tilde{\mathcal{X}}$ is still bounded by M from above.

If we take any data set $\tilde{\mathcal{X}}$ by replacing $(n-h)$ observations in $\mathcal{X}$, there still exists a subset $H_1 \in \mathcal{H}$ containing indices only corresponding to the data points of the original dataset $\mathcal{X}$. The determinant of $S_{H_1}(\lambda)$ is

$$\begin{aligned}\det\left[S_{H_1}(\lambda)\right] &= \prod_{k=1}^{p}\eta_k \leq \left(\frac{1}{p}\sum_{k=1}^{p}\eta_k\right)^p \\ &= \left[\frac{1}{hp}\sum_{k=1}^{p}\sum_{j\in H_1}\left\{x_{jk}-\hat{\mu}_k(H_1)\right\}^2+\lambda\right]^p \\ &\leq \left(4N^2+\lambda\right)^p,\end{aligned}$$

where $(\eta_1, \ldots, \eta_p)$ are the eigenvalues of the matrix $S_{H_1}(\lambda)$, $\hat{\mu}_k(H_1)$ denotes the kth component of $\bar{\boldsymbol{x}}_{H_1}$, and N is defined as $\max_{1\leq i\leq n, 1\leq j\leq p}|x_{ij}|$.

Let H_2 be the optimal subset corresponding to $\tilde{\mathcal{X}}$, then $\tilde{\boldsymbol{\mu}}_{\mathrm{RICD}} = \bar{\boldsymbol{x}}_{H_2}$. Since $h-(n-h) \geq 1$, the set H_2 contains one observation $\boldsymbol{x}_{i_0}$ from $\mathcal{X}$. Thus we have

$$\det\left[S_{H_2}(\lambda)\right] = \det[A+B] = \det(A)\cdot\det\left(I_p + A^{-1}B\right),$$

where

$$A = h^{-1}\left(\boldsymbol{x}_{i_0}-\bar{\boldsymbol{x}}_{H_2}\right)\left(\boldsymbol{x}_{i_0}-\bar{\boldsymbol{x}}_{H_2}\right)^\top + 2^{-1}\lambda I_p,$$

and

$$B = h^{-1}\sum_{i\in H_2, i\neq i_0}\left(\boldsymbol{x}_i-\bar{\boldsymbol{x}}_{H_2}\right)\left(\boldsymbol{x}_i-\bar{\boldsymbol{x}}_{H_2}\right)^\top + 2^{-1}\lambda I_p.$$

It follows that

$$\begin{aligned}\det\left[S_{H_2}(\lambda)\right] &> \det(A)\\ &= 2^{-p}\lambda^p \det\left[I_p + \frac{2}{h\lambda}\left(\boldsymbol{x}_{i_0} - \bar{\boldsymbol{x}}_{H_2}\right)\left(\boldsymbol{x}_{i_0} - \bar{\boldsymbol{x}}_{H_2}\right)^\top\right]\\ &= 2^{-p}\lambda^p + \frac{1}{h}2^{1-p}\lambda^{p-1}\left(\boldsymbol{x}_{i_0} - \bar{\boldsymbol{x}}_{H_2}\right)^\top\left(\boldsymbol{x}_{i_0} - \bar{\boldsymbol{x}}_{H_2}\right).\end{aligned}$$

Let

$$M = p^{1/2}\left[\left\{\left[\left(4N^2+\lambda\right)^p - 2^{-p}\lambda^p\right]2^{p-1}\lambda^{1-p}h\right\}^{1/2} + N\right].$$

If $\|\tilde{\boldsymbol{\mu}}_{\mathrm{RICD}}\|_F > M$, then there exists j_0 such that $|\hat{\mu}_{j_0}(H_2)| > M/p^{1/2}$. Thus,

$$\begin{aligned}\det\left[S_{H_2}(\lambda)\right] &> 2^{-p}\lambda^p + \frac{1}{h}2^{1-p}\lambda^{p-1}\left[x_{i_0j_0} - \hat{\mu}_{j_0}(H_2)\right]^2\\ &\geq 2^{-p}\lambda^p + \frac{1}{h}2^{1-p}\lambda^{p-1}\left[|x_{i_0j_0}| - |\hat{\mu}_{j_0}(\mathrm{H}_2)|\right]^2\\ &\geq 2^{-p}\lambda^p + \frac{1}{h}2^{1-p}\lambda^{p-1}\left[\frac{M}{p^{1/2}} - N\right]^2\\ &= \left(4N^2+\lambda\right)^p\end{aligned}$$

by the definition of M. This implies $\det\left[S_{H_2}(\lambda)\right] > \det\left[S_{H_1}(\lambda)\right]$, which contradicts the definition of $\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}$. So, we conclude that $\|\tilde{\boldsymbol{\mu}}_{\mathrm{RICD}}\|_F \leq M$. Since $\hat{\Sigma}_{\mathrm{RICD}}$ is obtained from $\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}$ based on the same subset H_{RICD}, we have $\varepsilon_n(\hat{\Sigma}_{\mathrm{RICD}}, \mathcal{X}) = \varepsilon_n(\hat{\boldsymbol{\mu}}_{\mathrm{RICD}}, \mathcal{X})$, which concludes the proof of Theorem 1.

Proof of Theorem 2. The conclusions of Theorem 2 can be derived from Theorem 1 of Boudt et al. (2019), which is briefly described below:

For a given H, Boudt et al. (2019) regularized the sample covariance matrix S_H as $K_H = \rho T + (1-\rho)S_H$, where $0 < \rho < 1$ is a scalar weight coefficient and T is a predetermined positive-definite target matrix. One can thus compute the distance $d_i^2(\bar{\boldsymbol{x}}_H, K_H) = \left(\boldsymbol{x}_i - \bar{\boldsymbol{x}}_H\right)^\top K_H^{-1}\left(\boldsymbol{x}_i - \bar{\boldsymbol{x}}_H\right)$. If we take $T = I_p$ and $\rho = \lambda/(1+\lambda)$, we have $S_H(\lambda) = (\lambda+1)K_H$, $d_i^2(\bar{\boldsymbol{x}}_H, S_H(\lambda)) = (\lambda+1)^{-1}d_i^2(\bar{\boldsymbol{x}}_H, K_H)$. Thus, Theorem 1 follows from Theorem 1 of Boudt et al. (2019).

Proof of Lemma A.4. First, let $V = \Sigma^{1/2}S(\lambda)^{-1}\Sigma^{1/2}$. By Condition A2 and the definition of $S(\lambda)$, the matrix V can be decomposed as $Q^\top \Lambda Q$, where Q is an orthogonal matrix and Λ is a diagonal matrix with positive diagonal elements $\zeta_{n,1} \leq \zeta_{n,2} \leq \cdots \leq \zeta_{n,p}$. It is obvious that for any n, the largest eigenvalue of $S(\lambda)^{-1}$ is bounded above by $1/\lambda$. On the other hand, Theorem 3.6 in Bai and Silverstein (2010) implies that $F^S(x)$ tends to the M-P law under Condition A1 (Bai and Silverstein (2010, Eq.(3.1.1))), and hence the largest eigenvalue of S is bounded away from infinity asymptotically. Therefore, we conclude that $\{\zeta_{n,i}\}$ are bounded away from both zero and infinity asymptotically.

Next, by the definition of V, we have

$$\tilde{d}^2(\boldsymbol{\mu}, S(\lambda)) = \tilde{\boldsymbol{Y}}^\top \Lambda \tilde{\boldsymbol{Y}} = \sum_{i=1}^{p} \zeta_{n,i} \tilde{y}_i^2 = \sum_{i=1}^{p} \zeta_{n,i} w_{n,i}, \tag{A.6}$$

where $\tilde{\boldsymbol{Y}} = Q\Sigma^{-1/2} T_p \tilde{\boldsymbol{Z}} = (\tilde{y}_1, \ldots, \tilde{y}_p)^\top$ with $\tilde{\boldsymbol{X}} = T_p \tilde{\boldsymbol{Z}} + \boldsymbol{\mu}$ (2.1), and $w_{n,i} = \tilde{y}_i^2$. Since $\tilde{\boldsymbol{X}}$ is independent of S thus independent of Q, by Conditions A2 and A5, $\mathrm{E}\, w_{n,i} = 1$, $\mathrm{E}\, w_{n,i}^2 = 3$.

Let $W_{n,i} = \zeta_{n,i}(w_{n,i} - 1)$, $\vartheta_p = \sqrt{2\sum_{i=1}^{p} \zeta_{n,i}^2}$. Denote the σ-field generated by $\{\zeta_{n,1}, \ldots, \zeta_{n,p}\}$ by $\mathcal{F}$. It is easy to see that $\sqrt{2p}\zeta_{n,1} \le \vartheta_p \le \sqrt{2p}\zeta_{n,p}$. Conditional on $\mathcal{F}$, we have $\mathrm{E}(W_{n,i} \mid \mathcal{F}) = 0$, $\mathrm{E}(W_{n,i}^2 \mid \mathcal{F}) = 2\zeta_{n,i}^2$, and $\sum_{i=1}^{p} \mathrm{E}((W_{n,i}/\vartheta_p)^2 \mid \mathcal{F}) = 1$. It follows that

$$\begin{aligned}
&\sum_{i=1}^{p} \mathrm{E}\left(\left| \frac{W_{n,i}}{\vartheta_p} \right|^2 ; \left| \frac{W_{n,i}}{\vartheta_p} \right| > \epsilon \mid \mathcal{F} \right) \\
&= \frac{1}{\vartheta_p^2} \sum_{i=1}^{p} \mathrm{E}\left(\zeta_{n,i}^2 (w_{n,i} - 1)^2 ; |w_{n,i} - 1| > \epsilon \frac{\vartheta_p}{\zeta_{n,i}} \mid \mathcal{F} \right) \\
&\le \frac{1}{\vartheta_p^2} \sum_{i=1}^{p} \mathrm{E}\left(\zeta_{n,p}^2 (w_{n,i} - 1)^2 ; |w_{n,i} - 1| > \epsilon \frac{\vartheta_p}{\zeta_{n,p}} \mid \mathcal{F} \right) \\
&\le \frac{p}{\vartheta_p^2} \mathrm{E}\left(\zeta_{n,p}^2 (w_{n,i} - 1)^2 ; |w_{n,i} - 1| > \epsilon \frac{\sqrt{2p}\zeta_{n,1}}{\zeta_{n,p}} \mid \mathcal{F} \right) \\
&\le \frac{1}{2\zeta_{n,1}^2} \mathrm{E}\left(W_{n,p}^2 ; |W_{n,p}| > \epsilon\sqrt{2p}\zeta_{n,1} \mid \mathcal{F} \right) \\
&\xrightarrow{\mathrm{p}} 0, \quad \text{as } p \to \infty.
\end{aligned}$$

Here $\mathrm{E}(x; a \mid b)$ denotes the expected value of x restricted to a while conditioned on b. Then, according to the Lindeberg-Feller central limit theorem, we have $(\sum_{i=1}^{p} W_{p,i}/\vartheta_p) \mid \mathcal{F} \xrightarrow{D} \mathrm{N}(0,1)$. Base on Lemma A.1 we have

$$\frac{\tilde{d}^2(\boldsymbol{\mu}, S(\lambda)) - \sum_{i=1}^{p} \zeta_{n,i}}{\sqrt{2\sum_{i=1}^{p} \zeta_{n,i}^2}} \xrightarrow{D} \mathrm{N}(0,1).$$

The proof is complete.

Proof of Lemma A.5. By (A.6), we have $\tilde{d}^2(\boldsymbol{\mu}, S(\lambda)) = \tilde{\boldsymbol{Y}}^\top \Lambda \tilde{\boldsymbol{Y}}$, where $\tilde{\boldsymbol{Y}} = (\tilde{y}_1, \ldots, \tilde{y}_p)^\top$. Similarly, for each $\boldsymbol{X}_i$, $i = 1, \ldots, n$, we can also define $\boldsymbol{Y}_i = Q\Sigma^{-1/2} T_p \boldsymbol{Z}_i$ with $\boldsymbol{X}_i = T_p \boldsymbol{Z}_i + \boldsymbol{\mu}$ and $\bar{\boldsymbol{Y}} = n^{-1} \sum_{i=1}^{n} \boldsymbol{Y}_i$, where $\boldsymbol{Y}_i = (y_{i1}, \ldots, y_{ip})^\top$. Then

$$\left| \tilde{d}^2(\bar{\boldsymbol{X}}, S(\lambda)) - \tilde{d}^2(\boldsymbol{\mu}, S(\lambda)) \right| = \left| \left(\tilde{\boldsymbol{Y}} - \bar{\boldsymbol{Y}} \right)^\top \Lambda \left(\tilde{\boldsymbol{Y}} - \bar{\boldsymbol{Y}} \right) - \tilde{\boldsymbol{Y}}^\top \Lambda \tilde{\boldsymbol{Y}} \right|$$

$$= \left|\bar{\boldsymbol{Y}}^\top \Lambda \bar{\boldsymbol{Y}} - 2\tilde{\boldsymbol{Y}}^\top \Lambda \bar{\boldsymbol{Y}}\right| \le \left|\bar{\boldsymbol{Y}}^\top \Lambda \bar{\boldsymbol{Y}}\right| + 2\left|\tilde{\boldsymbol{Y}}^\top \Lambda \bar{\boldsymbol{Y}}\right|.$$

As discussed in the proof of Lemma A.4, by Conditions A1 and A4 and the fact that the largest eigenvalue of $S(\lambda)^{-1}$ is bounded above by $1/\lambda$, the spectral norm of Λ, $\zeta_{n,p}$, is bounded above, say by ϖ. By Conditions A2 and the definition of $\boldsymbol{Y}_i$, we have

$$\mathrm{E}\, y_{ij} = 0, \quad \mathrm{E}\, y_{ij}^2 = 1.$$

Similar arguments also hold for $\tilde{y}_j$, $j = 1, \ldots, p$. Therefore, we have, for large n and p,

$$\mathrm{E}\left(\left|\bar{\boldsymbol{Y}}^\top \Lambda \bar{\boldsymbol{Y}}\right|\right) \le \varpi\, \mathrm{E}\left(\bar{\boldsymbol{Y}}^\top \bar{\boldsymbol{Y}}\right) \le \varpi\, \mathrm{E}\left[\sum_{j=1}^{p}\left(\frac{1}{n}\sum_{i=1}^{n} y_{ij}\right)^2\right] = \frac{\varpi p}{n} < 2c\varpi,$$

and

$$\mathrm{E}\left(\left|\tilde{\boldsymbol{Y}}^\top \Lambda \bar{\boldsymbol{Y}}\right|\right) \le \varpi\, \mathrm{E}\left(\tilde{\boldsymbol{Y}}^\top \bar{\boldsymbol{Y}}\right) \le \varpi\, \mathrm{E}\left[\sum_{j=1}^{p}\left(\frac{1}{n}\sum_{i=1}^{n} y_{ij}\tilde{y}_j\right)\right] < 2c\varpi,$$

which concludes the lemma.

Proof of Lemma A.6. Following steps of the truncation, centralization, and rescaling similar to those in Bai and Silverstein (2004), we may assume that the random variables $\{x_{ij}\}$ satisfy that

$$\mathrm{E}\, x_{ij} = 0, \quad \mathrm{E}\, x_{ij}^2 = 1, \quad \mathrm{E}\, x_{ij}^4 < \infty, \quad \text{and} \quad |x_{ij}| \le \eta_n \sqrt{n},$$

where $\{\eta_n\}$ is a deterministic sequence such that $\eta_n \downarrow 0$ whose convergence rate can be made arbitrarily slow. Under these assumptions, for any $\alpha > 4$, we have

$$\mathrm{E}\,|x_{ij}|^\alpha = O\left(\left(\eta_n\sqrt{n}\right)^{\alpha-4}\right).$$

Since

$$\bar{\boldsymbol{X}} = \bar{\boldsymbol{X}}_{k0} + \frac{1}{n}\boldsymbol{X}_k,$$

we have

$$\begin{aligned} S_n &= S_{n,k0} + a_n \boldsymbol{X}_k \boldsymbol{X}_k^\top - n^{-1}\boldsymbol{X}_k \bar{\boldsymbol{X}}_{k0}^\top - n^{-1}\bar{\boldsymbol{X}}_{k0}\boldsymbol{X}_k^\top \\ &= S_{n,k+} - n^{-1}\left(\boldsymbol{X}_k \bar{\boldsymbol{X}}_{k0}^\top + \bar{\boldsymbol{X}}_{k0}\boldsymbol{X}_k^\top\right), \end{aligned}$$

where $S_{n,k+} = S_{n,k0} + a_n \boldsymbol{X}_k \boldsymbol{X}_k^\top$ with $a_n = (n-1)/n^2$. For simplicity in writing, denote $R_n = S_n(\lambda)$, $R_0 = S_0(\lambda)$, and $R_1 = S_{n,k+} + \lambda I_p$. By the inverse matrix formula,

$$R_n^{-1} = R_1^{-1} + R_1^{-1}\left(n^{-1}\boldsymbol{X}_k, \bar{\boldsymbol{X}}_{k0}\right)\Delta^{-1}\begin{pmatrix} \bar{\boldsymbol{X}}_{k0}^\top \\ n^{-1}\boldsymbol{X}_k^\top \end{pmatrix} R_1^{-1}, \tag{A.7}$$

where

$$R_1^{-1} = R_0^{-1} - \frac{a_n R_0^{-1} \boldsymbol{X}_k \boldsymbol{X}_k^\top R_0^{-1}}{1 + a_n \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k},$$

and

$$\Delta = I_2 - \begin{pmatrix} n^{-1} \bar{\boldsymbol{X}}_{k0}^\top R_1^{-1} \boldsymbol{X}_k & \bar{\boldsymbol{X}}_{k0}^\top R_1^{-1} \bar{\boldsymbol{X}}_{k0} \\ n^{-2} \boldsymbol{X}_k^\top R_1^{-1} \boldsymbol{X}_k & n^{-1} \boldsymbol{X}_k^\top R_1^{-1} \bar{\boldsymbol{X}}_{k0} \end{pmatrix}.$$

Denote

$$\Upsilon_k = R_1^{-1} \left(n^{-1} \boldsymbol{X}_k, \bar{\boldsymbol{X}}_{k0} \right) \Delta^{-1} \begin{pmatrix} \bar{\boldsymbol{X}}_{k0}^\top \\ n^{-1} \boldsymbol{X}_k^\top \end{pmatrix} R_1^{-1}.$$

We have

$$\Upsilon_k = \frac{\Upsilon}{\left(1 - n^{-1} \boldsymbol{X}_k^\top R_1^{-1} \bar{\boldsymbol{X}}_{k0}\right)^2 - n^{-2} \boldsymbol{X}_k^\top R_1^{-1} \boldsymbol{X}_k \bar{\boldsymbol{X}}_{k0}^\top R_1^{-1} \bar{\boldsymbol{X}}_{k0}}, \tag{A.8}$$

where

$$\begin{aligned} \Upsilon = {} & n^{-1} R_1^{-1} \boldsymbol{X}_k \left(1 - n^{-1} \boldsymbol{X}_k^\top R_1^{-1} \bar{\boldsymbol{X}}_{k0}\right) \bar{\boldsymbol{X}}_{k0}^\top R_1^{-1} \\ & + n^{-2} R_1^{-1} \bar{\boldsymbol{X}}_{k0} \boldsymbol{X}_k^\top R_1^{-1} \boldsymbol{X}_k \bar{\boldsymbol{X}}_{k0}^\top R_1^{-1} \\ & + n^{-2} R_1^{-1} \boldsymbol{X}_k \bar{\boldsymbol{X}}_{k0}^\top R_1^{-1} \bar{\boldsymbol{X}}_{k0} \boldsymbol{X}_k^\top R_1^{-1} \\ & + n^{-1} R_1^{-1} \bar{\boldsymbol{X}}_{k0} \left(1 - n^{-1} \bar{\boldsymbol{X}}_{k0}^\top R_1^{-1} \boldsymbol{X}_k\right) \boldsymbol{X}_k^\top R_1^{-1}. \end{aligned} \tag{A.9}$$

Let $\beta_k = 1/(1 + a_n \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k)$. By applying the identity

$$R_1^{-1} = R_0^{-1} - a_n \beta_k R_0^{-1} \boldsymbol{X}_k \boldsymbol{X}_k^\top R_0^{-1}, \tag{A.10}$$

we obtain that

$$\boldsymbol{X}_k^\top \Upsilon \boldsymbol{X}_k := \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

where

$$\begin{aligned} \mathrm{I} = {} & n^{-1} \beta_k \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k \left(1 - n^{-1} \beta_k \boldsymbol{X}_k^\top R_0^{-1} \bar{\boldsymbol{X}}_{k0}\right) \\ & \times \left(\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1} \boldsymbol{X}_k - a_n \beta_k \bar{\boldsymbol{X}}_{k0}^\top R_0^{-1} \boldsymbol{X}_k \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k\right), \\ \mathrm{II} = {} & n^{-2} \beta_k \left(\boldsymbol{X}_k^\top R_0^{-1} \bar{\boldsymbol{X}}_{k0} - a_n \beta_k \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k \boldsymbol{X}_k^\top R_0^{-1} \bar{\boldsymbol{X}}_{k0}\right) \\ & \times \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k \left(\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1} \boldsymbol{X}_k - a_n \beta_k \bar{\boldsymbol{X}}_{k0}^\top R_0^{-1} \boldsymbol{X}_k \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k\right), \\ \mathrm{III} = {} & n^{-2} \beta_k^2 \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k \left(\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1} \bar{\boldsymbol{X}}_{k0}\right. \\ & \left. - a_n \beta_k \bar{\boldsymbol{X}}_{k0}^\top R_0^{-1} \boldsymbol{X}_k \boldsymbol{X}_k^\top R_0^{-1} \bar{\boldsymbol{X}}_{k0}\right) \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k, \end{aligned}$$

and

$$\begin{aligned} \mathrm{IV} = {} & n^{-1} \beta_k \left(\boldsymbol{X}_k^\top R_0^{-1} \bar{\boldsymbol{X}}_{k0} - a_n \beta_k \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k \boldsymbol{X}_k^\top R_0^{-1} \bar{\boldsymbol{X}}_{k0}\right) \\ & \times \left(1 - n^{-1} \beta_k \bar{\boldsymbol{X}}_{k0}^\top R_0^{-1} \boldsymbol{X}_k\right) \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k. \end{aligned}$$

For the first term I, we have

$$\begin{aligned} \mathrm{I} = {} & n^{-1} \beta_k \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k \bar{\boldsymbol{X}}_{k0}^\top R_0^{-1} \boldsymbol{X}_k \\ & - n^{-2} \beta_k^2 \boldsymbol{X}_k^\top R_0^{-1} \boldsymbol{X}_k \boldsymbol{X}_k^\top R_0^{-1} \bar{\boldsymbol{X}}_{k0} \bar{\boldsymbol{X}}_{k0}^\top R_0^{-1} \boldsymbol{X}_k \end{aligned}$$

$$-n^{-1}a_n\beta_k^2\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k$$
$$+n^{-2}a_n\beta_k^3\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\bar{\boldsymbol{X}}_{k0}\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k. \quad \text{(A.11)}$$

Note that β_k and $\|R_i\|$ for $i = n$, 0, or 1 are all bounded by some constant. It is easy to show that the order of the difference between $1/\left(1+\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k/n\right)$ and $\beta_k = 1/\left(1+a_n\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right)$ is $O_{L_1}\left(n^{-1}\right)$, say $\iota_n = O_{L_1}\left(n^{-1}\right)$, denoting that $\mathrm{E}\left|n\iota_n\right|$ is bounded by some constant. Thus, we simplify (A.11) by substituting β_k with $1/\left(1+\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k/n\right)$. Similarly, we substitute a_n with $1/n$ there. By applying Lemma A.3 and Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
&\mathrm{E}\left|n^{-1}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\right| \\
&\le \sqrt{\mathrm{E}\left|n^{-1}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right|^2\mathrm{E}\left|\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\right|^2} \\
&= O(1), \\
&\mathrm{E}\left|n^{-1}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\bar{\boldsymbol{X}}_{k0}\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\right| \\
&\le \sqrt{\mathrm{E}\left|n^{-1}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right|^2\mathrm{E}\left|\boldsymbol{X}_k^\top R_0^{-1}\bar{\boldsymbol{X}}_{k0}\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\right|^2} \\
&\le \sqrt{\mathrm{E}\left|n^{-1}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right|^2\sqrt{\mathrm{E}\left|\boldsymbol{X}_k^\top R_0^{-1}\bar{\boldsymbol{X}}_{k0}\right|^4\mathrm{E}\left|\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\right|^4}} \\
&= O(1), \\
&\mathrm{E}\left|n^{-2}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right| \\
&\le \sqrt{\mathrm{E}\left|n^{-1}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right|^2\mathrm{E}\left|n^{-1}\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right|^2} \\
&\le \sqrt{\mathrm{E}\left|n^{-1}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right|^2\sqrt{\mathrm{E}\left|\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\right|^4\mathrm{E}\left|n^{-1}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right|^4}} \\
&= O(1),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{E}\left|n^{-2}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\bar{\boldsymbol{X}}_{k0}\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right| \\
&\le \sqrt{\mathrm{E}\left|n^{-1}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\bar{\boldsymbol{X}}_{k0}\right|^2\mathrm{E}\left|n^{-1}\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right|^2} \\
&\le \sqrt{\mathrm{E}\left|n^{-1}\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\right|^4\mathrm{E}\left|\bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\boldsymbol{X}_k\right|^4} \\
&= O(1),
\end{aligned}$$

which imply that $\mathrm{I} = O_{L_1}(1)$. The orders of the other three terms, that is, II, III, and IV, can be derived similarly, from which one can verify that

$$\boldsymbol{X}_k^\top\Upsilon\boldsymbol{X}_k = O_{L_1}(1).$$

Furthermore, by (4.33) of Ha et al. (2021), the denominator of Υ_k in (A.8) has

the order of $1 + O_{L_1}(1)$, and hence it follows that

$$\boldsymbol{X}_k^\top \Upsilon_k \boldsymbol{X}_k = O_{L_1}(1). \tag{A.12}$$

Similarly, it can be shown that

$$\boldsymbol{X}_k^\top \Upsilon_k \bar{\boldsymbol{X}}_{k0} = O_{L_1}(1). \tag{A.13}$$

Returning to the first argument (A.3) of Lemma A.6, we have

$$\begin{aligned}
&d_k^2\left(\bar{\boldsymbol{X}}, S_n(\lambda)\right) - d_k^2\left(\bar{\boldsymbol{X}}_{k0}, S_0(\lambda)\right) \\
&= (\boldsymbol{X}_k - \bar{\boldsymbol{X}})^\top R_n^{-1}(\boldsymbol{X}_k - \bar{\boldsymbol{X}}) - (\boldsymbol{X}_k - \bar{\boldsymbol{X}}_{k0})^\top R_0^{-1}(\boldsymbol{X}_k - \bar{\boldsymbol{X}}_{k0}) \\
&= \boldsymbol{X}_k^\top R_n^{-1}\boldsymbol{X}_k + (\bar{\boldsymbol{X}}_{k0} + n^{-1}\boldsymbol{X}_k)^\top R_n^{-1}(\bar{\boldsymbol{X}}_{k0} + n^{-1}\boldsymbol{X}_k) + 2\boldsymbol{X}_k^\top R_0^{-1}\bar{\boldsymbol{X}}_{k0} \\
&\quad -2\boldsymbol{X}_k^\top R_n^{-1}(\bar{\boldsymbol{X}}_{k0} + n^{-1}\boldsymbol{X}_k) - \boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k - \bar{\boldsymbol{X}}_{k0}^\top R_0^{-1}\bar{\boldsymbol{X}}_{k0},
\end{aligned}$$

which, jointly with Lemma A.3, (A.7), (A.8), (A.10), (A.12) and (A.13), implies that

$$d_k^2\left(\bar{\boldsymbol{X}}, S_n(\lambda)\right) - d_k^2\left(\bar{\boldsymbol{X}}_{k0}, S_0(\lambda)\right) = -a_n\beta_k \boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k + O_{L_1}(1). \tag{A.14}$$

By the end of the proof of their Lemma 4.3 on Page 14 of Ha et al. (2021), we have that for any z_i satisfying the conditions of Lemma A.3,

$$\sum_{i=1}^{p} \mathrm{E}\left(\frac{1}{n}\left|z_i\right|^2\right)^v \leq \begin{cases} O\left(n^{-v+1}\right) & \text{if } v \leq 2, \\ O\left(\eta_n^{2v-4} n^{-1}\right) & \text{if } v > 2. \end{cases}$$

By replacing the coefficient $1/n$ of $|z_i|^2$ with $n^{-1/2}$ in the above inequality, and taking $v = 2$, it is obvious that

$$\sum_{i=1}^{p} \mathrm{E}\left(n^{-1/2}\left|z_i\right|^2\right)^2 \leq |O(1)|.$$

Thus, $\mathrm{E}(-a_n\beta_k(\boldsymbol{X}_k^\top R_0^{-1}\boldsymbol{X}_k)^2)$, the expectation of the first term of (A.14), has the order of $O(1)$, which concludes the first argument of Lemma A.6.

Next, we consider the third argument of Lemma A.6, that is, (A.5). We have

$$\mathrm{tr}\left(R_0^{-1}\Sigma\right) - \mathrm{tr}\left(R_n^{-1}\Sigma\right) = \mathrm{tr}\left(R_0^{-1} - R_0^{-1} + a_n\beta_k R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1} - \Upsilon_k\right)\Sigma.$$

As it has been shown above that

$$\mathrm{tr}\left(a_n\beta_k R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\right)\Sigma = a_n\beta_k \boldsymbol{X}_k^\top R_0^{-1}\Sigma R_0^{-1}\boldsymbol{X}_k = O_{L_1}(1),$$

we only need to find the order of $\mathrm{tr}(\Upsilon_k\Sigma)$. By the first term of $\mathrm{tr}(\Upsilon\Sigma)$ in (A.9),

we have

$$\begin{aligned}&\operatorname{tr}\left(n^{-1}R_1^{-1}\boldsymbol{X}_k\left(1-n^{-1}\boldsymbol{X}_k^\top R_1^{-1}\bar{\boldsymbol{X}}_{k0}\right)\bar{\boldsymbol{X}}_{k0}^\top R_1^{-1}\Sigma\right)\\&=n^{-1}\bar{\boldsymbol{X}}_{k0}^\top R_1^{-1}\Sigma R_1^{-1}\boldsymbol{X}_k\left(1-n^{-1}\boldsymbol{X}_k^\top R_1^{-1}\bar{\boldsymbol{X}}_{k0}\right)\\&=O_{L_1}(n^{-1}),\end{aligned}$$

and we can also show that the rest terms are also $O_{L_1}(n^{-1})$. Thus, we obtain that $\operatorname{tr}\left(R_0^{-1}\Sigma\right)-\operatorname{tr}\left(R_n^{-1}\Sigma\right)=O_{L_1}(1)$.

We now prove the second argument of Lemma A.6, that is, (A.4). By the fact that

$$\begin{aligned}&\operatorname{tr}\left(R_n^{-1}\Sigma R_n^{-1}\Sigma\right)-\operatorname{tr}\left(R_0^{-1}\Sigma R_0^{-1}\Sigma\right)\\&=\operatorname{tr}\left[-a_n\beta_k R_0^{-1}\Sigma R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\Sigma+R_0^{-1}\Sigma\Upsilon_k\Sigma\right.\\&\quad-a_n\beta_k R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\Sigma R_0^{-1}\Sigma+\Upsilon_k\Sigma\Upsilon_k\Sigma\\&\quad-a_n\beta_k R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\Sigma\Upsilon_k\Sigma+\Upsilon_k\Sigma R_0^{-1}\Sigma\\&\quad+a_n^2\beta_k^2 R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\Sigma R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\Sigma\\&\quad\left.-a_n\beta_k\Upsilon_k\Sigma R_0^{-1}\boldsymbol{X}_k\boldsymbol{X}_k^\top R_0^{-1}\Sigma\right],\end{aligned}$$

it follows that $\operatorname{tr}\left(R_n^{-1}\Sigma R_n^{-1}\Sigma\right)-\operatorname{tr}\left(R_0^{-1}\Sigma R_0^{-1}\Sigma\right)=O_{L_1}(1)$, which completes the proof of (A.4).

Proof of Theorem 3. In view of Lemma A.2 and Lemmas A.2–A.6, Theorem 3 is a natural extension by applying the Slutsky's Theorem, as

$$\frac{d_k^2(\bar{\boldsymbol{X}},S_n(\lambda))-\operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma\right)}{\sqrt{2\operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma\right)^2}}=(C_1+C_2+C_3+C_4)\times C_5,$$

where

$$\begin{aligned}&C_1=\frac{d_k^2\left(\bar{\boldsymbol{X}},S_n(\lambda)\right)-d_k^2\left(\bar{\boldsymbol{X}}_{k0},S_0(\lambda)\right)}{\sqrt{2\operatorname{tr}\left(S_0(\lambda)^{-1}\Sigma\right)^2}},\quad C_2=\frac{d_k^2\left(\bar{\boldsymbol{X}}_{k0},S_0(\lambda)\right)-d_k^2\left(\boldsymbol{\mu},S_0(\lambda)\right)}{\sqrt{2\operatorname{tr}\left(S_0(\lambda)^{-1}\Sigma\right)^2}},\\&C_3=\frac{d_k^2\left(\boldsymbol{\mu},S_0(\lambda)\right)-\operatorname{tr}\left(S_0(\lambda)^{-1}\Sigma\right)}{\sqrt{2\operatorname{tr}\left(S_0(\lambda)^{-1}\Sigma\right)^2}},\quad C_4=\frac{\operatorname{tr}\left(S_0(\lambda)^{-1}\Sigma\right)-\operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma\right)}{\sqrt{2\operatorname{tr}\left(S_0(\lambda)^{-1}\Sigma\right)^2}},\\&C_5=\frac{\sqrt{2\operatorname{tr}\left(S_0(\lambda)^{-1}\Sigma\right)^2}}{\sqrt{2\operatorname{tr}\left(S_n(\lambda)^{-1}\Sigma\right)^2}}.\end{aligned}$$

Proof of Proposition 1. We first consider the moment generating function,

$$M(\boldsymbol{T})=\mathrm{E}\left(e^{\boldsymbol{T}^\top\boldsymbol{X}_1}\mid w_1=1\right).\tag{A.15}$$

Following the discussion about $\tilde{d}^2(\boldsymbol{\mu}, S(\lambda))$ in the proof of Lemma A.4, we let $V = \Sigma^{1/2} S(\lambda)^{-1} \Sigma^{1/2}$. Assume that $V = Q^\top \Lambda Q$, where $Q^\top Q = I_p$ and $\Lambda = \operatorname{diag}(\zeta_1, \ldots, \zeta_p)$. We have

$$
\begin{aligned}
& M(\boldsymbol{T}) \\
&= \frac{1}{1-\delta} \mathrm{E}\left\{ e^{\boldsymbol{T}^\top \boldsymbol{X}_1} \mathrm{I}(w_1 = 1) \right\} \\
&= \frac{1}{1-\delta} \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \\
&\quad \int_{\{(\boldsymbol{X}_1 - \boldsymbol{\mu})^\top S(\lambda)^{-1} (\boldsymbol{X}_1 - \boldsymbol{\mu}) \le a_\delta\}} \exp\left\{ \frac{\boldsymbol{T}^\top \boldsymbol{X}_1 - (\boldsymbol{X}_1 - \boldsymbol{\mu})^\top \Sigma^{-1} (\boldsymbol{X}_1 - \boldsymbol{\mu})}{2} \right\} d\boldsymbol{X}_1 \\
&= \frac{1}{1-\delta} \frac{1}{(2\pi)^{p/2}} e^{\boldsymbol{T}^\top \boldsymbol{\mu} + \boldsymbol{T}^\top \Sigma \boldsymbol{T} 2} \\
&\quad \int_{\{\boldsymbol{z}^\top \Lambda \boldsymbol{z} \le a_\delta\}} \exp\left\{ -\frac{\left(\boldsymbol{z} - Q\Sigma^{1/2}\boldsymbol{T}\right)^\top \left(\boldsymbol{z} - Q\Sigma^{1/2}\boldsymbol{T}\right)}{2} \right\} d\boldsymbol{z} \\
&= \frac{1}{1-\delta} e^{\boldsymbol{T}^\top \boldsymbol{\mu} + \boldsymbol{T}^\top \Sigma \boldsymbol{T}/2} F_{\boldsymbol{T}}(a_\delta), \qquad \text{(A.16)}
\end{aligned}
$$

where $\boldsymbol{z} = Q\Sigma^{-1/2}(\boldsymbol{X}_1 - \boldsymbol{\mu})$, and $F_{\boldsymbol{T}}(a)$ is the cumulative distribution function of the non-negative definite quadratic form in non-central normal variables, that is

$$
F_{\boldsymbol{T}}(a) = \mathrm{P}\left(\boldsymbol{Z}_{\boldsymbol{v}}^\top \Lambda \boldsymbol{Z}_{\boldsymbol{v}} \le a\right), \quad \boldsymbol{Z}_{\boldsymbol{v}} \sim N(\boldsymbol{v}, I_p), \quad \boldsymbol{v} = Q\Sigma^{1/2}\boldsymbol{T}.
$$

Without loss of generality, we prove the proposition for $x_{11} \mid w_1 = 1$, whose moment generating function is

$$
m_1(t_1) = \mathrm{E}\left(e^{t_1 x_{11}} \mid w_1 = 1\right).
$$

In (A.15), let $\boldsymbol{T} = (t_1, 0, \ldots, 0)^\top$ with $p-1$ components of 0s. Then, it follows from (A.16) that

$$
m_1(t_1) = \frac{1}{1-\delta} e^{t_1 \mu_1 + \sigma_{11} t_1^2/2} F_{t_1}(a_\delta),
$$

where

$$
F_{t_1}(a_\delta) = \frac{1}{(2\pi)^{p/2}} \int_{\{\boldsymbol{z}^\top \Lambda \boldsymbol{z} \le a_\delta\}} \exp\left\{ \frac{-\left(\boldsymbol{z} - t_1 \boldsymbol{v}_1\right)^\top \left(\boldsymbol{z} - t_1 \boldsymbol{v}_1\right)}{2} \right\} d\boldsymbol{z},
$$

$\boldsymbol{v}_1$ is the first row of $Q\Sigma^{1/2}$ and $\boldsymbol{v}_1^\top \boldsymbol{v}_1 = \sigma_{11}$. Since a_δ is the upper δ-quantile of $d_k^2(\boldsymbol{\mu}, S(\lambda))$, by the Berry-Esseen inequality, we have

$$
\frac{a_\delta - \operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)}{\sqrt{2 \operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)^2}} = z_\delta + o(1).
$$

It is straightforward to show that

$$
\begin{aligned}
&F_{t_1}(a_\delta)|_{t_1=0} = \frac{1}{(2\pi)^{p/2}} \int_{\{\boldsymbol{z}^\top \Lambda \boldsymbol{z} \le a_\delta\}} \exp\left(-\frac{\boldsymbol{z}^\top \boldsymbol{z}}{2}\right) d\boldsymbol{z} = \mathrm{P}\left\{d_k^2(\boldsymbol{\mu}, S(\lambda)) \le a_\delta\right\}, \\
&\left.\frac{\partial F_{t_1}(a_\delta)}{\partial t_1}\right|_{t_1=0} \\
&= \frac{1}{(2\pi)^{p/2}} \int_{\{\boldsymbol{z}^\top \Lambda \boldsymbol{z} \le a_\delta\}} \left(\boldsymbol{v}_1^\top \boldsymbol{z} - \boldsymbol{v}_1^\top \boldsymbol{v}_1 t_1\right) \exp\left\{-\frac{(\boldsymbol{z} - t_1\boldsymbol{v}_1)^\top (\boldsymbol{z} - t_1\boldsymbol{v}_1)}{2}\right\} d\boldsymbol{z}\Bigg|_{t_1=0} \\
&= \frac{1}{(2\pi)^{p/2}} \int_{\{\boldsymbol{z}^\top \Lambda \boldsymbol{z} \le a_\delta\}} \left(\boldsymbol{v}_1^\top \boldsymbol{z}\right) \exp\left(-\frac{\boldsymbol{z}^\top \boldsymbol{z}}{2}\right) d\boldsymbol{z} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left.\frac{\partial^2 F_{t_1}(a_\delta)}{\partial t_1^2}\right|_{t_1=0} \\
&= \frac{1}{(2\pi)^{p/2}} \int_{\{\boldsymbol{z}^\top \Lambda \boldsymbol{z} \le a_\delta\}} \left\{\left(\boldsymbol{v}_1^\top \boldsymbol{z} - \boldsymbol{v}_1^\top \boldsymbol{v}_1 t_1\right)^2 - \boldsymbol{v}_1^\top \boldsymbol{v}_1\right\} \\
&\quad \exp\left\{-\frac{(\boldsymbol{z} - t_1\boldsymbol{v}_1)^\top (\boldsymbol{z} - t_1\boldsymbol{v}_1)}{2}\right\} d\boldsymbol{z}\Bigg|_{t_1=0} \\
&= \frac{1}{(2\pi)^{p/2}} \int_{\{\boldsymbol{z}^\top \Lambda \boldsymbol{z} \le a_\delta\}} \left(\sum_{j=1}^p v_{1j}^2 z_j^2\right) \exp\left(-\frac{\boldsymbol{z}^\top \boldsymbol{z}}{2}\right) d\boldsymbol{z} - \sigma_{11}\mathrm{P}\left(d_k^2(\boldsymbol{\mu}, S(\lambda)) \le a_\delta\right) \\
&= -\sigma_{11}\mathrm{P}\left\{d_k^2(\boldsymbol{\mu}, S(\lambda)) \le a_\delta\right\} \\
&\quad + \sum_{j=1}^p v_{1j}^2 \left[\Phi\left\{\frac{a_\delta - \operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)}{\sqrt{2\operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)^2}}\right\} - 2\phi\left\{\frac{a_\delta - \operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)}{\sqrt{2\operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)^2}}\right\}\right. \\
&\quad \left.\left\{\frac{\zeta_j}{\sqrt{2\operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)^2}} + \frac{a_\delta - \operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)}{\sqrt{2\operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)^2}} \frac{\zeta_j^2}{2\operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)^2}\right\} + o(1)\right].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mathrm{E}\left(x_{11} \mid w_1 = 1\right) &= \left.\frac{\partial m_1(t_1)}{\partial t_1}\right|_{t_1=0} \\
&= \frac{1}{1-\delta}\left\{\mu_1 F_{t_1}(a_\delta)|_{t_1=0} + \left.\frac{\partial F_{t_1}(a_\delta)}{\partial t_1}\right|_{t_1=0}\right\} \\
&= \mu_1,
\end{aligned}
$$

and

$$
\operatorname{Var}\left(x_{11} \mid w_1 = 1\right) = \left.\frac{\partial^2 m_1(t_1)}{\partial t_1^2}\right|_{t_1=0} - \mu_1^2 = \sigma_{11} + \frac{1}{1-\delta}\left.\frac{\partial^2 F_{t_1}(a_\delta)}{\partial t_1^2}\right|_{t_1=0}.
$$

Finally, we have

$$\begin{aligned}
&\operatorname{Var}(x_{11} \mid w_1 = 1) = \frac{1}{1-\delta} \\
&\quad \sum_{j=1}^{p} v_{1j}^2 \left\{ (1-\delta) - 2\phi(z_\delta) \left(\frac{\zeta_j}{\sqrt{2\operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)^2}} + z_\delta \frac{\zeta_j^2}{2\operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)^2} \right) + o(1) \right\} \\
&= \frac{1}{1-\delta} \sum_{j=1}^{p} v_{1j}^2 \left\{ (1-\delta) - 2\phi(z_\delta) \frac{\zeta_j}{\sqrt{2\operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)^2}} + o(1) \right\} \\
&= \sigma_{11} \left\{ 1 - \frac{2\phi(z_\delta)}{1-\delta} \frac{\left(\Sigma S(\lambda)^{-1}\Sigma\right)_{11}}{\sigma_{11}\sqrt{2\operatorname{tr}\left(S(\lambda)^{-1}\Sigma\right)^2}} + o(1) \right\} \\
&= \sigma_{11}\tau_1,
\end{aligned}$$

which completes the proof.

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