A COPULA-MODEL BASED SEMIPARAMETRIC INTERACTION TEST UNDER THE CASE-CONTROL DESIGN

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Supplementary Material

The consistency and the asymptotic normality of the pseudo-MLE is established in S1, and a proof of the second equation in (5.2) is given in S2.

S1 Consistency and asymptotic normality of pseudo-MLE

We provide the proof for the case that the marginal cumulative distribution functions F_X and F_Y are estimated with the empirical distribution functions. We focus on continuous-continuous scenario since the other two scenarios can be dealt with similarly. Let $\eta = (\beta, \gamma, \xi, \theta)$ and $\hat{\eta} = (\hat{\beta}, \hat{\gamma}, \hat{\xi}, \hat{\theta})$. Let the true values of η, F_X, and F_Y be η_0, F_{0X}, and F_{0Y}, respectively. Denote

$$c_\theta(u, v; \theta) = \frac{\partial c(u, v; \theta)}{\partial \theta}, c_X(u, v; \theta) = \frac{\partial c(u, v; \theta)}{\partial u}, c_Y(u, v; \theta) = \frac{\partial c(u, v; \theta)}{\partial v},$$

$$c_{\theta\theta}(u, v; \theta) = \frac{\partial^2 c(u, v; \theta)}{\partial \theta^2}, c_{\theta X}(u, v; \theta) = \frac{\partial^2 c(u, v; \theta)}{\partial \theta \partial u}, c_{\theta Y}(u, v; \theta) = \frac{\partial^2 c(u, v; \theta)}{\partial \theta \partial v}.$$

In the following we provide a proof of large sample properties for the pseudo-MLE of η, to this end, we assume the following conditions hold:

(c1) The logistic model and copula model hold true and $c(u, v; \theta)$ is identifiable in θ.

(c2) The sample size $n = n_0 + n_1$ goes to infinity with n_1/n_0 fixed at a value $\in (0, 1)$.

(c3) Some regularity conditions on the copula function $c(u, v; \theta)$ and the marginal distributions of X and Y are satisfied.

First mimic Gong and Samaniego (1981) we show that there exists a local maxima, say $\hat{\eta}$, of the pseudo-likelihood function that is consistent for η_0.

Under the regular condition (c3), we have that $\hat{F}_X$ and $\hat{F}_Y$ are strongly consistent for F_{0X} and F_{0Y}, respectively, uniformly in t. This implies that under further regular condition $n^{-1}l_n(\eta, \hat{F}_X, \hat{F}_Y)$ converges to $\bar{l}(\eta_0, F_{0X}, F_{0Y}) \doteq n^{-1}El_n(\eta, F_{0X}, F_{0Y})$ uniformly in η. Under condition (c1), for any $\epsilon > 0$, there exist $\epsilon' > 0$ such that $\bar{l}(\eta_0, F_{0X}, F_{0Y}) > \bar{l}(\eta, F_{0X}, F_{0Y}) + \epsilon'$ for $\eta \in O_\epsilon$, where $O_\epsilon = \{||\eta - \eta_0|| < \epsilon\}$. Therefore, for any $\delta, \epsilon > 0$, there exists a sufficiently large $N = N(\delta, \epsilon)$ such that for any $n > N$, $Pr\{l_n(\eta, \hat{F}_X, \hat{F}_Y) < l_n(\eta_0, \hat{F}_X, \hat{F}_Y) \text{ for any } \eta \in O_\epsilon\} > 1-\delta$. This shows that, with probability tending to 1, there exists in O_δ a local maximizer of $l_n(\eta, \hat{F}_X, \hat{F}_Y)$. The existence of a consistent pseudo-MLE is established.

Next we show that $n^{1/2}(\hat{\eta} - \eta_0)$ converges in distribution to a multivariate normal distribution.

Write

$$\begin{aligned} & l_n(\eta, F_X, F_Y) \\ = & \sum_{i=1}^{n_0} \log c(F_X(x_{0i}), F_Y(y_{0i}); \theta) \\ & + \sum_{i=1}^{n_1} \{\log c(F_X(x_{0i}), F_Y(y_{0i}); \theta) + (\beta x_{1i} + \gamma y_{1i} + \xi x_{1i} y_{1i}) - \alpha(\eta, F_X, F_Y)\}, \end{aligned}$$

where

$$\alpha(\eta, F_X, F_Y) = \log\left\{\int\int c(F_X(x), F_Y(y); \theta) \exp(\beta x + \gamma y + \xi xy) dF_X(x) dF_Y(y)\right\}.$$

The derivative of $l_n(\eta, F_X, F_Y)$ with respect to η is

$$\begin{aligned} & l_{n,\eta}(\eta, F_X, F_Y) \\ = & \sum_{i=1}^{n_0} \begin{pmatrix} \tilde{c}(\theta, F_X, F_Y)(x_{0i}, y_{0i}) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sum_{i=1}^{n_1} \left\{ \begin{pmatrix} \tilde{c}(\theta, F_X, F_Y)(x_{1i}, y_{1i}) \\ x_{1i} \\ y_{1i} \\ x_{1i} y_{1i} \end{pmatrix} - \alpha_\eta(\eta, F_X, F_Y) \right\} \quad \text{(S1.1)} \end{aligned}$$

where

$$\tilde{c}(\theta, F_X, F_Y)(x, y) = \frac{c_\theta(F_X(x), F_Y(y); \theta)}{c(F_X(x), F_Y(y); \theta)},$$

$$\alpha_\eta(\eta, F_X, F_Y) = \frac{\partial \alpha(\eta, F_X, F_Y)}{\partial \eta} = \frac{E_1(\eta, F_X, F_Y)}{E_0(\eta, F_X, F_Y)}$$

with

$$E_0(\eta, F_X, F_Y) = \int\int c(F_X(x), F_Y(y); \theta) \exp(\beta x + \gamma y + \xi xy) dF_X(x) dF_Y(y)$$

and

$$E_1(\eta, F_X, F_Y) = \int\int \begin{pmatrix} c_\theta(F_X(x), F_Y(y); \theta) \\ xc(F_X(x), F_Y(y); \theta) \\ yc(F_X(x), F_Y(y); \theta) \\ xyc(F_X(x), F_Y(y); \theta) \end{pmatrix} \exp(\beta x + \gamma y + \xi xy) dF_X(x) dF_Y(y).$$

Let $l_{n,\eta\eta}(\eta, F_X, F_Y) = \partial^2 l_n(\eta, F_X, F_Y)/\partial\eta\partial\eta'$, then Tayor's expansion gives

$$0 = l_{n,\eta}(\hat{\eta}, \hat{F}_X, \hat{F}_Y) = l_{n,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y) + l_{n,\eta\eta}(\eta^*, \hat{F}_X, \hat{F}_Y)(\hat{\eta} - \eta_0)$$

or

$$\hat{\eta} = \eta_0 - l_{n,\eta\eta}(\eta^*, \hat{F}_X, \hat{F}_Y)^{-1} l_{n,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y), \tag{S1.2}$$

where η^* lies between η_0 and $\hat{\eta}$. Under the regular condition (c3),

$$n^{-1} \left| l_{n,\eta\eta}(\hat{\eta}^*, \hat{F}_X, \hat{F}_Y) - l_{n,\eta\eta}(\eta_0, F_{0X}, F_{0Y}) \right| \to 0 \text{ in probability}$$

as n goes to infinity. By the Law of Large Number,

$$\begin{aligned} -n^{-1} l_{n,\eta\eta}(\eta_0, F_{0X}, F_{0Y}) &\to \Sigma_V(\eta_0, F_{0X}, F_{0Y}) \\ &\doteq -\pi_0 \begin{pmatrix} s_0(\eta_0,F_{0X},F_{0Y}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \pi_1 \begin{pmatrix} s_1(\eta_0,F_{0X},F_{0Y}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\quad + \pi_1 \alpha_{\eta\eta}(\eta_0, F_{0X}, F_{0Y}), \end{aligned} \tag{S1.3}$$

where

$$s_0(\eta, F_X, F_Y) = E\left[\frac{c_{\theta\theta}(F_X(x_{01}), F_Y(y_{01}); \theta)}{c(F_X(x_{01}), F_Y(y_{01}); \theta)} - \frac{c_\theta^2(F_X(x_{01}), F_Y(y_{01}); \theta)}{c^2(F_X(x_{01}), F_Y(y_{01}); \theta)} \right],$$

$$s_1(\eta, F_X, F_Y) = E\left[\frac{c_{\theta\theta}(F_X(x_{11}), F_Y(y_{11}); \theta)}{c(F_X(x_{11}), F_Y(y_{11}); \theta)} - \frac{c_\theta^2(F_X(x_{11}), F_Y(y_{11}); \theta)}{c^2(F_X(x_{11}), F_Y(y_{11}); \theta)} \right],$$

$$\alpha_{\eta\eta}(\eta, F_X, F_Y) = \frac{E_2(\eta, F_X, F_Y)}{E_0(\eta, F_X, F_Y)} - \frac{\{E_1(\eta, F_X, F_Y)\}\{E_1(\eta, F_X, F_Y)\}^T}{E_0^2(\eta, F_X, F_Y)}$$

with

$$E_2(\eta, F_X, F_Y) = \int\int \tilde{\tilde{c}}(\theta, F_X, F_Y)(x, y) \exp(\beta x + \gamma y + \xi xy) dF_X(x) dF_Y(y)$$

and

$$\begin{aligned} &\tilde{\tilde{c}}(\theta, F_X, F_Y)(x, y) \\ &= \begin{pmatrix} c_{\theta\theta}(F_X(x),F_Y(y);\theta) & xc_\theta(F_X(x),F_Y(y);\theta) & yc_\theta(F_X(x),F_Y(y);\theta) & xyc_\theta(F_X(x),F_Y(y);\theta) \\ xc_\theta(F_X(x),F_Y(y);\theta) & x^2c(F_X(x),F_Y(y);\theta) & xyc(F_X(x),F_Y(y);\theta) & x^2yc(F_X(x),F_Y(y);\theta) \\ yc_\theta(F_X(x),F_Y(y);\theta) & xyc(F_X(x),F_Y(y);\theta) & y^2c(F_X(x),F_Y(y);\theta) & xy^2c(F_X(x),F_Y(y);\theta) \\ xyc_\theta(F_X(x),F_Y(y);\theta) & x^2yc(F_X(x),F_Y(y);\theta) & xy^2c(F_X(x),F_Y(y);\theta) & x^2y^2c(F_X(x),F_Y(y);\theta) \end{pmatrix}. \end{aligned}$$

Combining (S1.2) and (S1.3) we have

$$n^{1/2}(\hat{\eta} - \eta_0) = \Sigma_V^{-1}(\eta_0, F_{0X}, F_{0Y}) \left\{ n^{-1/2} l_{n,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y) \right\} \{1 + o_p(1)\}. \tag{S1.4}$$

In the following we derive the limit distribution of $n^{-1/2} l_{n,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y)$.

Let $F_{0n_0}(x, y)$ and $F_{1n_1}(x, y)$ denote the empirical joint distribution functions of (x_{01}, y_{01}) and (x_{11}, y_{11}), respectively, and $F_0(x, y)$ and $F_1(x, y)$ denote the joint distributions of (x_{01}, y_{01}) and (x_{11}, y_{11}), respectively.

Noticing that $E\{l_{n,\eta}(\eta_0, F_{0X}, F_{0Y})\} = 0$ or equivalently

$$\pi_0 \int \begin{pmatrix} \tilde{c}(\theta_0, F_{0X}, F_{0Y})(x,y) \\ 0 \\ 0 \\ 0 \end{pmatrix} dF_0(x,y) + \pi_1 \int \begin{pmatrix} \tilde{c}(\theta_0, F_{0X}, F_{0Y})(x,y) \\ x \\ y \\ xy \end{pmatrix} dF_1(x,y)$$
$$-\pi_1 \alpha_\eta(\eta_0, F_{0X}, F_{0Y}) = 0,$$

we have that

$$\begin{aligned} n^{-1/2} l_{n,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y) &= \pi_0^{1/2} \int \begin{pmatrix} n_0^{1/2} \tilde{c}(\theta_0, \hat{F}_X, \hat{F}_Y)(x,y) \\ 0 \\ 0 \\ 0 \end{pmatrix} dF_{0n_0}(x,y) \\ &\quad + \pi_1^{1/2} \int \begin{pmatrix} n_1^{1/2} \tilde{c}(\theta_0, \hat{F}_X, \hat{F}_Y)(x,y) \\ n_1^{1/2} x \\ n_1^{1/2} y \\ n_1^{1/2} xy \end{pmatrix} dF_{1n_1}(x,y) \\ &\quad - \pi_1^{1/2} \left\{ n_1^{1/2} \alpha_\eta(\eta_0, \hat{F}_X, \hat{F}_Y) \right\} \\ &= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7, \end{aligned} \tag{S1.5}$$

where

$$T_1 = \pi_0^{1/2} \int \begin{pmatrix} n_0^{1/2} \{\tilde{c}(\theta_0, \hat{F}_X, \hat{F}_Y)(x,y) - \tilde{c}(\theta_0, F_{0X}, F_{0Y})(x,y)\} \\ 0 \\ 0 \\ 0 \end{pmatrix} dF_0(x,y),$$

$$T_2 = \pi_1 \pi_0^{-1/2} \int \begin{pmatrix} n_0^{1/2} \{\tilde{c}(\theta_0, \hat{F}_X, \hat{F}_Y)(x,y) - \tilde{c}(\theta_0, F_{0X}, F_{0Y})(x,y)\} \\ 0 \\ 0 \\ 0 \end{pmatrix} dF_1(x,y),$$

$$T_3 = -\pi_1 \pi_0^{-1/2} \left[n_0^{1/2} \left\{ \alpha_\eta(\eta_0, \hat{F}_X, \hat{F}_Y) - \alpha_\eta(\eta_0, F_{0X}, F_{0Y}) \right\} \right],$$

$$T_4 = \pi_0^{1/2} \int \begin{pmatrix} \tilde{c}(\theta_0, F_{0X}, F_{0Y})(x,y) \\ 0 \\ 0 \\ 0 \end{pmatrix} \{n_0^{1/2} d(F_{0n_0} - F_0)(x,y)\},$$

$$T_5 = \pi_1^{1/2} \int \begin{pmatrix} \tilde{c}(\theta_0, F_{0X}, F_{0Y})(x,y) \\ x \\ y \\ xy \end{pmatrix} \{n_1^{1/2} d(F_{1n_1} - F_1)(x,y)\},$$

$$T_6 = \pi_0^{1/2} \int \begin{pmatrix} \tilde{c}(\theta_0, \hat{F}_X, \hat{F}_Y)(x,y) - \tilde{c}(\theta_0, F_{0X}, F_{0Y})(x,y) \\ 0 \\ 0 \\ 0 \end{pmatrix} \{n_0^{1/2} d(F_{0n_0} - F_0)(x,y)\},$$

$$T_7 = \pi_1^{1/2} \int \begin{pmatrix} \tilde{c}(\theta_0, \hat{F}_X, \hat{F}_Y)(x,y) - \tilde{c}(\theta_0, F_{0X}, F_{0Y})(x,y) \\ 0 \\ 0 \\ 0 \end{pmatrix} \{n_1^{1/2} d(F_{1n_1} - F_1)(x,y)\},$$

Since ${n_0}^{1/2}(F_{0n_0} - F_0)(x,y) \to O_p(1)$, ${n_1}^{1/2}(F_{1n_1} - F_1)(x,y) \to O_p(1)$, $\hat{F}_X \to F_{0X}$, $\hat{F}_Y \to F_{0Y}$, T_6 and T_7 converges to 0 in probability under the regular condition (c3).

To prove the asymptotic normality of $n^{-1/2} l_{n,\eta}(\eta_0, \hat{\eta}_X, \hat{\eta}_Y)$, we shall show that T_1, T_2, T_3, T_4, and T_5 can be written as summations of i.i.d. random vectors.

Obviously, both T_4 and T_5 are summations of i.i.d. random vectors.

By von Mises expansion and integration by parts, the first element of T_1 is approximated by

$$\begin{aligned}
&\pi_0^{1/2}\Big[\int \tilde{c}_X(\theta_0, F_{0X}, F_{0Y})(x,y)\{n_0^{1/2}(\hat{F}_X(x) - F_{0X}(x))\}dF_0(x,y) \\
&+ \int \tilde{c}_Y(\theta_0, F_{0X}, F_{0Y})(x,y)\{n_0^{1/2}(\hat{F}_Y(y) - F_{0X}(y))\}dF_0(x,y)\Big] \\
= \;& \pi_0^{1/2}\Big[\int G_{1X}(x;\theta_0, F_{0X}, F_{0Y})d\{{n_0}^{1/2}(\hat{F}_X - F_{0X})(x)\} \\
&+ \int G_{1Y}(y;\theta_0, F_{0X}, F_{0Y})d\{{n_0}^{1/2}(\hat{F}_Y - F_{0Y})(y)\}\Big], \qquad \text{(S1.6)}
\end{aligned}$$

where

$$\tilde{c}_X(\theta_0, F_X, F_Y)(x,y) = \frac{c_{\theta X}(F_X(x), F_Y(y);\theta)}{c(F_X(x), F_Y(y);\theta)} - \frac{c_\theta(F_X(x), F_Y(y);\theta)c_X(F_X(x), F_Y(y);\theta)}{c^2(F_X(x), F_Y(y);\theta)},$$

$$\tilde{c}_Y(\theta_0, F_X, F_Y)(x,y) = \frac{c_{\theta Y}(F_X(x), F_Y(y);\theta)}{c(F_X(x), F_Y(y);\theta)} - \frac{c_\theta(F_X(x), F_Y(y);\theta)c_Y(F_X(x), F_Y(y);\theta)}{c^2(F_X(x), F_Y(y);\theta)},$$

$$G_{1X}(x;\theta_0, F_{0X}, F_{0Y}) = \int_{-\infty}^{x}\left\{\int \tilde{c}_X(\theta_0, F_{0X}, F_{0Y})(u,y)f_0(u,y)dy\right\}du,$$

$$G_{1Y}(y;\theta_0, F_{0X}, F_{0Y}) = \int_{-\infty}^{y}\left\{\int \tilde{c}_Y(\theta_0, F_{0X}, F_{0Y})(x,v)f_0(x,v)dx\right\}dv,$$

with $f_0(x,y) = \partial^2 F_0(x,y)/\partial x \partial y$. It is seen from (S1.6) that T_1 is approximated by the summation of i.i.d. random vectors. Similarly, the first element of T_2 can also be approximated by the summation of i.i.d. random vectors:

$$\begin{aligned}
&\pi_1\pi_0^{-1/2}\Big[\int G_{2X}(x;\theta_0, F_{0X}, F_{0Y})d\{{n_0}^{1/2}(\hat{F}_X - F_{0X})(x)\} \\
&+ \int G_{2Y}(y;\theta_0, F_{0X}, F_{0Y})d\{{n_0}^{1/2}(\hat{F}_Y - F_{0Y})(y)\}\Big], \qquad \text{(S1.7)}
\end{aligned}$$

where

$$\begin{aligned}
&G_{2X}(x;\theta_0, F_{0X}, F_{0Y}) \\
= \;& \int_{-\infty}^{x}\left\{\int \tilde{c}_X(\theta_0, F_{0X}, F_{0Y})(u,y)\exp(\beta_0 x + \gamma_0 y + \xi_0 xy)f_0(u,y)dy\right\}du, \\
&G_{2Y}(y;\theta_0, F_{0X}, F_{0Y}) \\
= \;& \int_{-\infty}^{y}\left\{\int \tilde{c}_Y(\theta_0, F_{0X}, F_{0Y})(x,v)\exp(\beta_0 x + \gamma_0 y + \xi_0 xy)f_0(x,v)dx\right\}dv.
\end{aligned}$$

Now we show that T_3 can be approximated by the summation of i.i.d. random vectors. Applying von Mises expansion, we get

$$\begin{aligned}
& {n_0}^{1/2}\{\alpha_\eta(\eta_0, \hat{F}_X, \hat{F}_Y) - \alpha_\eta(\eta_0, F_{0X}, F_{0Y})\} \\
= \quad & \frac{W_{1X}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_X)}{E_0(\eta_0, F_{0X}, F_{0Y})} - \frac{E_1(\eta_0, F_{0X}, F_{0Y}) W_{0X}(\eta_0, F_{0X}, F_{0X}, \hat{F}_X)}{E_0^2(\eta_0, F_{0X}, F_{0Y})} \\
& + \frac{W_{1Y}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_Y)}{E_0(\eta_0, F_{0X}, F_{0Y})} - \frac{E_1(\eta_0, F_{0X}, F_{0Y}) W_{0Y}(\eta_0, F_{0X}, F_{0X}, \hat{F}_Y)}{E_0^2(\eta_0, F_{0X}, F_{0Y})} \\
& + o_p(1). \qquad \text{(S1.8)}
\end{aligned}$$

Here

$$\begin{aligned}
& W_{1X}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_X) \\
= \quad & \int\int e_{1X}(\eta_0, F_{0X}, F_{0Y})(x, y)\{{n_0}^{1/2}(\hat{F}_X(x) - F_{0X}(x))\} dF_{0X}(x) dF_{0Y}(y) \\
& + \int\int e_1(\eta_0, F_{0X}, F_{0Y})(x, y)\{{n_0}^{1/2} d(\hat{F}_X - F_{0X})(x)\} dF_{0Y}(y),
\end{aligned}$$

$$\begin{aligned}
& W_{0X}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_X) \\
= \quad & \int\int e_{0X}(\eta_0, F_{0X}, F_{0Y})(x, y)\{{n_0}^{1/2}(\hat{F}_X(x) - F_{0X}(x))\} dF_{0X}(x) dF_{0Y}(y) \\
& + \int\int e_0(\eta_0, F_{0X}, F_{0Y})(x, y)\{{n_0}^{1/2} d(\hat{F}_X - F_{0X})(x)\} dF_{0Y}(y),
\end{aligned}$$

$$\begin{aligned}
& W_{1Y}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_Y) \\
= \quad & \int\int e_{1Y}(\eta_0, F_{0X}, F_{0Y})(x, y)\{{n_0}^{1/2}(\hat{F}_Y(y) - F_{0Y}(y))\} dF_{0X}(x) dF_{0Y}(y) \\
& + \int\int e_1(\eta_0, F_{0X}, F_{0Y})(x, y) dF_{0X}(x)\{{n_0}^{1/2} d(\hat{F}_Y - F_{0Y})(y)\},
\end{aligned}$$

$$\begin{aligned}
& W_{0Y}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_Y) \\
= \quad & \int\int e_{0Y(}\eta_0, F_{0X}, F_{0Y})(x, y)\{{n_0}^{1/2}(\hat{F}_Y(y) - F_{0Y}(y))\} dF_{0X}(x) dF_{0Y}(y) \\
& + \int\int e_0(\eta_0, F_{0X}, F_{0Y})(x, y) dF_{0X}(x)\{{n_0}^{1/2} d(\hat{F}_Y - F_{0Y})(y)\},
\end{aligned}$$

with

$$\begin{aligned}
e_0(\eta, F_X, F_Y)(x, y) &= c(F_X(x), F_Y(y); \theta) \exp(\beta x + \gamma y + \xi xy), \\
e_{0X}(\eta, F_X, F_Y)(x, y) &= c_X(F_X(x), F_Y(y); \theta) \exp(\beta x + \gamma y + \xi xy), \\
e_{0Y}(\eta, F_X, F_Y)(x, y) &= c_Y(F_X(x), F_Y(y); \theta) \exp(\beta x + \gamma y + \xi xy, \\
e_1(\eta, F_X, F_Y)(x, y) &= \begin{pmatrix} c_\theta(F_X(x), F_Y(y); \theta) \\ xc(F_X(x), F_Y(y); \theta) \\ yc(F_X(x), F_Y(y); \theta) \\ xyc(F_X(x), F_Y(y); \theta) \end{pmatrix} \exp(\beta x + \gamma y + \xi xy),
\end{aligned}$$

$$e_{1X}(\eta, F_X, F_Y)(x,y) = \begin{pmatrix} c_{\theta X}(F_X(x),F_Y(y);\theta) \\ xc_X(F_X(x),F_Y(y);\theta) \\ yc_X(F_X(x),F_Y(y);\theta) \\ xyc_X(F_X(x),F_Y(y);\theta) \end{pmatrix} \exp(\beta x + \gamma y + \xi xy),$$

$$e_{1Y}(\eta, F_X, F_Y)(x,y) = \begin{pmatrix} c_{\theta Y}(F_X(x),F_Y(y);\theta) \\ xc_Y(F_X(x),F_Y(y);\theta) \\ yc_Y(F_X(x),F_Y(y);\theta) \\ xyc_Y(F_X(x),F_Y(y);\theta) \end{pmatrix} \exp(\beta x + \gamma y + \xi xy).$$

Using integration by parts, we have

$$W_{1X}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_X) = \int H_{1X}(x; \eta_0, F_{0X}, F_{0Y})\{n_0^{1/2} d(\hat{F}_X - F_{0X})(x)\} \tag{S1.9}$$

with

$$\begin{aligned} H_{1X}(x; \eta, F_X, F_Y) &= \int_{-\infty}^{x} \left\{ \int e_{1X}(\eta, F_X, F_Y)(u,y) dF_Y(y) \right\} dF_X(u) \\ &\quad + \int e_1(\eta, F_X, F_Y)(x,y) dF_Y(y), \end{aligned}$$

$$W_{1Y}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_Y) = \int H_{1Y}(y; \eta_0, F_{0X}, F_{0Y}))\{n_0^{1/2} d(\hat{F}_Y - F_{0Y})(y)\} \tag{S1.10}$$

with

$$\begin{aligned} H_{1Y}(y; \eta, F_X, F_Y) &= \int_{-\infty}^{y} \left\{ \int e_{1Y}(\eta, F_X, F_Y)(x,v) dF_X(x) \right\} dF_Y(v) \\ &\quad + \int e_1(\eta, F_X, F_Y)(x,y) dF_X(x), \end{aligned}$$

$$W_{0X}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_X) = \int H_{0X}(x; \eta_0, F_{0X}, F_{0Y})\{n_0^{1/2} d(\hat{F}_X - F_{0X})(x)\} \tag{S1.11}$$

with

$$\begin{aligned} H_{0X}(x; \eta, F_X, F_Y) &= \int_{-\infty}^{x} \left\{ \int e_{0X}(\eta, F_X, F_Y)(u,y) dF_Y(y) \right\} dF_X(u) \\ &\quad + \int e_0(\eta, F_X, F_Y)(x,y) dF_Y(y), \end{aligned}$$

$$W_{0Y}(\eta_0, F_{0X}, F_{0Y}, \hat{F}_Y) = \int H_{0Y}(y; \eta_0, F_{0X}, F_{0Y}))\{n_0^{1/2} d(\hat{F}_Y - F_{0Y})(y)\} \tag{S1.12}$$

with

$$\begin{aligned} H_{0Y}(y; \eta, F_X, F_Y) &= \int_{-\infty}^{y} \left\{ \int e_{0Y}(\eta, F_X, F_Y)(x,v) dF_X(x) \right\} dF_Y(v) \\ &\quad + \int e_0(\eta, F_X, F_Y)(x,y) dF_X(x). \end{aligned}$$

It is seen that (S1.9), (S1.10), (S1.11), and (S1.12) are summations of i.i.d. random variables, so is T_3 from (S1.8).

From (S1.6), (S1.7), and (S1.8), we have the following asymptotic expression:

$$
\begin{aligned}
& n^{-1/2} l_{n,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y) \\
= & \int \Big[\pi_0^{1/2} G_{1X}(x;\eta_0, F_{0X}, F_{0Y}) + \pi_1 \pi_0^{-1/2} \Big\{ G_{2X}(x;\eta_0, F_{0X}, F_{0Y}) \\
& \quad - \frac{H_{1X}(x;\eta_0, F_{0X}, F_{0Y})}{E_0(\eta_0, F_{0X}, F_{0Y})} + \frac{E_1(\eta_0, F_{0X}, F_{0Y})\{H_{0X}(x;\eta_0, F_{0X}, F_{0Y})\}}{E_0^2(\eta_0, F_{0X}, F_{0Y})} \Big\} \Big] \\
& \quad \times \{ n_0^{1/2} d(\hat{F}_X - F_{0X})(x) \} \\
& + \int \Big[\pi_0^{1/2} G_{1Y}(y;\eta_0, F_{0X}, F_{0Y}) + \pi_1 \pi_0^{-1/2} \Big\{ G_{2Y}(y;\eta_0, F_{0X}, F_{0Y}) \\
& \quad - \frac{H_{1Y}(y;\eta_0, F_{0X}, F_{0Y})}{E_0(\eta_0, F_{0X}, F_{0Y})} + \frac{E_1(\eta_0, F_{0X}, F_{0Y})\{H_{0Y}(y;\eta_0, F_{0X}, F_{0Y})\}}{E_0^2(\eta_0, F_{0X}, F_{0Y})} \Big\} \Big] \\
& \quad \times \{ n_0^{1/2} d(\hat{F}_Y - F_{0Y})(y) \} \\
& + \int \pi_0^{1/2} \begin{pmatrix} \tilde{c}(\theta_0, F_{0X}, F_{0Y})(x,y) \\ 0 \\ 0 \\ 0 \end{pmatrix} \{ n_0^{1/2} d(F_{0n_0} - F_0)(x,y) \} \\
& + \int \pi_1^{1/2} \begin{pmatrix} \tilde{c}(\theta_0, F_{0X}, F_{0Y})(x,y) \\ x \\ y \\ xy \end{pmatrix} \{ n_1^{1/2} d(F_{1n_1} - F_1)(x,y) \} \\
& + o_p(1).
\end{aligned}
$$

We can see that $n^{-1/2} l_{n,\eta}(\eta_0, \hat{F}_X, \hat{F}_Y)$ is approximated by the summation of several i.i.d. random vector summations, thus it is asymptotically normally distributed with expectation 0 and a variance-covariance, say $\Sigma_S(\eta_0, F_{0X}, F_{0Y})$, which is quite complicated because the first two terms are correlated with the later two terms and it does not have a close form.

Now the limit distribution of $n^{1/2}(\hat{\eta} - \eta_0)$ is the multivariate normal with expectation 0 and variance-covariance $\Sigma_V^{-1}(\eta_0, F_{0X}, F_{0Y}) \Sigma_S(\eta_0, F_{0X}, F_{0Y}) \Sigma_V^{-1}(\eta_0, F_{0X}, F_{0Y})$.

S2 Derivation of the second equation in (5.2)

The Gaussian copula function is

$$C(u, v; \theta) = \Phi_\theta(\Phi^{-1}(u), \Phi^{-1}(v)).$$

The derivative of $C(u, v; \theta)$ with respect to v is

$$C_2(u, v; \theta) = \Phi_\theta^{(2)}\{\Phi^{-1}(u), \Phi^{-1}(v)\} \frac{\partial \Phi^{-1}(v)}{\partial v}.$$

Here

$$\Phi_\theta^{(2)}(x, y) = \frac{\partial \Phi_\theta(x, y)}{\partial y} = \Phi \left\{ \frac{x - \theta y}{(1 - \theta^2)^{1/2}} \right\} \phi(y),$$

which is the marginal density function of Y at y times the cumulative distribution function of X given $Y = y$ at x. Therefore,

$$\begin{aligned} C_2(u, v; \theta) &= \Phi\left\{\frac{\Phi^{-1}(u) - \theta\Phi^{-1}(v)}{(1-\theta^2)^{1/2}}\right\}\phi(\Phi^{-1}(v))\frac{\partial\Phi^{-1}(v)}{\partial v} \\ &= \Phi\left\{\frac{\Phi^{-1}(u) - \theta\Phi^{-1}(v)}{(1-\theta^2)^{1/2}}\right\}. \end{aligned}$$