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NONPARAMETRIC TESTS OF INDEPENDENCE FOR CIRCULAR DATA BASED ON TRIGONOMETRIC MOMENTS

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Abstract: We introduce nonparametric tests of independence for bivariate circular data based on trigonometric moments. Our contributions lie in (i) proposing nonparametric tests that are locally and asymptotically optimal against bivariate cosine von Mises alternatives and (ii) extending these tests, via the empirical characteristic function, to obtain consistent tests against broader sets of alternatives, eventually being omnibus. In particular, one such omnibus test is a circular version of the celebrated distance-covariance test. Thus, we provide a collection of trigonometric-based tests of varying generality and known optimalities. We obtain the large-sample behavior of the tests under the null and alternative hypotheses, and use simulations to show that the new tests are competitive against previous proposals. Lastly, we demonstrate the proposed tests with two data applications in astronomy and forest science.

Key words and phrases: Characteristic function, directional data, independence, trigonometric moments.

1. Introduction

The goal of this paper is to propose new tests of independence between two circular random variables $\vartheta^{(1)}$ and $\vartheta^{(2)}$ that are supported on $\mathbb{T} := [-\pi, \pi)$. Given an independent and identically distributed (i.i.d.) sample $\big(\vartheta_1^{(1)}, \vartheta_1^{(2)}\big), \ldots, \big(\vartheta_n^{(1)}, \vartheta_n^{(2)}\big)$, we wish to test the null hypothesis $\mathcal{H}_0$ of independence between $\vartheta^{(1)}$ and $\vartheta^{(2)}$ against the general alternative $\mathcal{H}_1$ consisting of the negation of $\mathcal{H}_0$. This fundamental testing problem has relevant applications in fields in which circular data is common, such as in astronomy, biology, geology, and forest science, to name just a few.

Several statistical methods for the analysis of data comprised by directions, such as circular data, have been developed in the last decades; see the general treatments of Mardia and Jupp (1999), Jammalamadaka and SenGupta (2001),

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and Ley and Verdebout (2017), as well as the recent review of Pewsey and García-Portugués (2021). In particular, the analysis of data on $\mathbb{T}^2$ that is generated by a pair of angular variables, referred to as "circular-circular" or "toroidal" data, has attracted a sizable number of modeling proposals in the recent years (Pewsey and García-Portugués (2021, Sec. 3.2)). This interest has been notably boosted by applications in bioinformatics, where a sequence of dihedral angles characterizes a protein's three-dimensional backbone (e.g., Boomsma et al. (2008)). In addition, toroidal distributions are closely related to the design of models for circular time series (Wehrly and Johnson (1980)) that naturally appear in a variety of other fields, such as astronomy and forest science; see Section 5.

Much of the modeling effort for toroidal data has been dominated by the search for bivariate extensions of the von Mises distribution, often regarded as the "circular Gaussian" distribution. The first of such proposals was the bivariate von Mises density of Mardia (1975), considered as an overparametrized model due to its eight parameters. This motivated the six-parameter submodel of Rivest (1988) and the five-parameter "sine" (Singh, Hnizdo and Demchuk (2002)), "cosine" (Mardia, Taylor and Subramaniam (2007)), and "hybrid" (Kent, Mardia and Taylor (2008)) submodels. The properties of the last three models were compared in Kent, Mardia and Taylor (2008) and Mardia and Frellsen (2012). A different modeling pathway was initiated with the family of copula-structured toroidal densities by Wehrly and Johnson (1980), whose most successful representative is the bivariate wrapped Cauchy distribution (Kato and Pewsey (2015)).

Investigating relationships between variables is central to many scientific studies, and tests of independence typically precede any attempt at modeling association. Consequently, many contributions in directional statistics have been dealing with correlation, dependence, and tests of independence. Measures of circular correlation have been put forward by Watson and Beran (1967), Jupp and Mardia (1980), Shieh, Johnson and Frees (1994), and more recently by Zhan et al. (2019). In a different direction, Rothman (1971) introduced a version of the Cramér–von Mises test of independence. In parametric contexts related with the models of the previous paragraph, one may resort to the likelihood-based tests suggested by Mardia and Puri (1978), Puri and Rao (1977), and Shieh and Johnson (2005). Finally, for testing independence in data with mixed directional/linear components, smoothing-based tests have been proposed by García-Portugués, Crujeiras and González-Manteiga (2015). Unlike standard independence tests, the aforementioned tests honor the circular/directional nature of the random variables involved by being rotation invariant on them. A non-rotation-invariant test provides spurious decisions for assessing the independence of $(\vartheta^{(1)}, \vartheta^{(2)})$, as its p-value is dependent of the sample coordinates (e.g., representations on $[-\pi, \pi)^2$ or $[0, 2\pi)^2$ might yield different test decisions); see Section D in the Supplementary Material (SM) for specific examples.

When testing for independence, nonparametric methods based on the characteristic function have also been employed as alternatives to non-omnibus tests based on association coefficients, and to smoothing-based tests that exhibit the familiar drawbacks of bandwidth selection and slow convergence. These tests exploit the factorization characterization of the joint characteristic function of independent random variables. This property propagated “Fourier”-type tests in the past, going back as far as Csörgő and Hall (1982) and Csörgő (1985). Since then, Fourier methods have enjoyed increasing popularity, finally reaching some sort of climax with the introduction of the novel notions of “distance covariance” and “distance correlation” (Székely, Rizzo and Bakirov (2007)), and beyond. Indicatively, we refer to the contributions by Gretton et al. (2005), Székely, Rizzo and Bakirov (2007), Meintanis and Iliopoulos (2008), Hlávka, Hušková and Meintanis (2011), Fan et al. (2017), Chen, Meintanis and Zhu (2019), and Chakraborty and Zhang (2019), all of which propose tests of independence in varying settings and different levels of generality, but always with the characteristic function being the underlying notion. This popularity notwithstanding, and despite the fact that testing based on characteristic functions is not unfamiliar to circular data (Meintanis and Verdebout (2019)), the use of characteristic functions for testing independence of nonlinear data remains substantially unexplored.

We introduce in this paper nonparametric tests of independence for toroidal data based on trigonometric moments. We first propose nonparametric tests using joint cosine moments that are locally and asymptotically optimal against sequences of bivariate cosine von Mises alternatives, and for which the powers of the tests are explicitly obtained. We then extend these tests, via the empirical characteristic function, to more general multiple-orders tests that merge cosine and sine moments, and that are consistent against broader sets of alternatives. We obtain usable asymptotic null distributions for all the test statistics, thus avoiding calibration using resampling methods. We then propose two characteristic function-based omnibus tests with tractable computational forms that can be calibrated efficiently using permutations. The second one is a kind of circular distance-covariance test. Simulations corroborate the adequate finite-sample null and non-null behavior of the tests, as well as their competitiveness against other testing approaches based on association coefficients and smoothing. Two data applications are provided, one on the study on the serial dependence of long-period comet records and another on the evaluation of the dependence between the orientations of wildfires in Portugal.

2. A Cosine Test of Independence

2.1. Genesis and null asymptotic distribution

Our objective is to test the null hypothesis $\mathcal{H}_0$ of independence between $\vartheta^{(1)}$ and $\vartheta^{(2)}$. Without loss of generality (see Proposition 3), we assume that $\vartheta^{(j)}$ is circularly centered, i.e., such that its circular mean $\mu^{(j)} := \mathrm{atan2}\big(\mathbb{E}\big[\sin\big(\vartheta^{(j)}\big)\big], \mathbb{E}\big[\cos\big(\vartheta^{(j)}\big)\big]\big)$ is zero, $j = 1, 2$, where $\mathrm{atan2}(y, x) \in \mathbb{T}$ is the argument of the complex number $x + \mathrm{i}y$. Given an iid sample $\big(\vartheta_1^{(1)}, \vartheta_1^{(2)}\big), \ldots, \big(\vartheta_n^{(1)}, \vartheta_n^{(2)}\big)$ from $(\vartheta^{(1)}, \vartheta^{(2)})$, we consider the empirical versions

$$\hat{\mathcal{J}}_{jc}(r) := n^{-1}\sum_{i=1}^{n}\cos\big(r\vartheta_i^{(j)}\big), \quad \hat{\mathcal{J}}_{js}(r) := n^{-1}\sum_{i=1}^{n}\sin\big(r\vartheta_i^{(j)}\big), \quad j = 1, 2,$$
$$\hat{\mathcal{J}}_{c}(r_1, r_2) := n^{-1}\sum_{i=1}^{n}\cos\big(r_1\vartheta_i^{(1)} + r_2\vartheta_i^{(2)}\big),$$
$$\hat{\mathcal{J}}_{s}(r_1, r_2) := n^{-1}\sum_{i=1}^{n}\sin\big(r_1\vartheta_i^{(1)} + r_2\vartheta_i^{(2)}\big),$$

of the respective marginal "cosine" and "sine" population moments (as well as their "addition" forms) given by

$$\mathcal{J}_{jc}(r) := \mathbb{E}\big[\cos\big(r\vartheta^{(j)}\big)\big], \quad \mathcal{J}_{js}(r) := \mathbb{E}\big[\sin\big(r\vartheta^{(j)}\big)\big], \quad j = 1, 2,$$
$$\mathcal{J}_{c}(r_1, r_2) := \mathbb{E}\big[\cos\big(r_1\vartheta^{(1)} + r_2\vartheta^{(2)}\big)\big], \quad \mathcal{J}_{s}(r_1, r_2) := \mathbb{E}\big[\sin\big(r_1\vartheta^{(1)} + r_2\vartheta^{(2)}\big)\big].$$

Here, r, r_1, and r_2 are reals, although we later restrict them to be integer numbers; see (3.1).

Based on the form of the "cosine addition moment", we have that, under the null hypothesis of independence,

$$\mathcal{J}_{c}(r_1, r_2) = \mathcal{J}_{1c}(r_1)\mathcal{J}_{2c}(r_2) - \mathcal{J}_{1s}(r_1)\mathcal{J}_{2s}(r_2). \tag{2.1}$$

Based on (2.1), it is natural to consider tests that reject $\mathcal{H}_0$ for large absolute values of the statistic

$$D_c^{(n)}(r_1, r_2) := \hat{\mathcal{J}}_{c}(r_1, r_2) - \hat{\mathcal{J}}_{1c}(r_1)\hat{\mathcal{J}}_{2c}(r_2) + \hat{\mathcal{J}}_{1s}(r_1)\hat{\mathcal{J}}_{2s}(r_2), \tag{2.2}$$

because, for any $(r_1, r_2) \in \mathbb{R}^2$, $D_c^{(n)}(r_1, r_2)$ is close to zero under $\mathcal{H}_0$. The following proposition provides the asymptotic distribution of $D_c^{(n)}(r_1, r_2)$ under $\mathcal{H}_0$. Its proof is relegated to Section A in the SM, where all the results of the paper are proved.

Proposition 1. *Fix $(r_1, r_2) \in \mathbb{R}^2$. Under $\mathcal{H}_0$, $\sqrt{n}D_c^{(n)}(r_1, r_2)$ converges weakly as $n \to \infty$ to a Gaussian random variable with mean zero and variance $V(r_1, r_2) := \mathbb{E}\big[\big\{\cos\big(r_1\vartheta^{(1)} + r_2\vartheta^{(2)}\big) - \mathcal{J}_{2c}(r_2)\cos\big(r_1\vartheta^{(1)}\big) - \mathcal{J}_{1c}(r_1)\cos\big(r_2\vartheta^{(2)}\big) + \mathcal{J}_{2s}(r_2)\sin\big(r_1\vartheta^{(1)}\big) + \mathcal{J}_{1s}(r_1)\sin\big(r_2\vartheta^{(2)}\big)\big\}^2\big]$.*

The asymptotic normality of $\sqrt{n}D_c^{(n)}(r_1, r_2)$ does not require any assumption on the distribution of the pair of random angles $(\vartheta^{(1)}, \vartheta^{(2)})$. A purely nonparametric test of independence can therefore be obtained on the basis of the Proposition 1. Indeed, we can consider tests $\phi_c^{(n)}(r_1, r_2)$ rejecting the null hypothesis of independence at the asymptotic level α when

$$T_n(r_1, r_2) := \frac{n\big(D_c^{(n)}(r_1, r_2)\big)^2}{\hat{V}_n(r_1, r_2)} > \chi^2_{1;1-\alpha}, \tag{2.3}$$

where $\chi^2_{1;\nu}$ denotes the νth (lower) quantile of the chi-square distribution with one degree of freedom and $\hat{V}_n(r_1, r_2)$ is a consistent estimator of the variance term $V(r_1, r_2)$ defined in Proposition 1, such as its direct empirical version. Although being purely nonparametric, as no assumption on the data-generating process is imposed, the tests $\phi_c^{(n)}(r_1, r_2)$ with $r_1 = 1$ and $r_2 = \pm 1$ enjoy certain local and asymptotic optimality properties.

2.2. Optimality and power against bivariate von Mises alternatives

Consider the bivariate cosine von Mises model of Mardia, Taylor and Subramaniam (2007), characterized by densities of the form

$$\begin{aligned}&\big(\vartheta^{(1)}, \vartheta^{(2)}\big) \mapsto \\ &C(\kappa_1, \kappa_2, \kappa_3) \exp\big\{\kappa_1 \cos\big(\vartheta^{(1)}\big) + \kappa_2 \cos\big(\vartheta^{(2)}\big) + \kappa_3 \cos\big(\vartheta^{(1)} - \vartheta^{(2)}\big)\big\},\end{aligned} \tag{2.4}$$

where $\kappa_1, \kappa_2 \geq 0$ are concentration parameters, $\kappa_3 \in \mathbb{R}$ is a parameter controlling the dependence, and $C(\kappa_1, \kappa_2, \kappa_3)$ is a normalizing constant. Note that, for the ease of our derivations, we flip the sign of $\kappa_3 \in \mathbb{R}$ in (2.4) with respect to the original model parametrization. Following the terminology in Mardia and Frellsen (2012), the density (2.4) is called the bivariate cosine model with *positive interaction*. The same model with *negative interaction* is obtained by replacing $\cos\big(\vartheta^{(1)} - \vartheta^{(2)}\big)$ with $\cos\big(\vartheta^{(1)} + \vartheta^{(2)}\big)$ in (2.4). As stated in Mardia, Taylor and Subramaniam (2007), both models capture the correlations between the cosines and sines of the circular variables, although neither is strictly associated with positive or negative correlations between angles. Indeed, the signs of "angular correlations" depend on κ_3, which affects asymmetrically the kind of dependence induced by (2.4). Positive values of κ_3 guarantee unimodality, with a positive/negative angular correlation that depends on the positive/negative interaction (Theorem 6.2 in Mardia and Frellsen (2012); third column of Figure 1). A negative κ_3 may generate bimodality distributed in an opposite correlation pattern to that of $\kappa_3 > 0$. Shifting of (2.4) can be achieved by replacing $\vartheta^{(j)}$ with $\vartheta^{(j)} - \mu^{(j)}$, for $\mu^{(j)} \in \mathbb{T}$, $j = 1, 2$. The location parameters do not affect the dependence form of (2.4), but they do make it more cumbersome.

When $\kappa_3 = 0$, the marginals of (2.4) are independent and centered von Mises distributions with concentrations κ_1 and κ_2. Thus, testing for independence in this model reduces to testing $\mathcal{H}_0 : \kappa_3 = 0$ against $\mathcal{H}_1 : \kappa_3 \neq 0$. We show in Proposition 2 that the tests $\phi_c^{(n)}(1,1)$ and $\phi_c^{(n)}(1,-1)$ are locally and asymptotically maximin (see (Ley and Verdebout, 2017, Sec. 5, for a definition)) for testing $\mathcal{H}_0 : \kappa_3 = 0$ against $\mathcal{H}_1 : \kappa_3 \neq 0$ within sequences of bivariate cosine models with negative and positive interactions, respectively. Recall that a test ϕ^* is called maximin in the class $\mathcal{C}_\alpha$ of level-α tests for some null hypothesis $\mathcal{H}_0$ against the alternative $\mathcal{H}_1$ if (i) ϕ^* has level α and (ii) the power of ϕ^* is such that

$$\inf_{\mathrm{P}\in\mathcal{H}_1} \mathbb{E}_{\mathrm{P}}[\phi^*] \geq \sup_{\phi\in\mathcal{C}_\alpha} \inf_{\mathrm{P}\in\mathcal{H}_1} \mathbb{E}_{\mathrm{P}}[\phi].$$

We denote by $\mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,\kappa_3);-}$ and $\mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,\kappa_3);+}$ the joint distributions of an iid sample $\big(\vartheta_1^{(1)},\vartheta_1^{(2)}\big),\ldots,\big(\vartheta_n^{(1)},\vartheta_n^{(2)}\big)$ from distribution (2.4), respectively with negative and positive interactions. Obviously, $\mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,0);-} = \mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,0);+}$, which is simply denoted as $\mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,0)}$.

Proposition 2. *Letting τ_n be a bounded real sequence, the test $\phi_c^{(n)}(1,1)$ is locally and asymptotically maximin for testing $\mathcal{H}_0 : \cup_{\kappa_1\geq 0}\cup_{\kappa_2\geq 0} \mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,0)}$ against $\mathcal{H}_1 : \cup_{\kappa_1\geq 0}\cup_{\kappa_2\geq 0} \mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,n^{-1/2}\tau_n);-}$, while the test $\phi_c^{(n)}(1,-1)$ is locally and asymptotically maximin for testing $\mathcal{H}_0 : \cup_{\kappa_1\geq 0}\cup_{\kappa_2\geq 0} \mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,0)}$ against $\mathcal{H}_1 : \cup_{\kappa_1\geq 0}\cup_{\kappa_2\geq 0} \mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,n^{-1/2}\tau_n);+}$.*

The nonparametric tests $\phi_c^{(n)}(1,1)$ and $\phi_c^{(n)}(1,-1)$ therefore enjoy some parametric optimality properties for testing $\mathcal{H}_0 : \kappa_3 = 0$ against $\mathcal{H}_1 : \kappa_3 \neq 0$. Although the tests $\phi_c^{(n)}(r_1,r_2)$, $(r_1,r_2)\in\mathbb{R}^2$ do not have such local and asymptotic optimality, it is easy to show that they exhibit non-trivial power against the contiguous alternatives $\mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,n^{-1/2}\tau_n);+}$ and $\mathrm{P}^{(n)}_{(\kappa_1,\kappa_2,n^{-1/2}\tau_n);-}$, and can therefore be considered as reasonable tests for such alternatives.

Hitherto, we have assumed the sample comes from a circularly-centered random vector. Otherwise, the test statistic $T_n(r_1,r_2)$ in (2.3) has to be computed from the centered data $\vartheta_i^{(j)} - \mu^{(j)}$, $i = 1,\ldots,n$, $j = 1,2$. Then, Proposition 1 holds if we replace $\vartheta_i^{(j)}$ and $\vartheta^{(j)}$ with $\vartheta_i^{(j)} - \mu^{(j)}$ and $\vartheta^{(j)} - \mu^{(j)}$, respectively, $i = 1,\ldots,n$, $j = 1,2$. Moreover, the local and asymptotic optimality obtained in Proposition 2 also holds in the unspecified location case. Of course, the location parameters $\mu^{(1)}$ and $\mu^{(2)}$ are rarely known in practice, so they have to be estimated. This can be done using the sample circular means

$$\hat{\mu}^{(j)} := \mathrm{atan2}\left(\frac{1}{n}\sum_{i=1}^n \sin\big(\vartheta_i^{(j)}\big), \frac{1}{n}\sum_{i=1}^n \cos\big(\vartheta_i^{(j)}\big)\right), \quad j = 1,2.$$

This estimation produces the centered sample

$$\big(\vartheta_1^{(1)} - \hat{\mu}^{(1)}, \vartheta_1^{(2)} - \hat{\mu}^{(2)}\big), \ldots, \big(\vartheta_n^{(1)} - \hat{\mu}^{(1)}, \vartheta_n^{(2)} - \hat{\mu}^{(2)}\big). \tag{2.5}$$

When computed from this centered sample, the test statistic $T_n(r_1, r_2)$ in (2.3) is rotation invariant, which is a highly desirable property in the present toroidal context. We moreover have the following result.

Proposition 3. *Denote by* $\hat{D}_c^{(n)}(r_1, r_2)$ *and* $D_c^{(n)}(r_1, r_2)$ *the quantities defined in* (2.2)*, but computed from the samples* (2.5) *and*

$$\big(\vartheta_1^{(1)} - \mu^{(1)}, \vartheta_1^{(2)} - \mu^{(2)}\big), \ldots, \big(\vartheta_n^{(1)} - \mu^{(1)}, \vartheta_n^{(2)} - \mu^{(2)}\big),$$

respectively. Then, provided that $\sqrt{n}\big(\hat{\mu}^{(j)} - \mu^{(j)}\big) = O_{\mathrm{P}}(1)$ *as* $n \to \infty$, $j = 1, 2$, $\sqrt{n}\big(\hat{D}_c^{(n)}(r_1, r_2) - D_c^{(n)}(r_1, r_2)\big)$ *is* $o_{\mathrm{P}}(1)$ *as* $n \to \infty$.

Classical arguments similarly show that, provided that the data-generating process ensures that $\sqrt{n}\big(\hat{\mu}^{(j)} - \mu^{(j)}\big) = O_{\mathrm{P}}(1)$ as $n \to \infty$, $j = 1, 2$, the centering has no asymptotic effect on $\hat{V}_n(r_1, r_2)$ in (2.3). Consequently, the centering step does not affect the asymptotic null distribution of $T_n(r_1, r_2)$ in (2.3). Note that the same holds under contiguous alternatives. Since the centering of the sample is innocuous in terms of the asymptotic behavior of (2.3) and it makes the test rotation invariant, this centering is implicitly assumed henceforth when applying the $\phi_c^{(n)}(r_1, r_2)$ test.

We conclude the section by pointing out that, while being of a nonparametric nature, the tests $\phi_c^{(n)}(r_1, r_2)$ are clearly designed to detect certain types of dependence (and not any kind of dependence): as seen in Proposition 2, the tests $\phi_c^{(n)}(1, \pm 1)$ are particularly well-adapted to bivariate cosine von Mises alternatives that feature reflective symmetric marginal distributions. Working along the same lines, one could consider tests based on the sine empirical moments, and show that some of their versions are locally and asymptotically optimal within specific parametric models. Rather than moving in this direction, in the following section we proceed towards tests of independence that are able to detect *arbitrary* types of dependence.

3. Omnibus Tests

The well-known factorization property of characteristic functions entails that the null hypothesis of independence may equivalently be stated as

$$\varphi(r_1, r_2) = \varphi_1(r_1)\varphi_2(r_2), \quad \text{for all } (r_1, r_2) \in \mathbb{Z}^2, \tag{3.1}$$

where $\varphi(r_1, r_2) := \mathbb{E}\big[e^{\mathrm{i}(r_1\vartheta^{(1)} + r_2\vartheta^{(2)})}\big]$, $\mathrm{i} := \sqrt{-1}$, is the joint characteristic function and $\varphi_j(r_j) := \mathbb{E}\big[e^{\mathrm{i}r_j\vartheta^{(j)}}\big]$ is the marginal characteristic function of $\vartheta^{(j)}$, $j = 1, 2$. Recall that, for random variables on the real line, (3.1) needs to be considered for all $(r_1, r_2) \in \mathbb{R}^2$ while, due to periodicity, in the case of circular random variables, it is sufficient to consider the characteristic functions only for integer arguments. This is because the joint distribution of $(\vartheta^{(1)}, \vartheta^{(2)})$ is identical to that of $(\vartheta^{(1)} + 2\pi, \vartheta^{(2)})$ and thus we have $\varphi(r_1, r_2) = e^{\mathrm{i}2\pi r_1}\varphi(r_1, r_2)$, hence r_1 must be an integer, and likewise for r_2 (Jammalamadaka and SenGupta (2001, Sec. 2.1)).

Based on $(\vartheta_1^{(1)}, \vartheta_1^{(2)}), \ldots, (\vartheta_n^{(1)}, \vartheta_n^{(2)})$, the classical estimator of the joint characteristic function is

$$\hat{\varphi}(r_1, r_2) := \frac{1}{n}\sum_{i=1}^{n} e^{\mathrm{i}(r_1\vartheta_i^{(1)} + r_2\vartheta_i^{(2)})}. \tag{3.2}$$

The corresponding empirical marginals, say $\hat{\varphi}_1$ (respectively, $\hat{\varphi}_2$), can be obtained by setting $r_2 = 0$ ($r_1 = 0$) in (3.2). Then, in view of (3.1), it is natural to consider the test statistics

$$D^{(n)}(r_1, r_2) := \hat{\varphi}(r_1, r_2) - \hat{\varphi}_1(r_1)\hat{\varphi}_2(r_2), \quad (r_1, r_2) \in \mathbb{Z}^2, \tag{3.3}$$

as diagnostic components for independence. Note that the quantity $D_c^{(n)}(r_1, r_2)$ defined in (2.2) is just the real part of $D^{(n)}(r_1, r_2)$, and consequently an extension of the tests studied in Section 2 may be obtained by considering both the real and imaginary parts of $D^{(n)}(r_1, r_2)$ for multiple arguments $(r_1, r_2) \in \mathbb{Z}^2$. To this end, we define the vector

$$\begin{aligned}\boldsymbol{\Delta}_n(\boldsymbol{r}^{(c)}, \boldsymbol{r}^{(s)}) :=& \Big(D_c^{(n)}(r_{11}^{(c)}, r_{12}^{(c)}), \ldots, D_c^{(n)}(r_{J1}^{(c)}, r_{J2}^{(c)}),\\ & D_s^{(n)}(r_{11}^{(s)}, r_{12}^{(s)}), \ldots, D_s^{(n)}(r_{K1}^{(s)}, r_{K2}^{(s)})\Big)',\end{aligned}$$

where $D_c^{(n)}(r_1, r_2)$ and $D_s^{(n)}(r_1, r_2)$ denote the real and imaginary parts, respectively, of $D^{(n)}(r_1, r_2)$. Using similar arguments to those in Section 2, it may be shown that $\sqrt{n}\boldsymbol{\Delta}_n(\boldsymbol{r}^{(c)}, \boldsymbol{r}^{(s)})$ is asymptotically a zero-mean multivariate Gaussian with some covariance matrix $\boldsymbol{\Sigma}$ that is easily computable; see Section B in the SM. As a result, letting $\hat{\boldsymbol{\Sigma}}$ be an invertible and consistent estimator of $\boldsymbol{\Sigma}$, the natural test $\phi^{(n)}(\boldsymbol{r}^{(c)}, \boldsymbol{r}^{(s)})$ rejects $\mathcal{H}_0$ for large values of $n(\boldsymbol{\Delta}_n(\boldsymbol{r}^{(c)}, \boldsymbol{r}^{(s)}))'\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Delta}_n(\boldsymbol{r}^{(c)}, \boldsymbol{r}^{(s)})$. Note that some choices of $\boldsymbol{r}^{(c)} = (r_{11}^{(c)}, r_{12}^{(c)}, \ldots, r_{J1}^{(c)}, r_{J2}^{(c)})' \in \mathbb{Z}^{2J}$ and $\boldsymbol{r}^{(s)} = (r_{11}^{(s)}, r_{12}^{(s)}, \ldots, r_{K1}^{(s)}, r_{K2}^{(s)})' \in \mathbb{Z}^{2K}$ yield matrices $\boldsymbol{\Sigma}$ that are invertible, while others do not. Also, note that the particular case obtained by putting $J = 2$ with $(r_{11}^{(c)}, r_{12}^{(c)}, r_{21}^{(c)}, r_{22}^{(c)}) = (1, -1, 1, 1)$ and $K = 0$ (so that there is no "sine part" in $\boldsymbol{\Delta}_n(\boldsymbol{r}^{(c)}, \boldsymbol{r}^{(s)})$) yields a test that combines the two test statistics that are locally and asymptotically optimal against contiguous cosine von Mises alternatives with positive and negative dependence. An implicit centering of the sample is also assumed when applying $\phi^{(n)}(\boldsymbol{r}^{(c)}, \boldsymbol{r}^{(s)})$ as, analogously to the $\phi_c^{(n)}(r_1, r_2)$ test, this centering step has no effect on the asymptotic behavior of the test and makes it rotation invariant.

Although the tests $\phi^{(n)}(\boldsymbol{r}^{(c)}, \boldsymbol{r}^{(s)})$, with $\boldsymbol{r}^{(c)} \in \mathbb{Z}^{2J}$ and $\boldsymbol{r}^{(s)} \in \mathbb{Z}^{2K}$, are expected to have good power properties beyond the class of von Mises distributions for which $\phi_c^{(n)}(1, \pm 1)$ is locally and asymptotically maximin, these tests are not "omnibus"; i.e., their power is potentially trivial against certain alternatives. In order to have an omnibus test, the uniqueness property of characteristic functions dictates that we must take into account all possible pairs $(r_1, r_2) \in \mathbb{Z}^2$. Consequently, we define a test criterion that rejects $\mathcal{H}_0$ for large values of

$$T_{n,w} := n \sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} \left| D^{(n)}(r_1, r_2) \right|^2 w(r_1, r_2), \tag{3.4}$$

where $|\cdot|$ denotes the modulus of a complex number and $w : \mathbb{Z}^2 \to [0, \infty)$ is a weight function specified below. The following proposition formalizes the limit behavior of $T_{n,w}$ against arbitrary deviations from the null hypothesis of independence.

Proposition 4. *Assume that w in* (3.4) *satisfies $\sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} w(r_1, r_2) < \infty$. Then,*

$$\frac{T_{n,w}}{n} \to \mathcal{T}_w := \sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} |\varphi(r_1, r_2) - \varphi_1(r_1)\varphi_2(r_2)|^2 \, w(r_1, r_2) \tag{3.5}$$

almost surely as $n \to \infty$. Moreover, $\mathcal{T}_w$ is strictly positive unless $\mathcal{H}_0$ holds, a fact which entails strong consistency of the test that rejects $\mathcal{H}_0$ for large values of $T_{n,w}$.

Although L_2-type test statistics such as $T_{n,w}$ are omnibus, they typically have highly non-trivial asymptotic null distributions that essentially prevent their use as test criteria. We refer to Puri and Rao (1977), Shieh, Johnson and Frees (1994), and Watson and Beran (1967) for analogous results; see also Jammalamadaka and SenGupta (2001, Section 8.9). Nevertheless, it is straightforward to implement a permutation version of a test based on $T_{n,w}$.

The applicability of the test statistic would be further advanced if $T_{n,w}$ could be computed analytically. To this end, consider a weight function decomposed as $w(r_1, r_2) = v(r_1)v(r_2)$, with v being a symmetric function about zero. Then, (3.4) may be rewritten as (see Section A in the SM)

$$\begin{aligned} T_{n,w} = {} & \frac{1}{n} \sum_{j,k=1}^{n} \mathcal{J}_c^{(v)}\big(\vartheta_{jk}^{(1)}\big) \mathcal{J}_c^{(v)}\big(\vartheta_{jk}^{(2)}\big) + \frac{1}{n^3} \left[\sum_{j,k=1}^{n} \mathcal{J}_c^{(v)}\big(\vartheta_{jk}^{(1)}\big) \right] \left[\sum_{j,k=1}^{n} \mathcal{J}_c^{(v)}\big(\vartheta_{jk}^{(2)}\big) \right] \\ & - \frac{2}{n^2} \sum_{j,k,\ell=1}^{n} \mathcal{J}_c^{(v)}\big(\vartheta_{jk}^{(1)}\big) \mathcal{J}_c^{(v)}\big(\vartheta_{j\ell}^{(2)}\big), \end{aligned} \tag{3.6}$$

where

$$\mathcal{J}_c^{(v)}(\vartheta) := \sum_{r=-\infty}^{\infty} \cos(r\vartheta)v(r), \tag{3.7}$$

with $\vartheta_{jk}^{(m)} := \vartheta_j^{(m)} - \vartheta_k^{(m)}$, $j,k = 1,\ldots,n$, $m = 1,2$. Since $T_{n,w}$ depends only on the distances between observations, it is rotation invariant without requiring a prior centering of the sample.

Moreover, if we consider any probability mass function on the nonnegative integers and set v equal to the symmetrized version of this function, then the series in (3.7) equals the real part of the characteristic function of that probability mass function, evaluated at ϑ. A standard option is to choose the Poisson distribution, in which case

$$\mathcal{J}_c^{(v)}(\vartheta) = \cos(\lambda \sin\vartheta)e^{\lambda(\cos\vartheta - 1)}, \tag{3.8}$$

where λ is the Poisson parameter. Choosing $\lambda \in (0, \pi/2]$ guarantees the nonnegativity of the kernel (3.8) for any $\vartheta \in \mathbb{T}$ (and also if $0 < |\lambda| \leq \pi/2$). We denote by $T_{n,\lambda}$ the statistic (3.6) based on (3.8). The test $\phi^{(n)}(\lambda)$ that rejects $\mathcal{H}_0$ for large values of $T_{n,\lambda}$ is implemented with a permutation approach that is described in Section C in the SM.

The weight specification $w_{\mathrm{dc}}(r_1, r_2) = (r_1 r_2)^{-2} 1_{\{r_1 \neq 0, r_2 \neq 0\}}$ in (3.4) yields the "distance-covariance" test statistic

$$S_{n,\mathrm{dc}} := n \sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} \left| D^{(n)}(r_1, r_2) \right|^2 w_{\mathrm{dc}}(r_1, r_2); \tag{3.9}$$

its definition is driven by the connection between a distance covariance statistic and the characteristic function. Note that in (3.9) we exclude the origin, which is anyway uninformative regarding independence. Furthermore, $S_{n,\mathrm{dc}}$ clearly satisfies the global consistency property of Proposition 4.

Carrying out analogous computations as in (3.6) and denoting

$$\mathcal{I}(\vartheta) := \sum_{\substack{r=-\infty,\\ r\neq 0}}^{\infty} \frac{\cos(r\vartheta)}{r^2}, \tag{3.10}$$

we obtain from (3.9) that

$$\begin{aligned} S_{n,\mathrm{dc}} = {} & \frac{1}{n} \sum_{j,k=1}^{n} \mathcal{I}(\vartheta_{jk}^{(1)})\mathcal{I}(\vartheta_{jk}^{(2)}) + \frac{1}{n^3} \left[\sum_{j,k=1}^{n} \mathcal{I}(\vartheta_{jk}^{(1)}) \right] \left[\sum_{j,k=1}^{n} \mathcal{I}(\vartheta_{jk}^{(2)}) \right] \\ & - \frac{2}{n^2} \sum_{j,k,\ell=1}^{n} \mathcal{I}(\vartheta_{jk}^{(1)})\mathcal{I}(\vartheta_{j\ell}^{(2)}). \end{aligned}$$

However, unlike $T_{n,\lambda}$, the computation of $S_{n,\mathrm{dc}}$ is less straightforward, as it requires evaluating (3.10), which can also be expressed as $\mathcal{I}(\vartheta) = \mathrm{Li}_2(e^{-\mathrm{i}\vartheta}) +$

$\mathrm{Li}_2(e^{\mathrm{i}\vartheta})$, with $\mathrm{Li}_2(x) := \sum_{m=1}^{\infty} m^{-2}x^m$ being the dilogarithm function. Because $\mathcal{O}(Bn^2)$ evaluations of the kernel (3.10) are required to evaluate the test based on $S_{n,\mathrm{dc}}$ (henceforth denoted as $\phi_{\mathrm{dc}}^{(n)}$) with B permutations, the increased computational burden with respect to the $\phi^{(n)}(\lambda)$ test is significant.

As all the tests introduced in this paper, that based on $S_{n,\mathrm{dc}}$ honors the circularity of the variables $\vartheta^{(1)}$ and $\vartheta^{(2)}$ for testing their independence. Section D in the SM exemplifies the important practical issues of applying an independence test that is unaware of the circular nature of $\vartheta^{(1)}$ and $\vartheta^{(2)}$, such as the standard distance-covariance test.

Remark 1. The test statistic in (3.4) can be, heuristically, further scrutinized with regards to correlations between $\vartheta^{(1)}$ and $\vartheta^{(2)}$. Consider, for simplicity, its population counterpart from Proposition 4 and write it as

$$\mathcal{T}_w = \sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} \left|\mathbb{C}\mathrm{ov}\left[e^{\mathrm{i}r_1\vartheta^{(1)}}, e^{\mathrm{i}r_2\vartheta^{(2)}}\right]\right|^2 w(r_1, r_2),$$

where $\mathbb{C}\mathrm{ov}\left[\cdot,\cdot\right]$ denotes covariance. Now use the exponential function expansion $e^z = 1 + (z/1!) + (z^2/2!) + \cdots$, compute a few terms of the covariance thereof and, after some simplification, write

$$\begin{aligned}\mathbb{C}\mathrm{ov}\left[e^{\mathrm{i}r_1\vartheta^{(1)}}, e^{\mathrm{i}r_1\vartheta^{(2)}}\right] =& -r_1r_2\mathbb{C}\mathrm{ov}\left[\vartheta^{(1)}, \vartheta^{(2)}\right] \\ & -\frac{\mathrm{i}}{2}\left(r_1r_2^2\mathbb{C}\mathrm{ov}\left[\vartheta^{(1)}, {\vartheta^{(2)}}^2\right] + r_1^2r_2\mathbb{C}\mathrm{ov}\left[({\vartheta^{(1)}}^2, \vartheta^{(2)}\right]\right) \\ & + \frac{r_1^2r_2^2}{4}\mathbb{C}\mathrm{ov}\left[{\vartheta^{(1)}}^2, \vartheta^{(2)}\right] + \cdots .\end{aligned}$$

Expanding $\left|\mathbb{C}\mathrm{ov}\left[e^{\mathrm{i}r_1\vartheta^{(1)}}, e^{\mathrm{i}r_2\vartheta^{(2)}}\right]\right|^2$, using $w(r_1, r_2) = v(r_1)v(r_2)$, and letting $v(\cdot)$ be a probability function symmetric around zero, it follows that

$$\begin{aligned}\mathcal{T}_w =& \mu_2^2\mathbb{C}\mathrm{ov}^2\left[\vartheta^{(1)}, \vartheta^{(2)}\right] + \frac{\mu_4^2}{16}\mathbb{C}\mathrm{ov}^2\left[{\vartheta^{(1)}}^2, {\vartheta^{(2)}}^2\right] \\ & + \frac{\mu_2\mu_4}{4}\left(\mathbb{C}\mathrm{ov}^2\left[\vartheta^{(1)}, {\vartheta^{(2)}}^2\right] + \mathbb{C}\mathrm{ov}^2\left[{\vartheta^{(1)}}^2, \vartheta^{(2)}\right]\right) + \cdots ,\end{aligned}$$

where μ_m denotes the mth moment of $v(\cdot)$, which is assumed to exist. Consequently, $\mathcal{T}_w$ may be written as a weighted sum of the classical (squared) covariances between the powers of $\vartheta^{(1)}$ and $\vartheta^{(2)}$. In this regard, the role of $v(\cdot)$ is to assign weights to these covariances via its population moments.

4. Simulation Study

4.1. Toroidal distributions considered

To explore various shapes of dependence between $\vartheta^{(1)}$ and $\vartheta^{(2)}$, with the strength of dependence controlled by the value of a single parameter, we consider the following four joint parametric distributions of $\big(\vartheta^{(1)}, \vartheta^{(2)}\big)$, all supported on $\mathbb{T}^2$:

(i) The ParaBolic distribution PB(p), defined by $\vartheta^{(1)} \sim \text{Unif}\,(\mathbb{T})$ and $\vartheta^{(2)} = 2\big[p\big(\vartheta^{(1)}\big)^2 + (1-p)U^2\big]/\pi - \pi$, where $U \sim \text{Unif}\,(\mathbb{T})$ is independent of $\vartheta^{(1)}$ and $p \in [0,1]$.

(ii) The (centered) Bivariate Wrapped Cauchy distribution as given in Pewsey and Kato (2016), denoted as BWC(ρ_1, ρ_2, ρ), and with density

$$\begin{aligned}\big(\vartheta^{(1)}, \vartheta^{(2)}\big) \mapsto c_0\big\{c_1 - c_2\cos\big(\vartheta^{(1)}\big) - c_3\cos\big(\vartheta^{(2)}\big) \\ - c_4\cos\big(\vartheta^{(1)}\big)\cos\big(\vartheta^{(2)}\big) - c_5\sin\big(\vartheta^{(1)}\big)\sin\big(\vartheta^{(2)}\big)\big\}^{-1},\end{aligned}$$

where c_j, $j = 0, \ldots, 5$, are closed-form constants depending on $\rho_1, \rho_2, |\rho| \in [0,1)$.

(iii) The (centered) Bivariate Cosine von Mises model with *positive* interaction, denoted as BCvM($\kappa_1, \kappa_2, \kappa_3$), and with density described in Equation (2.4).

(iv) The (centered) Bivariate von Mises distribution by Shieh and Johnson (2005), denoted as BvM($\kappa_1, \kappa_2, \mu_g, \kappa_g$), and with density

$$\big(\vartheta^{(1)}, \vartheta^{(2)}\big) \mapsto f_1\big(\vartheta^{(1)}\big)f_2\big(\vartheta^{(2)}\big)f_g\big(2\pi\big\{F_1\big(\vartheta^{(1)}\big) - F_2\big(\vartheta^{(2)}\big)\big\}\big),$$

where f_j and F_j are respectively the marginal density and distribution functions of a zero-mean von Mises with concentration $\kappa_j \geq 0$, $j = 1, 2$, and the link density f_g is that of a von Mises with circular mean $\mu_g \in \mathbb{T}$ and concentration $\kappa_g \geq 0$.

The last parameter in each distribution controls the degree of dependence, with $p = \rho = \kappa_3 = \kappa_g = 0$ producing independence between $\vartheta^{(1)}$ and $\vartheta^{(2)}$.

Sampling from (i) is straightforward. For (iii), we used the function `rvmcos` from the `BAMBI` (v. 2.3.0) package (Chakraborty and Wong (2019)). One can simulate from (iv) using Algorithm A for von Mises marginals in Shieh and Johnson (2005). The R code for sampling (ii) and (iv) uses the package `circular` (v. 0.4-93) (Agostinelli and Lund (2017)), and was kindly provided by Arthur Pewsey. Replicating code is available from the authors. Figure 1 shows scatterplots obtained from the considered distributions.

4.2. Empirical powers

Here, we investigate the empirical sizes and powers of our three families of tests. More specifically, we consider tests based on the statistics $T_n(\boldsymbol{r}_1)$ and $T_n(\boldsymbol{r}_2)$ with $\boldsymbol{r}_1 = (1,1)$ and $\boldsymbol{r}_2 = (1,-1)$, $\boldsymbol{\Delta}_n \equiv \boldsymbol{\Delta}_n(\boldsymbol{r}^{(c)}, \boldsymbol{r}^{(s)})$ with $\boldsymbol{r}^{(c)} = (1,-1,1,1)$ and $K = 0$, and $T_{n,\lambda}$ for $\lambda \in \{0.1, 0.5, 1.0, 2.0\}$, as well as $S_{n,\text{dc}}$. We also

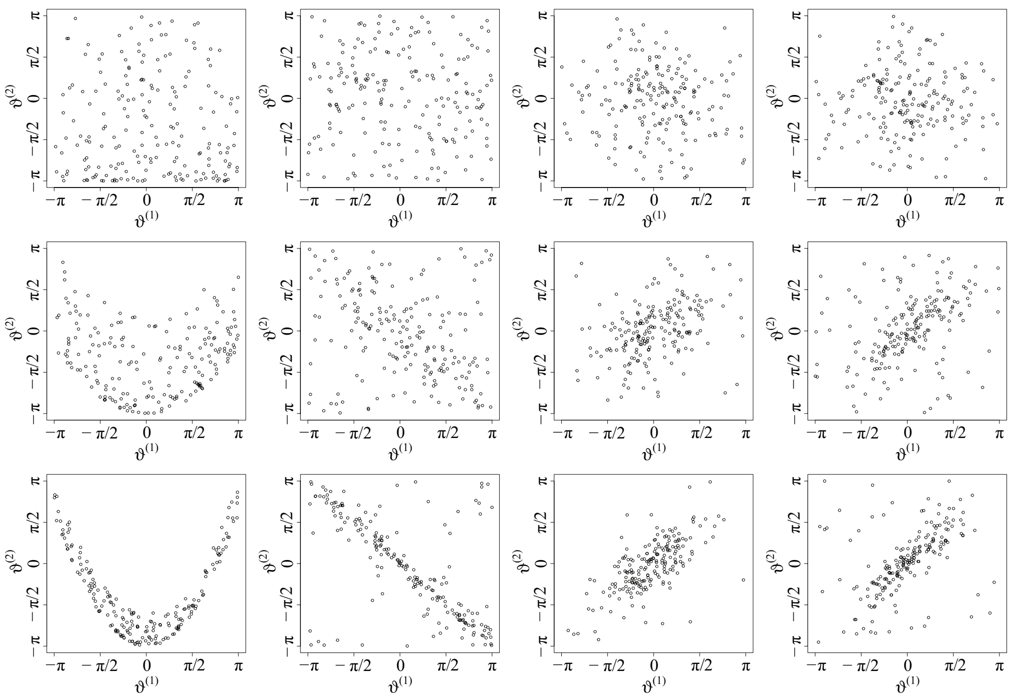

Figure 1. Scatterplots generated from the simulation scenarios considered in the simulation study. From left to right, columnwise: (i) PB(p) for $p = 0, 0.4, 0.8$ (top to bottom); (ii) BWC$(0.1, 0.1, -\rho)$ for $\rho = 0, 0.4, 0.8$; (iii) BCvM$(1, 1, \kappa_3)$ for $\kappa_3 = 0, 1, 2$; and (iv) BvM$(1, 1, 0, \kappa_g)$ for $\kappa_g = 0, 1, 2$. The sample size is $n = 200$.

consider the smoothing-based test of García-Portugués, Crujeiras and González-Manteiga (2015, Sec. C.2), denoted by G_n, the test based on the weighted U-statistic of Shieh, Johnson and Frees (1994, p.737), denoted by U_n, the correlation test of Zhan et al. (2019, p.1835) based on the statistic $\hat{\rho}_0$, and the omnibus test of Rothman (1971) based on the integrated empirical independence process denoted by C_n. For G_n, we set the bandwidths (h_1, h_2) respectively to $(1.00, 0.70)$, $(1.00, 1.00)$, $(0.50, 0.50)$, and $(0.55, 0.55)$ for the four scenarios of dependence considered. These bandwidths are sensible, as they are the empirical medians of 10^3 marginal "rule-of-thumb" bandwidths (García-Portugués (2013)) for $n = 20, 50$ and for each of the considered scenarios.

The empirical power of these tests is compared by generating $M = 10^5$ independent samples of sizes $n = 20$ and $n = 50$ from the distributions (i)–(iv), for varying dependence strengths. The results for a significance level $\alpha = 5\%$ are summarized in Table 1 for $n = 50$ and in Table E.2 in the SM for $n = 20$. In these tables, the first row in each panel corresponds to the independence case, while subsequent rows represent an increasing strength of dependence. The extreme cases $p = 1$ and $\rho = 1$ give functional dependence. We compute the critical values under $\mathcal{H}_0$ as follows. For a given sample size n

Table 1. Empirical level and power (in %) for the distributions PB(p), BWC($0.1, 0.1, -\rho$), BCvM($1, 1, \kappa_3$), and BvM($1, 1, 0, \kappa_g$) (top to bottom), for $\alpha = 5\%$ and $n = 50$. In each row, the largest power value is indicated in bold, as are any other power values falling in Wilson (1927)'s 95% binomial confidence interval for the theoretical power of this best test.

		$T_n(\boldsymbol{r}_1)$	$T_n(\boldsymbol{r}_2)$	$\boldsymbol{\Delta}_n$	$T_{n,0.1}$	$T_{n,0.5}$	$T_{n,1.0}$	$T_{n,2.0}$	$S_{n,\mathrm{dc}}$	G_n	U_n	$\hat{\rho}_0$	C_n
p	0.0	5.00	5.05	4.95	4.95	4.97	4.99	5.00	4.97	4.98	4.74	4.86	4.79
	0.2	8.87	8.88	10.47	12.54	16.37	22.81	**30.48**	19.31	15.04	11.17	12.91	12.17
	0.4	39.03	38.88	47.93	75.72	82.01	**85.90**	79.04	83.18	79.30	18.07	25.01	21.26
	0.6	69.00	68.85	87.13	**99.98**	**99.98**	**99.98**	99.66	**99.99**	**99.98**	33.51	46.85	52.92
	0.8	73.27	72.84	95.66	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	56.63	62.09	99.89
	1.0	66.87	66.93	83.73	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	71.16	70.48	**100.00**
ρ	0.0	5.04	4.85	4.94	4.96	5.01	5.03	5.06	5.03	4.97	5.03	5.01	4.93
	0.2	30.15	4.74	23.26	**30.82**	30.03	26.60	14.84	26.78	**30.82**	18.63	20.68	3.53
	0.4	74.85	4.96	69.17	**91.26**	90.70	87.91	66.01	90.55	91.23	77.59	80.21	28.50
	0.6	91.11	5.21	89.59	**99.98**	**99.98**	**99.97**	99.40	**99.98**	**99.98**	99.71	99.77	95.59
	0.8	97.55	5.14	97.17	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**
κ_3	0.0	4.86	4.93	4.91	4.98	4.99	5.11	5.04	5.04	5.09	5.09	4.88	4.98
	0.5	10.07	**39.46**	27.43	32.74	30.37	24.92	13.22	30.06	26.57	13.20	30.75	20.10
	1.0	21.70	**88.71**	77.86	81.41	78.24	69.57	40.01	78.70	72.55	41.25	78.11	57.48
	1.5	35.22	**99.11**	96.89	97.35	96.42	93.08	71.47	97.02	94.39	72.01	96.65	84.89
	2.0	45.13	**99.94**	99.62	99.61	99.45	98.77	90.13	99.70	99.06	88.74	99.63	95.55
	3.0	56.22	**100.00**	**100.00**	99.99	99.98	99.96	99.23	**100.00**	99.97	98.47	99.99	99.66
κ_g	0.0	4.79	4.87	4.79	4.94	4.82	4.84	4.93	4.77	4.81	4.94	4.95	5.05
	0.5	6.49	**43.85**	31.75	34.98	40.05	42.11	32.66	40.87	41.60	27.76	43.16	41.95
	1.0	8.84	93.57	88.53	89.49	94.65	**96.08**	90.66	95.41	95.72	88.02	94.06	95.20
	1.5	10.85	99.74	99.41	99.68	99.96	**99.98**	99.86	**99.98**	99.97	99.71	99.90	99.96
	2.0	12.77	99.97	99.96	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**
	3.0	15.27	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**

and a given bivariate parametric alternative distribution $\mathcal{D}(\theta)$, we generate two independent samples, $\big(\vartheta_1^{(1)}, \vartheta_1^{(2)}\big), \ldots, \big(\vartheta_n^{(1)}, \vartheta_n^{(2)}\big)$ and $\big(\tilde{\vartheta}_1^{(1)}, \tilde{\vartheta}_1^{(2)}\big), \ldots, \big(\tilde{\vartheta}_n^{(1)}, \tilde{\vartheta}_n^{(2)}\big)$, from $\mathcal{D}(\theta)$. Critical values are then obtained by computing empirical quantiles from the sample $\big(\vartheta_1^{(1)}, \tilde{\vartheta}_1^{(2)}\big), \ldots, \big(\vartheta_n^{(1)}, \tilde{\vartheta}_n^{(2)}\big)$. While this necessitates generating two samples, it is much faster than relying on a permutation approach. Moreover, this ensures that our empirical power values measure the ability to detect dependence by completely disregarding any potential marginal effect, since the marginal distributions of $\big(\vartheta^{(1)}, \tilde{\vartheta}^{(2)}\big)$ are the same as those of $\big(\vartheta^{(1)}, \vartheta^{(2)}\big)$ under the null and the alternative. Section E of the SM provides an extensive simulation study showing that this much faster approach is equivalent, in terms of comparing the power values of the 12 tests under scrutiny, to obtaining by permutations the critical values. Both approaches lead to very close power values for all four scenarios considered.

An empirical level outside the interval $[4.86, 5.14]$ indicates that the nominal level (5%) does not fall within the corresponding realized 95% confidence interval. Given that 96 empirical levels were computed, a Bonferroni correction permits to extend the acceptable range to $[4.76, 5.24]$. The only marked discrepancy between the nominal and the empirical levels (i.e., 4.74%) occurs for the U_n-based test, the reason being that its statistic is a discrete random variable.

The following conclusions can be drawn from Table 1 and Table E.2 in the SM:

1. The optimality of $\phi_c^{(n)}(1,-1)$ is corroborated for alternative (*iii*). In general, $\phi_c^{(n)}(1,-1)$ has reasonable power against positive-correlation alternatives (*iii*) and (*iv*), but has very low power against the negative-correlation alternative (*ii*). An opposite behavior for $\phi_c^{(n)}(1,1)$ is evidenced.

2. The $\phi^{(n)}((1,-1,1,1))$ test behaves as expected in terms of merging the benefits of $\phi_c^{(n)}(1,-1)$ and $\phi_c^{(n)}(1,1)$, providing competitive power (particularly, against the tests based on U_n, $\hat{\rho}_0$, and C_n) in all scenarios and against positive/negative correlations. The test suffers a moderate loss of power with respect to the best-performing test for $\phi_c^{(n)}(1,-1)$ and $\phi_c^{(n)}(1,1)$.

3. $\phi^{(n)}(\lambda)$ is a competitive test overall. For at least one choice of $\lambda \in \{0.1, 0.5, 1.0\}$ per simulation scenario, it dominates the other tests for alternatives (*i*), (*ii*), and (*iv*), or offers competitive power for alternative (*iii*). In particular, $\phi^{(n)}(\lambda)$ dominates the three competing tests based on U_n, $\hat{\rho}_0$, and C_n for at least one choice of $\lambda \in \{0.1, 0.5, 1.0\}$ and for all scenarios. This dominance is more marked when compared with the omnibus test based on C_n.

4. The choice of λ affects the power of $\phi^{(n)}(\lambda)$. The choice $\lambda = 2$ appears to be systematically worse, which might be explained by the fact that, in this case, the kernel (3.8) can be negative. Therefore, the power of $\phi^{(n)}(\lambda)$ might be drained by reducing the value of $T_{n,\lambda}$ for certain pairwise angles $\vartheta_{jk}^{(\ell)}$, $j, k = 1, \ldots, n$, $\ell = 1, 2$.

5. $\phi_{\mathrm{dc}}^{(n)}$ performs similarly to $\phi^{(n)}(\lambda)$ and, depending on the value of λ, its power is above or below that of $\phi^{(n)}(\lambda)$.

6. The three competing tests based on U_n, $\hat{\rho}_0$, and C_n perform comparatively poorly in scenario (*i*), which does not have a positive/negative-dependence pattern. In this case, our four tests clearly outperform the other tests by a large margin.

7. The test based on G_n offers comparable power to that of characteristic-function tests when the dependence is moderate to strong.

Overall, we recommend using the test $\phi^{(n)}(\lambda)$ for $\lambda \in \{0.1, 1.0\}$ given its similar performance to $\phi_{\mathrm{dc}}^{(n)}$ and its faster application. In particular, we corroborated that applying $\phi^{(n)}(\lambda)$ is five times faster than $\phi_{\mathrm{dc}}^{(n)}$ when $n = 50$ and $B = 10^4$ permutations are used.

5. Data Applications

5.1. Wildfires

Barros, Pereira and Lund (2012) identified the existence of preferential orientations of wildfires on 102 characteristic watersheds of Portugal (see Figure 2) that were determined in a data-driven fashion. Their analysis quantified annual wildfire orientations through the axial direction (e.g., North–South) of the first principal component of a wildfire perimeter. These perimeters were obtained from Landsat imagery of Portugal after the end of the wildfire season and were then assigned to watersheds according to the position of their centroids. Wildfire orientation is likely explained by dominant weather during the Portuguese wildfire season (Barros, Pereira and Lund (2012)) and is significantly associated with the size of the burnt area (García-Portugués et al. (2014)).

We aim to formally address the existence of significant long-term and short-term temporal patterns in wildfire orientations in Portugal. As in García-Portugués et al. (2014), we focus on the 26,870 wildfires mapped during the period 1985–2005, owing to the higher resolution of the satellite imagery for this period (minimum mapping unit of 5 hectares). We then perform two data preprocessing steps. First, because a wildfire (axial) orientation is a π-periodic angular variable ϑ supported in $[0, \pi)$, we consider 2ϑ, a standard circular variable supported in $[0, 2\pi)$. With this simple transformation, the angles $\{0, \pi/2, \pi, 3\pi/2\}$ represent the $\{\text{E–W}, \text{NE–SW}, \text{N–S}, \text{NW–SE}\}$ orientations, respectively. Second, we summarize the preferred orientation of the wildfires in each watershed by their weighted circular sample mean, with weights being the product of the proportion of the explained variance and the burnt area of the wildfire perimeter. The resulting dataset contains 102 representative wildfire orientations, shown in Figure 2 for the periods 1986–1995 and 1996–2005.

When applied to the datasets displayed in Figure 2, the tests $\phi_c^{(n)}(1,1)$, $\phi_c^{(n)}(1,-1)$, $\phi^{(n)}((1,-1,1,1))$, $\phi^{(n)}(0.1)$, and $\phi^{(n)}(1)$ yielded p-values 0.0593, $0.0798, 0.0730, 0$, and 0.0003 (using 10^4 permutations for $\phi^{(n)}(\lambda)$), respectively. Therefore, significant long-term dependence is present in the orientation of wildfires. Short-term temporal dependence was also investigated by testing the null hypotheses of independence associated with 20 consecutive pairs of years in 1985–2005 and applying the correction procedure of Benjamini and Yekutieli (2001). None of the (corrected) p-values of the five tests were below the 5% significance level. For the 10% significance level, only three $\phi_c^{(n)}(1,-1)$ tests and one $\phi^{(n)}(0.1)$ test were significant. To investigate mid-term temporal dependence, we repeated the analysis for pairs of consecutive periods of five years (12 pairs) and three years (16 pairs). The proportion of (corrected) 5%-significant $\phi_c^{(n)}(1,-1)$ tests increased to 0.5 and 0.1875, respectively, while again no $\phi_c^{(n)}(1,1)$ tests were significant at any usual significance level. The corresponding proportions for the tests $\phi^{(n)}(0.1)$ and $\phi^{(n)}(1)$ were 0.75 and 0.8333

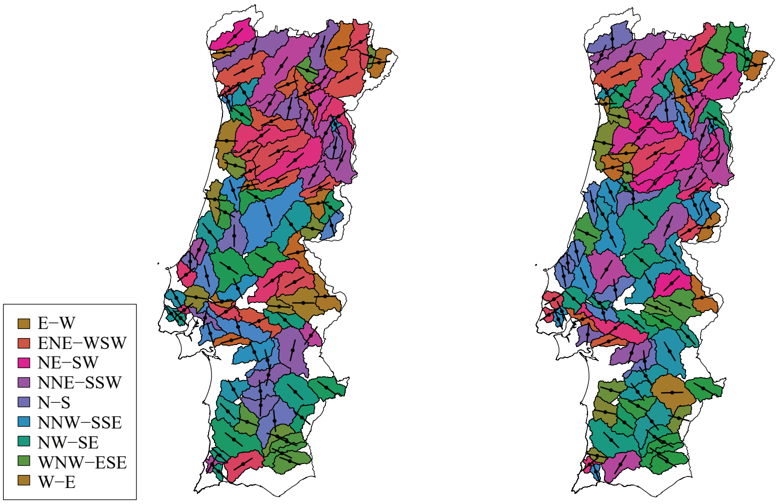

Figure 2. Weighted average orientations of wildfires from 1986–1995 (left) and 1996–2005 (right) for each of the 102 watersheds determined in Barros, Pereira and Lund (2012).

(five years), and 0 and 0.125 (three years). In conclusion, significant positive dependence of the orientations of wildfires is present among spans of ten and five years, while no significant dependence is found on consecutive years. Both conclusions support the existence of long-term drivers of the orientations of wildfires, such as the dominant weather during the wildfire season (Barros, Pereira and Lund, 2012).

5.2. Long-period comets

Long-period comets are thought to originate in the Oort cloud, a widely-accepted model posing the existence of a roughly spherical reservoir of icy planetesimals in the limits of the Solar System. It is believed that these icy planetesimals become long-period comets when randomly captured in heliocentric orbits due to the effect of several gravitational forces (see, e.g., Sec. 5 and 7.2 in Dones et al. (2015)). This conjectured origin explains the highly-characteristic nearly-isotropic distribution of the orbits of such comets (e.g., Wiegert and Tremaine (1999)). This distribution is markedly different from that of short-period comets, which originate in the flattened Kuiper belt and have orbits that cluster about the ecliptic plane.

An orbit with *inclination* $i \in [0, \pi]$ and *longitude of the ascending node* $\Omega \in [0, 2\pi)$ has the directed normal vector $(\sin(i)\sin(\Omega), -\sin(i)\cos(\Omega), \cos(i))'$ to the orbit's plane (e.g., Jupp et al. (2003)). Using this parametrization, the projected Cramér–von Mises, projected Rothman, and projected Anderson–Darling tests (García-Portugués, Navarro-Esteban and Cuesta-Albertos (2023)) reject the uniformity of the orbits of long-period comets (p-values smaller than

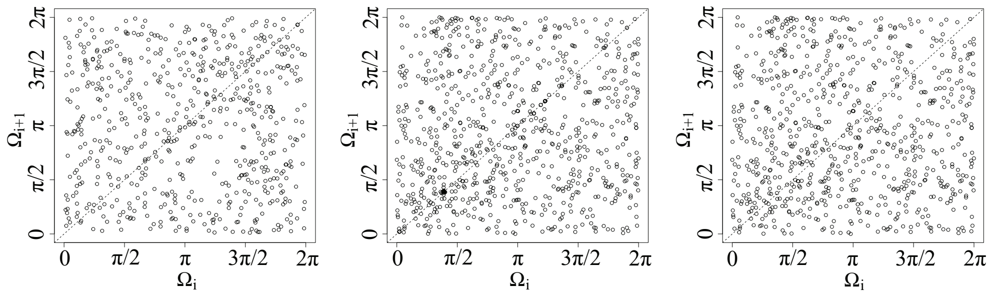

Figure 3. Scatterplots of (Ω_i, Ω_{i+1}) for long-period comets (left) and short-period comets (center and right). The clusters appearing on the diagonal of the central plot disappear once the fragments of disintegrating comets are removed from the dataset (right plot).

0.0197) using the records of the JPL Small-Body Database Search Engine (`https://ssd.jpl.nasa.gov/tools/sbdb_query.html`) as of May 2022. The rejection may be driven by a truly non-uniform population or, according to the analysis in Jupp et al. (2003), by the existence of significant observational bias in the available records. As Jupp et al. (2003) explain, bias is induced by how comet search programs maximize success detection chances by preferentially exploring regions about the ecliptic plane, because these are where most asteroids and short-period comets cluster.

A possible manifestation of observational bias, both in long- and short-period comets, is in the appearance of serial dependence in the orbits of the observed comets. To assess the existence of such serial dependence in a nonparametric way, we investigated the lag-1 dependence of the time series of Ω. We used the lagged samples (Ω_i, Ω_{i+1}), $i = 1, \ldots, n-1$, with $n = 623$ for long-period comets and $n = 905$ for short-period comets (see Figure 3), and applied to them several of the new tests of independence. The dataset is available in the `comets` object of the `sphunif` R package (v. 1.0.2) (García-Portugués and Verdebout (2022)), and is sorted using JPL's database ID, which is assigned chronologically based on the discovery of new comets.

The tests $\phi_c^{(n)}(1,1)$, $\phi_c^{(n)}(1,-1)$, $\phi^{(n)}((1,-1,1,1))$, $\phi^{(n)}(0.1)$, and $\phi^{(n)}(1)$ yielded p-values 0.6063, 0.3710, 0.8941, 0.1745, and 0.3767, respectively, for the lagged sample of long-period comets. Therefore, no evidence against lag-1 independence on the series $\{\Omega_i\}_{i=1}^n$ for long-period comets is found. Thus, if significant observational bias is present, it does not significantly induce the most obvious form of serial dependence on Ω. For short-period comets, the p-values were 0.0085, 4.8×10^{-8}, 2.9×10^{-7}, 0, and 0, signaling significant lag-1 dependence on the series of longitudes. A data inspection reveals that this rejection is a consequence of the clusters formed by fragments of disintegrating comets (see the central plot of Figure 3). For example, there is a sequence of 68 records corresponding to fragments of the "73P/Schwassmann–Wachmann 3" comet. After removing 121 fragment records, the tests gave p-values 0.6702, 0.5066, 0.9609, 0.5608, and 0.6207, hence not rejecting lag-1 independence on the longitudes of non-disintegrating short-period comets. The same test decisions at the 5% significance level were obtained when using lags of order two and three in the whole analysis, and when first sorting the database records according to the dates of the first observations used in the fit of the orbits (this yields a different chronological ordering) and then repeating the whole analysis while applying the correction procedure of Benjamini and Yekutieli (2001).

Supplementary Material

The Supplementary Material (SM) contains the proofs of the stated results. In addition, it provides the derivation of the covariance matrix $\mathbf{\Sigma}$, details the permutation algorithm applied to $\phi^{(n)}(\lambda)$, and gives further simulation results.

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