# Sequential Analysis of Cox Model under Response Dependent Allocation

Xiaolong Luo, Gongjun Xu and Zhiliang Ying

*Celgene Corporation, Columbia University and Columbia University*

**Supplementary Material**

## S1 Proof of Lemma 6

To simplify notation, we let $f_n(t,s,u) = \bar{Z}_n(\beta_0; t, s-u)$ and

$$g(t,s,u) = \frac{\bar{E}_1(t-(s-u), s-u)}{\bar{E}_0(t-(s-u), s-u)}.$$

For a subject who enrolled into the study at time $u$, define, for $s \in [u, \tau]$, counting measure

$$p_{n,u}(ds) = I(u + \tilde{T}_u = s).$$

Under the $\sigma$-filtration $\mathcal{F}_{n,t}$, it is easy to see the compensator for $p_{n,u}(ds)$ is

$$q_{n,u}(ds) = I_{(\tilde{T}_u \geq s-u)} \exp\{\beta' Z_u\} \lambda_0(s-u) ds.$$

Thus,

$$M_{n,u}(ds) = I_{(u<s)}[p_{n,u}(ds) - q_{n,u}(ds)]$$

is a martingale measure. Comparing this with (4), it follows that $M_i(t,s)$ is a martingale as a process in $s$, since for the $i$th subject with entry time $u = U_i$, $M_{n,U_i}(ds) = I(U_i + \tilde{T}_i \in ds) - q_{n,U_i}(ds) = M_i(t, d(s-U_i))$ for $s > U_i$. Define martingale integral

$$M_{n,u}(t) = \int_u^t M_{n,u}(ds),$$

which is the total measure on interval $[u,t]$. Let

$$M_n(t,du) = \left[\int_u^t M_{n,u}(ds)\right] dR(u) = M_{n,u}(t) dR(u),$$

which defines a random measure along entry time for subjects who enrolled into the study before time $t$.

Under the above notation, for $M_n(t,\vartheta)$ defined as in (9), we have the following identity

$$M_n(t,\vartheta) = \int_0^t \int_0^\vartheta M_n(ds\,du) = \int_0^\vartheta M_n(t,du).$$

Note that from Lemma 1, $M_n(t,\vartheta)$ is a martingale along both calendar and entry times, i.e., $M_n(t,\vartheta)$ is a martingale in $t$ for any fixed $\vartheta$ and a martingale in $\vartheta$ for any $t$. When $\vartheta = t$, we have $M_n(t,t) = \int_0^t \int_0^t M_n(ds\,du)$, which is $M_n(t)$. Similarly, define random integral $\tilde{M}_n(w,\vartheta)$ with respect to survival time $w$ and entry time $\vartheta$ by

$$\tilde{M}_n(w,\vartheta) = \int_0^\vartheta M_n(w+u,du) \left( = \int_0^\vartheta M_{n,u}(w+u)dR(u) \right). \tag{S1.1}$$

Note that $\tilde{M}_n(w,\vartheta)$ is defined on the information observed before entry time $\vartheta$ and survival time $w$.

To prove Lemma 6, we need the following two propositions, whose proofs are given in Sections S2 and S2, respectively. Proposition 1 shows that $M_n(t,\vartheta)/\sqrt{n}$ is tight along calendar and entry times while Proposition 2 shows the tightness property for $\tilde{M}_n(w,\vartheta)/\sqrt{n}$ along survival and entry times.

**Proposition 1** *Under Conditions C1 and C2, for any $\epsilon > 0$, there exist constant $n_0 < \infty$ and partition $0 = u_{n,0} \leq u_{n,1} \leq \cdots \leq u_{n,n_0} = \tau$ such that for all large $n$,*

$$P\left( \max_{0\leq j<n_0} \sup_{\substack{\vartheta\in[u_{n,j},u_{n,j+1}];\\ 0\leq t\leq\tau}} |W_{n,t,\vartheta} - W_{n,t,u_{n,j}}| \geq \epsilon \right) \leq \epsilon,$$

*where $W_{n,t,\vartheta} = M_n(t,\vartheta)/\sqrt{n}$.*

**Proposition 2** *Under Conditions C1 and C2, for any $\epsilon > 0$, there exist partitions $0 = w_0 < w_1 < \cdots < w_{N_0} = \tau$ and $0 = u_{n,0} \leq u_{n,1} \leq \cdots \leq u_{n,n_0} = \tau$ such that for all large $n$,*

$$P\left( \max_{\substack{0\leq j<n_0\\ 0\leq k<N_0}} \sup_{\substack{\vartheta\in[u_{n,j},u_{n,j+1}]\\ w\in[w_k,w_{k+1}]}} |\tilde{W}_{n,w,\vartheta} - \tilde{W}_{n,w_k,u_{n,j}}| \geq \epsilon \right) \leq \epsilon,$$

*where $\tilde{W}_{n,w,\vartheta} = \tilde{M}_n(w,\vartheta)/\sqrt{n}$.*

**Proof of Lemma 6.** For any $(\vartheta,t)$ such that $0 \leq \vartheta \leq t \leq \tau$, by changing the integration order, we have that

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\int_0^t\int_0^{\vartheta}(f_n(t,s,u)-g(t,s,u))M_n(ds\,du)\\
=\;&\frac{1}{\sqrt{n}}\int_0^t\int_0^{s\wedge\vartheta}(f_n(t,s,u)-g(t,s,u))(p_n(ds\,du)-q_n(ds\,du))\\
=\;&\frac{1}{\sqrt{n}}\int_0^{\vartheta}\int_u^t(f_n(t,s,u)-g(t,s,u))(p_n(du\,ds)-q_n(du\,ds))\\
=\;&\frac{1}{\sqrt{n}}\int_0^{\vartheta}\left[\int_u^t(f_n(t,s,u)-g(t,s,u))M_{n,u}(ds)\right]dR(u)\\
=\;&\frac{1}{\sqrt{n}}\int_0^{\vartheta}\Big[M_{n,u}(t)(f_n(t,t,u)-g(t,t,u))-M_{n,u}(u)(f_n(t,u,u)-g(t,u,u))\\
&-\int_u^t M_{n,u}(s)(f_n(t,ds,u)-g(t,ds,u))\Big]dR(u)+o_p(1), \qquad \text{(S1.2)}
\end{aligned}$$

where the last equation follows from the integration-by-parts formula. Inclusion of $o_p(1)$ is due to the discontinuity of both the integrand and the integrator functions when the integration-by-parts formula is used. Therefore, by the definition of $M_n(t,du)$ and the fact that $M_{n,u}(u)=0$, we get

$$\begin{aligned}
\text{(S1.2)}\;=\;&\frac{1}{\sqrt{n}}\int_0^{\vartheta}(f_n(t,t,u)-g(t,t,u))M_n(t,du)\\
&-\frac{1}{\sqrt{n}}\int_0^{\vartheta}\left[\int_u^t M_{n,u}(s)(f_n(t,ds,u)-g(t,ds,u))\right]dR(u)+o_p(1). \qquad \text{(S1.3)}
\end{aligned}$$

In view of (S1.3), it remains to show that the two leading terms in (S1.3) are negligible.

For the first term, taking integration by parts, we have that

$$\begin{aligned}
&\left|\frac{1}{\sqrt{n}}\int_0^{\vartheta}(f_n(t,t,u)-g(t,t,u))M_n(t,du)\right|\\
=\;&\Big|\frac{1}{\sqrt{n}}(f_n(t,t,\vartheta)-g(t,t,\vartheta))M_n(t,\vartheta)-\frac{1}{\sqrt{n}}(f_n(t,t,0)-g(t,t,0))M_n(t,0)\\
&-\frac{1}{\sqrt{n}}\int_0^{\vartheta}M_n(t,u)(f_n(t,t,du)-g(t,t,du))\Big|+o_p(1)\\
\le\;&\left|\frac{1}{\sqrt{n}}(f_n(t,t,\vartheta)-g(t,t,\vartheta))M_n(t,\vartheta)\right|\\
&+\frac{1}{\sqrt{n}}\left|\int_0^{\vartheta}M_n(t,u)(f_n(t,t,du)-g(t,t,du))\right|+o_p(1). \qquad \text{(S1.4)}
\end{aligned}$$

From Proposition 1, we have, for any $\epsilon>0$, there exists a partition $0=u_{n,0}\le u_{n,1}\le$

$\cdots \le u_{n,n_0} = \tau$ such that for all large $n$,

$$P\left(\sup_{i;u\in(u_{n,i},u_{n,i+1}]} \frac{1}{\sqrt{n}}|M_n(t,u_{n,i+1}) - M_n(t,u)| < \epsilon\right) \ge 1-\epsilon.$$

Combining this with (S1.4), for all large $n$, the following result holds uniformly on $0 \le \vartheta \le t \le \tau$ with probability at least $1-2\epsilon$:

$$\begin{aligned}
\text{(S1.4)} \quad &\le \left|\frac{1}{\sqrt{n}}(f_n(t,t,\vartheta) - g(t,t,\vartheta))M_n(t,\vartheta)\right| \\
&\quad + \frac{1}{\sqrt{n}}\sum_{i=1}^{n_0}\left|\int_{u_{n,i-1}}^{u_{n,i}} M_n(t,u_{n,i})(f_n(t,t,du) - g(t,t,du))\right| + 2\epsilon K \\
&\le 3\epsilon K, \qquad \text{(S1.5)}
\end{aligned}$$

where $K$ is the total variation bound for $f_n(t,s,u)(= f_n(t,s-u,0))$, and the last inequality follows from Lenglart's inequality (Lemma 3). Since $\epsilon$ can be arbitrarily small, the first term is negligible.

For the second term, by the definitions of $f_n$ and $g$, we have $f_n(t,s,u) = f_n(t,s-u,0)$ and $g(t,s,u) = g(t,s-u,0)$. Therefore,

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\int_0^{\vartheta}\int_u^t M_{n,u}(s)(f_n(t,ds,u) - g(t,ds,u))dR(u) \\
= &\frac{1}{\sqrt{n}}\int_0^{\vartheta}\int_u^t M_{n,u}(s)(f_n(t,d(s-u),0) - g(t,d(s-u),0))dR(u) \\
= &\frac{1}{\sqrt{n}}\int_0^{t}\left[\int_0^{(t-w)\wedge\vartheta} M_{n,u}(w+u)dR(u)\right](f_n(t,dw,0) - g(t,dw,0)) \\
= &\frac{1}{\sqrt{n}}\int_0^{t}\tilde{M}_n(w,(t-w)\wedge\vartheta)\cdot(f_n(t,dw,0) - g(t,dw,0)), \qquad \text{(S1.6)}
\end{aligned}$$

where the last equality follows from the definition of $\tilde{M}_n(w,\vartheta)$ in (S1.1). Then, by Proposition 2, there exist partitions $0 = w_0 < w_1 < \cdots < w_{N_0} = \tau$ and $0 = u_{n,0} \le u_{n,1} \le \cdots \le u_{n,n_0} = \tau$ such that for all large $n$,

$$P\left(\sup_{\substack{i,j;w\in[w_i,w_{i+1}),\\ u\in[u_{n,j},u_{n,j+1})}} \frac{1}{\sqrt{n}}|\tilde{M}_n(w,u) - \tilde{M}_n(w_i,u_{n,j})| < \epsilon\right) > 1-\epsilon.$$

Then, similarly to the derivation of (S1.5), we have that for all large $n$, the following holds with probability bigger than $1-2\epsilon$:

$$\begin{aligned}
\text{(S1.6)} \quad &\le \frac{1}{\sqrt{n}}\sum_{i=1}^{N_0}\sum_{j=1}^{n_0}\left|\tilde{M}_n(w_i,u_{n,j})\int_{w_{i-1}}^{w_i}(f_n(t,dw,0) - g(t,dw,0))\right| + 2\epsilon K \\
&\le 3\epsilon K. \qquad \text{(S1.7)}
\end{aligned}$$

Therefore the second term is also negligible. ■

# S2 Proof of Proposition 1

For the proof of Proposition 1, we shall make use of certain martingale inequalities as given in the following lemma, which is due to Lenglart, Lepingle, and Pratelli (1980).

**Lemma 3** *Let $\{W(s), 0 \le s \le \tau\}$ be a square integrable martingale process whose sample paths are right continuous with left limits. Then, for any $q > 1$, there exists a constant $C_q$ depending only on $q$, such that*

$$E\left(\sup_{s\le\tau} |W(s)|^q\right) \le C_q \left(E[\langle W\rangle(\tau)]^{q/2} + E(\sup_{s\le\tau} |\triangle W(s)|^q)\right), \qquad \text{(S2.8)}$$

*where $\langle W\rangle(s)$ denotes the predictable variation process of the martingale $\{W(s)\}$ and $\Delta W(s) = W(s) - W(s-)$.*

*Moreover, if $\sup_{s\le\tau} |\triangle W(s)| \le c$, then for any $a, b > 0$*

$$P\left(\sup_{s\le\tau} |W(s)| \ge a, \langle W\rangle(\tau) \le b\right) \le 2\exp\left(-\frac{a^2}{2b}\psi(ac/b)\right),$$

*where $\psi(x) = 2x^{-2}\{(1+x)[\log(1+x) - 1] + 1\}$.*

**Proof of Proposition 1.** Choose positive numbers $p$, $q > 1$ such that $pq/2 - p - q > 1$. Let $u_0 = 0$ and define $u_{n,j}$ recursively by

$$u_{n,j+1} = \inf\{\vartheta : \vartheta > u_{n,j}, 2\tau\tilde{K}_\tau(R(\vartheta) - R(u_{n,j})) \ge \epsilon^p n\} \wedge (u_{n,j} + \epsilon^p) \wedge \tau,$$

where $\tilde{K}_\tau$ is a constant satisfying

$$\int_A\int_I q_n(ds\,du) < \tilde{K}_\tau \int_A\int_I ds\,dR(u), \qquad \text{for any } A, I \subset [0,\tau].$$

It is easy to see from the above partition that there are at most $O(\epsilon^{-p})$ many, say $n_0$, distinct points in $[0,\tau]$. From Lemma 1, $\{W_{n,t,\vartheta},\ \mathcal{F}_{n,t},\ t \ge 0\}$ is a martingale, and we know that $u_{n,j}$, $j = 1, \cdots, n_0$, are $\{\mathcal{F}_{n,t}, 0 < t \le \tau\}$ predictable. Thus, $\{\sup_{\vartheta\in[u_{n,j},u_{n,j+1}]} |W_{n,t,\vartheta} - W_{n,t,u_{n,j}}|, \mathcal{F}_{n,t}, t \ge 0\}$ is a nonnegative submartingale. By the Morkov inequality and Doob's maximal inequality (Doob, 1953),

$$\begin{aligned}
& P\left(\max_{0\le j<n_0} \sup_{\substack{\vartheta\in[u_{n,j},u_{n,j+1}];\\ 0\le t\le\tau}} |W_{n,t,\vartheta} - W_{n,t,u_{n,j}}| \ge \epsilon\right) \\
\le\ & \frac{1}{\epsilon^q}\sum_{j=0}^{n_0-1} E\left(\sup_{\substack{\vartheta\in[u_{n,j},u_{n,j+1}];\\ 0\le t\le\tau}} |W_{n,t,\vartheta} - W_{n,t,u_{n,j}}|^q\right) \\
\le\ & \frac{1}{\epsilon^q}\sum_{j=0}^{n_0-1} \left(\frac{q}{q-1}\right)^q E\left(\sup_{\vartheta\in[u_{n,j},u_{n,j+1}]} |W_{n,\tau,\vartheta} - W_{n,\tau,u_{n,j}}|^q\right).
\end{aligned}$$

Since $\{W_{n,\tau,\vartheta}, \mathcal{F}_{n,\tau,\vartheta}, \vartheta \geq 0\}$ is a martingale and

$$\sup_{\vartheta\in[u_{n,j},u_{n,j+1}]} \triangle|W_{n,\tau,\vartheta} - W_{n,\tau,u_{n,j}}| \leq \frac{1+\tilde{K}_\tau\tau}{\sqrt{n}},$$

it follows from (S2.8) that

$$\begin{aligned} & \frac{1}{\epsilon^q}\sum_{j=0}^{n_0-1}\left(\frac{q}{q-1}\right)^q E\Big(\sup_{\vartheta\in[u_{n,j},u_{n,j+1}]}|W_{n,\tau,\vartheta} - W_{n,\tau,u_{n,j}}|^q\Big) \\ \leq\ & \frac{1}{\epsilon^q}\sum_{j=0}^{n_0-1}\left(\frac{q}{q-1}\right)^q C_q\left(E[\langle W_{n,\tau,u_{n,j}+\cdot}\rangle(u_{n,j+1}-u_{n,j})]^{q/2} + \frac{(1+\tilde{K}_\tau\tau)^q}{n^{q/2}}\right) \\ \leq\ & C_q^*(\epsilon)^{pq/2-p-q} \leq \epsilon, \end{aligned}$$

where $C_q^*$ is a constant depending only on $q$ and the last inequality holds when $\epsilon$ is sufficiently enough. Hence the desired result follows. ∎

## S3 Proof of Proposition 2

To prove Proposition 2, we need the following lemma; see Lemma 5 in Gu and Lai (1991).

**Lemma 4** *Let $q > 0$ and $r > 1$. Let $\{W_n, n \geq 1\}$ be a sequence of random variables defined in the same probability space and let $\{g_n\}$ be a sequence of nonnegative integrable functions on a measure space $(\mathcal{X}, \mathcal{B}, \mu)$. Suppose that for every fixed $x \in \mathcal{X}$, $g(x)$ is nondecreasing in $n \leq N$ and that*

$$E|W_i - W_j|^q \leq \left(\int_{\mathcal{X}}[g_i(x) - g_j(x)]d\mu(x)\right)^r \text{ for all } 1 \leq j \leq i \leq N.$$

*Then there exists a universal constant $C_{q,r}$ depending only on $q$ and $r$ such that*

$$E\left(\sup_{n\leq N}|W_n - W_1|\right)^q \leq C_{q,r}\left(\int_{\mathcal{X}}[g_N(x) - g_1(x)]d\mu(x)\right)^r.$$

**Proof of Proposition 2.** Choose positive numbers $p, q > 1$ such that $pq/2-p-2q > 1$. Let $w_0 = 0$, and define $w_j$ recursively by $w_{j+1} = j\epsilon^p/\tilde{K}_\tau$, where $\tilde{K}_\tau$ is a constant satisfying

$$\int_A\int_I q_n(ds\,du) < \tilde{K}_\tau\int_A\int_I ds\,dR(u), \text{ for any } A, I \subset [0,\tau].$$

Denote $N_0 = \left\lfloor \tilde{K}_\tau\tau/\epsilon^p \right\rfloor + 1$, and redefine $w_{N_0} = \tau$.

Let $w_{n,i} = i\sqrt{\epsilon}/(nM)$ and $\mathcal{N}_w = \{w_{n,i} : i = 0, 1, \cdots, \lfloor \tau Mn/\sqrt{\epsilon}\rfloor + 1\}$ for some constant $M$. Then

$$P\left(\int_0^\tau\int_{u+w_{n,i}}^{u+w_{n,i+1}} p_n(du\,ds) \geq 2\right) = O(n^2)\cdot\epsilon/(nM)^2 \leq \epsilon/2$$

when $M$ is large enough. By the definition of $\tilde{K}_\tau$, for $\tilde{W}_{n,w,\vartheta} = \tilde{M}_n(w,\vartheta)/\sqrt{n}$, we have that

$$\begin{aligned} & P\left(\sup_{i, w_{n,i}\le w\le w_{n,i+1}} |\tilde{W}_{n,w,\tau} - \tilde{W}_{n,w_{n,i},\tau}| \ge 2n^{-1/2} + \tilde{K}_\tau n^{-1/2}\right) \\ \le\ & P\left(\sup_i \int_0^\tau \int_{u+w_{n,i}}^{u+w_{n,i+1}} p_n(du\, ds) \ge 2\right) \\ & + P\left(\sup_i \int_0^\tau \int_{u+w_{n,i}}^{u+w_{n,i+1}} q_n(du\, ds) \ge \tilde{K}_\tau\right) \\ \le\ & \epsilon/2. \end{aligned} \tag{S3.9}$$

Therefore, to prove Proposition 2, by (S3.9) and the martingale property for $\{\tilde{W}_{n,w,\vartheta}, \mathcal{F}_{n,\tau,\vartheta}, 0 < \vartheta \le \tau\}$ along entry time, we only need to show that for any $\epsilon > 0$,

$$P\left(\max_{0\le j<N_0} \sup_{\substack{0\le\vartheta\le\tau \\ w\in[w_j,w_{j+1}]\cap\mathcal{N}_w}} |\tilde{W}_{n,w,\vartheta} - \tilde{W}_{n,w_j,\vartheta}| \ge \epsilon\right) < \epsilon/2,$$

for all large $n$. Then, by Doob's inequality and (S2.8), similarly as in the proof of Proposition 1,

$$\begin{aligned} & P\left(\max_{0\le j<N_0} \sup_{\substack{0\le\vartheta\le\tau \\ w\in[w_j,w_{j+1}]\cap\mathcal{N}_w}} |\tilde{W}_{n,w,\vartheta} - \tilde{W}_{n,w_j,\vartheta}| \ge \epsilon\right) \\ \le\ & \frac{1}{\epsilon^q} \sum_{j=0}^{N_0-1} E\left(\sup_{\substack{0\le\vartheta\le\tau \\ w\in[w_j,w_{j+1}]\cap\mathcal{N}_w}} |\tilde{W}_{n,w,\vartheta} - \tilde{W}_{n,w_j,\vartheta}|^q\right) \\ \le\ & \frac{1}{\epsilon^q} \sum_{j=0}^{N_0-1} \left(\frac{q}{q-1}\right)^q E\left(\sup_{w\in[w_j,w_{j+1}]\cap\mathcal{N}_w} |\tilde{W}_{n,w,\tau} - \tilde{W}_{n,w_j,\tau}|^q\right). \end{aligned}$$

For any $w_{n,i}, w_{n,k} \in [w_j, w_{j+1}] \cap \mathcal{N}_w$, since $\tilde{W}_{n,w_{n,k},\vartheta} - \tilde{W}_{n,w_{n,i},\vartheta}$ is a $\{\mathcal{F}_{n,\tau,\vartheta}, \vartheta \ge 0\}$ martingale, from (S2.8) we have

$$\begin{aligned} & E\left(|\tilde{W}_{n,w_{n,k},\tau} - \tilde{W}_{n,w_{n,i},\tau}|^q\right) \\ \le\ & C_q\left(E\left[\langle \tilde{W}_{n,w_{n,k},\cdot} - \tilde{W}_{n,w_{n,i},\cdot}\rangle(\tau)\right]^{q/2} + \left(\frac{1+\tilde{K}_\tau(w_{n,k}-w_{n,i})}{n^{1/2}}\right)^q\right) \\ \le\ & C\epsilon^{-q/4}\left(\int_0^\tau \left[\tilde{K}_\tau \cdot 1(x \le w_{n,k}) - \tilde{K}_\tau \cdot 1(x \le w_{n,i})\right] dx\right)^{q/2}, \end{aligned}$$

where $C$ is a constant. Then from Lemma 4, there exists constant $C^* > 0$ such that for all large $n$,

$$
\begin{aligned}
& \frac{1}{\epsilon^q} \sum_{j=0}^{N_0-1} \left(\frac{q}{q-1}\right)^q E\left( \sup_{w\in[w_j, w_{j+1}]\cap\mathcal{N}_w} |\tilde{W}_{n,w,\tau} - \tilde{W}_{n,w_j,\tau}|^q \right) \\
\leq \;& \frac{1}{\epsilon^q} \sum_{j=0}^{N_0-1} \left(\frac{q}{q-1}\right)^q C\epsilon^{-q/4} \left( \int_0^\tau \tilde{K}_\tau \cdot 1(w_{n,i_{w_j}} < x \leq w_{n,i_{w_{j+1}+1}})\, dx \right)^{q/2} \\
\leq \;& C^*(2\epsilon)^{pq/2-p-5q/4}, \qquad \text{(S3.10)}
\end{aligned}
$$

where $i_{w_j} = \max\{i : w_{n,i} \leq w_j\}$. By choosing $\epsilon$ sufficiently small, we have that the last term in (S3.10) must be smaller than $\epsilon$. Hence the desired conclusion follows. ■

## S4 Lemma 5

Lemma 5 is used in the proof of Theorem 8. It is a restatement of Lemma A.5 in Bilias *et al.* (1997).

**Lemma 5** *Consider a set of functions $\{f_{n,\alpha} : n \geq 1, \alpha \in A\}$ from $R^d$ to $R^d$. Suppose that (i) $\frac{\partial}{\partial\theta} f_{n,\alpha}(\theta)$ are nonnegative definite for all $n$, $\alpha$, $\theta$; (ii) $\sup_\alpha |f_{n,\alpha}(\theta_0)| \to 0$ as $n \to \infty$; (iii) there exists a neighborhood of $\theta_0$, denoted by $\mathcal{N}(\theta_0)$, such that*

$$
\liminf_{n\to\infty} \inf_{\theta\in\mathcal{N}(\theta_0)} \inf_{a\in A} \lambda_{min} \left( \frac{\partial f_{n,\alpha}(\theta)}{\partial\theta} \right) > 0,
$$

*where $\lambda_{min}$ is the minimum eigenvalue as defined in C4. Then there exists $n_0$ such that for every $n > n_0$ and $\alpha \in A$, $f_{n,\alpha}$ has a unique root $\theta_{n,\alpha}$ and $\sup_{\alpha\in A} |\theta_{n,\alpha} - \theta_0| \to 0$.*

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Xiaolong Luo

Celgene Corporation

E-mail: xluo@celgene.com

Gongjun Xu and Zhiliang Ying

Department of Statistics, Columbia University

E-mail: gongjun, zying@stat.columbia.edu